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Certain New Robust Properties of Invariant Sets and Attractors of Dynamical Systems*

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§1. Introduction

In this paper, new robust properties of (partially hyperbolic) invariant sets and attractors of diffeomorphisms are obtained, including the coexistence of dense sets of periodic points with different indices and the existence of a dense orbit with zero Lyapunov exponent. The precise statements of these properties are given below in Theorem A. Partially hyperbolic invariant sets with the property of robust transitivity were intensively studied [3–7]. A review of this field with a vast bibliography can be found in [8].

1.1. Main results.

Theorem A. *For a given finite interval $I \subset \mathbb{R}$, $0 \in I$, and for a closed manifold M, $\dim M \geqslant 3$, there exists an open set $U \subset \mathrm{Diff}^2(M)$ such that, for any $f \in U$, there is a locally maximal invariant set $\Delta \subset M$ with the following properties*:

(i) *there exist two numbers l_1 and $l_2 = l_1 + 1$ such that the hyperbolic periodic orbits with stable manifolds of dimension l_i are dense in Δ*;

(ii) *for any $\lambda \in I$, there exists an orbit dense in Δ for which one of the intermediate Lyapunov exponents is equal to λ.*

Addendum. *For $\dim M \geqslant 4$, for the set Δ in Theorem* A *we can take a partially hyperbolic attractor.*

The existence of an open set of mappings with property (i) was conjectured in [9]. Property (ii) implies the absence of the shadowing property, as was shown in [2, 10].

Systems described in Theorem A can occur under special nonlocal bifurcations.

Theorem B. *There exists an open set of one-parameter families of vector fields in $\mathbb{R}^n$, $n \geqslant 4$, such that, for each family from this set, the following assertions hold*:

(1) *the zero value of the parameter corresponds to a vector field on the boundary of the Morse–Smale set with a saddlenode cycle*;

(2) *there is an open subset $\mathscr{B} \subset \mathbb{R}$ with zero on its boundary such that the Poincaré mappings of the vector fields that correspond to the values of the parameter in this subset have a partially hyperbolic set described in Theorem* A;

(3) *the subset $\mathscr{B}$ has positive density at the origin.*

The main ingredients of the construction are random dynamical systems realized as subsystems (restrictions to invariant subsets) of smooth dynamical systems. In fact, we could use the notion of skew product instead of the notion of random dynamical system, but we use the latter to stress the promising relationship that can be a subject of subsequent investigations.

1.2. Step and mild random dynamical systems. A simple example of a random dynamical system is as follows. Consider two diffeomorphisms of a circle, f_0 and f_1, and a random bi-infinite sequence of zeros and ones,

$$\omega = \dots \omega_{-n} \dots \omega_{-1} \omega_0 \omega_1 \dots \omega_n \dots .$$

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A random dynamical system of diffeomorphisms of a circle generated by this data is the sequence $\{F_n\}$ of products of two given diffeomorphisms, where the choice of a factor at the kth place is determined by the kth element of the random sequence. For $n > 0$, this gives

$$F_n = f_{\omega_n} \circ F_{n-1}, \qquad F_0 = f_{\omega_0}.$$

This system can be included into a skew product as follows. Let Σ^2 be a space of all bi-infinite sequences of 0 and 1 and let σ be the Bernoulli shift $\Sigma^2 \to \Sigma^2$. The mapping

$$F\colon \Sigma^2 \times S^1 \to \Sigma^2 \times S^1, \qquad (\omega, \varphi) \mapsto (\sigma\omega, f_{\omega_0}(\varphi)), \tag{1.1}$$

is called a universal random dynamical system. In fact, the mapping F is determinate. However, any orbit of a random dynamical system defined at the beginning of the subsection is an orbit of some point under the action of F. A more general example can be obtained by replacing Σ^2 and S^1 by Σ^N and by a closed Riemannian manifold M, respectively:

$$F\colon \Sigma^N \times M \to \Sigma^N \times M, \qquad (\omega, \varphi) \mapsto (\sigma\omega, f_{\omega_0}(\varphi)). \tag{1.2}$$

The first factor of the product $\Sigma^N \times M$ is called the *base* and the other the *fiber*. The metric d_M on the fiber is induced by the Riemannian structure. The metric on Σ^N is defined as follows:

$$d_{\Sigma^N}(\omega, \omega') = 2^{-l}, \qquad l = \min\{j \in \mathbb{Z}^+ \mid \omega_j \neq \omega'_j \text{ or } \omega_{-j} \neq \omega'_{-j}\}. \tag{1.3}$$

Note that the restriction of the mapping F given by (1.2) to the fiber over a point ω depends not on the entire sequence ω but on its element ω_0 only. We refer to such systems as *step* random dynamical systems. In the theory of random dynamical systems, this corresponds to the case of independent and identically distributed random mappings [1]. We also consider the mappings

$$G\colon \Sigma^N \times M \to \Sigma^N \times M, \qquad (\omega, \varphi) \mapsto (\sigma\omega, f_\omega(\varphi)), \tag{1.4}$$

where $f_\omega\colon M \to M$ depends on the entire sequence ω. We refer to these mappings as *mild* random dynamical systems.

