# ERGODIC THEORY, HW # 3

**Homework will not be collected or graded. Nevertheless, please, make sure that you understand how to solve the problems. Interaction between students is strongly encouraged.**

## Problem 1.

Is it possible to find a topological conjugacy between $\sigma_2 : \Sigma_2 \to \Sigma_2$ and $\sigma_3 : \Sigma_3 \to \Sigma_3$? Semiconjugacy?

## Problem 2.

Set $B = \{(x, y) | \, |x| < 1, |y| < 1\}$ and consider a map $h : B \times S^1 \to B \times S^1$, $h(x, y, \phi) = (10x, \frac{y}{10}, \phi)$. Let $k : B \times S^1 \to B \times S^1$ be $C^1$-close to $h$.

a) Prove that $\cap_{n\in\mathbb{Z}} k^n(B \times S^1)$ is homeomorphic to a circle.

b) Prove that $\cap_{n\in\mathbb{Z}} k^n(B \times S^1)$ is a smooth closed curve.

## Problem 3.

Consider the following map $\alpha : \Sigma_2^+ \to \Sigma_2^+$,

$$(\alpha(\omega))_i = \begin{cases} 1 - \omega_i, & if\ \omega_i = 1\ for\ \ all\ \ j < i, \\ \omega_i, & otherwise. \end{cases}$$

a) Prove that $\alpha : \Sigma_2^+ \to \Sigma_2^+$ is a homeomorphism of $\Sigma_2^+$.

b) Prove that $\alpha : \Sigma_2^+ \to \Sigma_2^+$ is uniquely ergodic (i.e. has only one Borel probability invariant measure), and describe the invariant measure.

## Problem 4.

Which of the following Bernoulli shifts are isomorphic?

a) $\sigma_1 : \left(\Sigma_4, \mu_{\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)}\right) \to \left(\Sigma_4, \mu_{\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)}\right)$;

b) $\sigma_2 : \left(\Sigma_3, \mu_{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)}\right) \to \left(\Sigma_3, \mu_{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)}\right)$;

c) $\sigma_3 : \left(\Sigma_5, \mu_{\left(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right)}\right) \to \left(\Sigma_5, \mu_{\left(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right)}\right)$.

## Problem 5.

Let $S \subset \Sigma_2$ be the set of all sequences of zero and ones that do not contain two subsequent zeros.

a) Show that $S$ is an invariant set for the topological Bernoulli shift $\sigma : \Sigma_2 \to \Sigma_2$.

b) Prove that periodic points of $\sigma$ are dense in $S$.

c) Find the number of periodic points of period $N$.

d) Prove that $\sigma : S \to S$ is transitive.