ERGODIC THEORY, MIDTERM

This is a take-home midterm. Solve these problems and turn it in either in class or via email on or before May 23, 2018. You can use any textbooks, notes, etc., but not direct help of other people.

Problem 1.

Let $f : M \to M$ be a measure preserving transformation preserving a finite measure μ. Given $k \geq 1$ and a positive measure set $A \subset M$, show that for almost every $x \in A$ there exists $n \in \mathbb{N}$ such that $f^{jn}(x) \in A$ for every $1 \leq j \leq k$.

Problem 2.

We have seen that two irrational circle rotations R_α and R_β, $\alpha, \beta \notin \mathbb{Q}$, are ergodic equivalent if and only if $\alpha = \beta$ or $\alpha = -\beta$. Under what condition two *rational* circle rotations R_p and R_q, $p, q \in \mathbb{Q}$, are ergodic equivalent?

Problem 3.

Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x - 1, & \text{if } x \geq 0; \\ x + \sqrt{2}, & \text{if } x < 0. \end{cases}$$

Prove that for any $x_0 \in \mathbb{R}$ the sequence

$$\frac{1}{n} \sum_{j=0}^{n-1} \left(f^j(x_0) \right)^2$$

converges to the limit that does not depend on x_0. Find that limit.

Problem 4.

a) In order to find a decimal representation of a number $x \in (0, 1)$ we divide the interval $(0, 1)$ into ten equal pieces, and determine the first digit $a_1 = a_1(x) \in \{0, 1, \ldots, 9\}$ depending on the small interval that contains x. Then we divide that small interval into ten equal pieces, etc.
If we had, say, two fingers on the left hand and three fingers on the right, we could developed the following analog of the decimal representation. First divide the interval $(0, 1)$ into two equal pieces, and set $a_1 = a_1(x) \in \{0, 1\}$ depending on which smaller interval contains x. Then divide that smaller interval into three equal pieces, and set $a_2 = a_2(x) \in \{0, 1, 2\}$ depending on which of those contains x. Then divide into two intervals and set $a_3 \in \{0, 1\}$, then again into three, and set $a_4 \in \{0, 1, 2\}$, etc.
Prove that for Lebesgue almost every x the limit of the sequence $\frac{1}{n}(a_1 + \ldots + a_n)$ is well defined. Find that limit.
What would be that limit if we had 5 fingers on the left hand and 7 on the right?

b) What if we had only three fingers in total, but of different length? Suppose that on n-th step we divide the interval into three subinterval of lengthes proportional to α, β, and γ, $\alpha + \beta + \gamma = 1$, and choose the next digit $a_n \in \{0, 1, 2\}$ depending on which interval contains x. Prove that in this case we would also have well defined limit of the sequence $\frac{1}{n}(a_1 + \ldots + a_n)$. Find that limit.

Problem 5.

Let μ be a probability measure invariant under some invertible transformation $f : M \to M$. Assume that (f^n, μ) is ergodic for every $n \geq 1$. Show that if ϕ is a non-constant eigenfunction of the Koopman operator U_f, then the corresponding eigenvalue is *not* a root of unity.