The general plan of the investigation and the addendum to Theorem A are due to the second author. The results of Sec. 2 were proved by the authors jointly. All the other results of the paper were proved by the first author. The detailed proofs (which are rather technical and lengthy) will be published elsewhere.

§2. Properties of Step Random Dynamical Systems

We show here that some properties similar to properties (i), (ii) of Theorem A manifest themselves in step dynamical systems.

2.1. Locally typical properties of step random dynamical systems. Consider the mapping F given by (1.2). It is completely determined by the choice of N diffeomorphisms $f_0, \dots, f_{N-1}$ of S^1 into itself.

Theorem 1. *For any $N \in \mathbb{N}$, there exist an interval $I \subset \mathbb{R}$, $0 \in I$, and N open sets $U_0, \dots, U_{N-1} \subset \mathrm{Diff}^1(S^1)$ such that, for any $f_0, \dots, f_{N-1}$, $f_j \in U_j$, the mapping F given by (1.2) has the following properties*:

(i) *the periodic orbits of the mapping F with the multiplier along the circle whose modulus is greater than one are dense in $\Sigma^N \times S^1$, and the same holds for the periodic orbits with the multiplier whose modulus is less than one*;

(ii) *for any $\lambda \in I$, there exists an orbit dense in $\Sigma^N \times S^1$ with the Lyapunov exponent along the circle that is equal to λ*;

(iii) *if the orbit of a sequence $\omega \in \Sigma^N$ under the Bernoulli shift σ is dense in Σ^N, then, for any $\varphi \in S^1$, the orbit of the point (ω, φ) under F is dense in $\Sigma^N \times S^1$.*

2.2. Example. Set $N = 2$. As an example of sets U_0 and U_1 in Theorem 1, one can take small C^1-neighborhoods of the following diffeomorphisms g_0 and g_1.

Let g_0 be a rotation of $S^1 = \mathbb{R}/\mathbb{Z}$ about a small angle, say, 0.1. Let g_1 be a diffeomorphism with two fixed points, namely, an attractor q and a repeller p. All other points wander from p to q under the positive iterates of g_1. We assume that $q = 0$, $p = \frac{1}{2}$, and g_1 is linear in some neighborhoods of q and p. Namely, on $O(q) = (-\frac{1}{4}, \frac{1}{4}) \subset S^1$, the mapping g_1 is an expansion $\varphi \mapsto a\varphi$, and, on $O(p) = (\frac{1}{4}, \frac{3}{4}) \subset S^1$, the mapping g_1^{-1} is similar, $\varphi \mapsto a(\varphi - \frac{1}{2}) + \frac{1}{2}$. Let $\max_{S^1}|Dg_1| \leqslant a$ and $\max_{S^1}|Dg_1^{-1}| \leqslant a$. Then the statement of Theorem 1 holds (with the interval $I = (-\frac{1}{4}\ln 2; \frac{1}{4}\ln 2)$ for $a = \sqrt{2}$).

2.3. Periodic orbits. To illustrate Theorem 1, we prove the following assertion.

Proposition 1. *Statement* (i) *of Theorem* 1 *holds for the mapping* F *given by formula* (1.1) *with* $f_0 = g_0$ *and* $f_1 = g_1$.

Proof. Let Θ be an arbitrary open set in $\Sigma^2 \times S^1$. There exists a finite sequence $\alpha_{-n} \dots \alpha_0 \dots \alpha_{n-1}$ of 0 and 1 and a small arc $J \subset S^1$, $a^n|J| < 0.1$, such that $U \times J \subset \Theta$, where $U = \{\omega \mid \omega_j = \alpha_j,\ j \in \{-n, \dots, n-1\}\}$ and $|J|$ denotes the arc length of J. Let us find a periodic point in $U \times J$ with the multiplier along the circle that is greater than one.

Let J_ω be the arc $\{\omega\} \times J$ and let $\pi\colon \Sigma^2 \times S^1 \to S^1$ be the projection along the first factor. Then, for any $\omega \in U$,

$$\pi \circ F^n J_\omega = f^+ J, \qquad \pi \circ F^{-n} J_\omega = f^- J,$$

where $f^+ = f_{\omega_{n-1}} \circ \dots \circ f_{\omega_0}$ and $f^- = f^{-1}_{\omega_{-n}} \circ \dots \circ f^{-1}_{\omega_{-1}}$. Let $J^\pm = f^\pm J$. According to our choice of J, we have $|J^-| < 0.1$. If we could find a finite composition g of the mappings g_0 and g_1 such that the image of J^+ covers J^- with strong expansion, then, under the mapping $f = (f^-)^{-1} \circ g \circ f^+\colon S^1 \to S^1$, the arc J would be covered by its image, and the restriction of f to J would be an expansion. This would imply the existence of a periodic point in J for this mapping with positive multiplier. Considering the corresponding periodic sequence in Σ^2, we could find the required periodic point in $U \times J \subset \Sigma^2 \times S^1$ for the mapping F given by (1.1). Hence, to complete the proof, it suffices to obtain the following lemma.

Lemma 1. *For any two arcs* $J^-, J^+ \subset S^1$, $|J^-| < 0.1$, *and for any* $K > 0$, *there exist a subarc* $J^{++} \subset J^+$ *and a finite composition* g *of the mappings* g_0 *and* g_1 *such that* $g(J^{++}) \supset J^-$ *and* $Dg|_{J^{++}} > K$.

Proof of Lemma 1. Let us choose $J^{++} \subset J^+$ so small that $K < 0.1/|J^{++}|$. By means of at most ten rotations, the arc J^{++} can be mapped into an expanding neighborhood $O(q)$. The application of g_1 expands the arc with multiplier a. After m applications of this process, we obtain an arc of length $a^m|J^{++}|$. Thus, for sufficiently large m, we obtain an arc with arc length greater than 0.1. Then, by at most ten rotations, this arc covers the arc J^-. By the choice of J^{++} and m, we have $a^m > K$, that is, $Dg|_{J^{++}} > K$, where g is the above composition. This completes the proof.

§3. Smooth Realization

We seek a smooth dynamical system such that its restriction to some locally maximal invariant subset is conjugate to a step random dynamical system F given by (1.2).

3.1. Skew products as smooth realizations of random dynamical systems. Consider a standard Smale horseshoe realized as a mapping of pairwise disjoint rectangles. Namely, let $D = D_0 \cup \dots \cup D_{N-1}$ and $D' = D'_0 \cup \dots \cup D'_{N-1}$, and let $D_i \cap D_j = \varnothing$ and $D'_i \cap D'_j = \varnothing$ for $i \neq j$. We denote a domain and the differential of a mapping by the same letter, but the meaning is always clear from the context. Let $S\colon D \to D'$ be a mapping such that $S|_{D_i}\colon D_i \to D'_i$ is the restriction to D_i of a linear hyperbolic map with the constants of contraction and expansion equal to $k \in (0,1)$ and $k^{-1} > 1$, respectively. Note that the coefficient k can be chosen to be arbitrarily small by a suitable choice of rectangles D and D'.

As is well known, the mapping S has an invariant set Λ homeomorphic to Σ^N, and $S|_\Lambda$ is conjugate to the Bernoulli shift $\sigma\colon \Sigma^N \to \Sigma^N$.

Consider a locally constant function $i\colon D \to \{0, \dots, N-1\}$ defined by the condition $i(x) = j \iff x \in D_j$. The skew product

$$\mathfrak{F}\colon D \times M \to D' \times M, \qquad \mathfrak{F}(x, \varphi) = (S(x), f_{i(x)}(\varphi))$$

is called a *k-realization* of system (1.2).

We can readily see that $\mathfrak{F}|_{\Lambda\times M}$ is conjugate to the mapping F in formula (1.2). The set $\Lambda \times M$ is partially hyperbolic for $\mathfrak{F}$, and its central leaves are fibers of the projection onto the first factor, $\Lambda \times M \to \Lambda$, $\Lambda \simeq \Sigma^N$.

The natural question is: What happens under a small perturbation of the mapping $\mathfrak{F}$?

3.2. Smooth perturbations of skew products. Let

$$L = \max_{i\in\{0,\dots,N-1\}} \max_{\varphi\in M}(\|Df_i(\varphi)\|, \|Df_i^{-1}(\varphi)\|).$$

Theorem 6.1 in [11] readily yields the following assertion.

Theorem 2. *Let $\mathfrak{F}\colon D \times M \to D' \times M$ be a k-realization of the mapping F. Let $kL^r < 1$ and $r \geqslant 1$. Then any C^r-diffeomorphism $\mathfrak{G}$ that is C^r-close to $\mathfrak{F}$ has an invariant subset Δ homeomorphic to $\Sigma^N \times M$, the projection $\Phi\colon (\Delta, \mathfrak{G}) \to (\Sigma^N, \sigma)$ is a semiconjugacy, and the leaves $\Phi^{-1}(\omega)$ are C^r-smooth.*

In order to investigate the properties of the mapping $\mathfrak{G}$, we need more information on the dependence of the central leaves $\Phi^{-1}(\omega)$ on the point ω in the base Σ^N.

§4. Hölder Dependence of Central Leaves

Here we sketch the proof of the fact that the central leaves in Theorem 2 depend Hölder continuously on the point in the base (that is, in Σ^N) in the C^r-norm.

4.1. Statement of the Hölder dependence theorem.

Theorem 3. *Let $\mathfrak{F}$ be the same k-realization as in Theorem 2. Then any $C^{r+\mathrm{Lip}}$-diffeomorphism $\mathfrak{G}$ that is C^r-close to $\mathfrak{F}$ has an invariant subset Δ homeomorphic to $\Sigma^N \times M$. The projection $\Phi\colon (\Delta, \mathfrak{G}) \to (\Sigma^N, \sigma)$ is a semiconjugacy, and the leaves $\Phi^{-1}(\omega)$ are C^r-smooth and depend Hölder continuously on the point $\omega \in \Sigma^N$ in the C^r-norm.*

4.2. C^r-norm on the space of central leaves. The central leaves $\Phi^{-1}(\omega)$ which arise in Theorem 2 are graphs of C^r-smooth mappings $E_\omega\colon M \to D$ in the Cartesian product $D \times M$. The C^r-distance between two central leaves $\Phi^{-1}(\omega)$ and $\Phi^{-1}(\omega')$ is defined as the C^r-distance between the corresponding mappings E_ω and $E_{\omega'}$.

Let $\mathfrak{L}^r$ be the metric space of all central leaves of the mapping $\mathfrak{G}$ with C^r-metric on it. A single central leaf $\Phi^{-1}(\omega)$ is a point of this space. Note that the mapping $\mathfrak{G}$ sends a leaf to a leaf, and hence $\mathfrak{G}$ induces a well-defined mapping $\mathfrak{f}\colon \mathfrak{L}^r \to \mathfrak{L}^r$. Since every point $\omega \in \Sigma^N$ corresponds to the leaf $\Phi^{-1}(\omega)$, it follows that the mapping $H\colon \Sigma^N \to \mathfrak{L}^r$ is also well defined. The projection $\Phi\colon (\Delta, \mathfrak{G}) \to (\Sigma^N, \sigma)$ is a semiconjugacy, and hence the mapping $H\colon \Sigma^N \to \mathfrak{L}^r$, $\omega \mapsto \Phi^{-1}(\omega)$, is a conjugacy, $H \circ \sigma = \mathfrak{f} \circ H$. To study the dependence of the central leaves in the C^r-norm on the point of the base is to study the properties of the mapping $H\colon \Sigma^N \to \mathfrak{L}^r$.

4.3. Hölder continuity of conjugacies: a classical result. We claim that the conjugacy $H\colon \Sigma^N \to \mathfrak{L}^r$, $H \circ \sigma = \mathfrak{f} \circ H$, is Hölder continuous. There is a theorem that looks very similar to the one we need.

Theorem 4 [12]. *Let Λ and Λ' be compact hyperbolic sets for diffeomorphisms f and f', respectively, and let $h\colon \Lambda \to \Lambda'$ be a topological conjugacy, $h = f'hf^{-1}$. Then both h and h^{-1} are Hölder continuous.*

Unfortunately, our spaces Σ^N and $\mathfrak{L}^r$ are not hyperbolic sets (there is no ambient space for the first space, and the ambient space is infinite-dimensional for the second one). One can replace Σ^N by a hyperbolic set Λ, but there is no hope of doing that for $\mathfrak{L}^r$. Therefore, we must generalize Theorem 4 to obtain the desired result.

4.4. Intrinsically hyperbolic sets. Here we introduce the notion of intrinsically hyperbolic set, which needs no ambient space at all but still allows us to prove an analog of Theorem 4.

Let (Λ, d) be a compact metric space and let $f\colon \Lambda \to \Lambda$ be a Lipschitz-morphism (i.e., f is a homeomorphism, and f and f^{-1} are Lipschitz continuous).

Let us define two equivalence relations $\sim_u$ and $\sim_s$ on Λ as follows:

$$x \sim_u y \iff \lim_{n\to+\infty} d(f^{-n}(x), f^{-n}(y)) = 0,$$

$$x \sim_s y \iff \lim_{n\to+\infty} d(f^{n}(x), f^{n}(y)) = 0.$$

Let us also define local manifolds, stable and unstable:

$$W^s_\varepsilon(x) = \{z \mid z \sim_s x \text{ and } d(f^n(x), f^n(z)) \leqslant \varepsilon \text{ for any } n \in \mathbb{Z}^+\},$$
$$W^u_\varepsilon(x) = \{z \mid z \sim_u x \text{ and } d(f^{-n}(x), f^{-n}(z)) \leqslant \varepsilon \text{ for any } n \in \mathbb{Z}^+\}.$$

Definition. A Lipschitz-morphism f of a compact metric space (Λ, d) is *intrinsically hyperbolic* if and only if there exist $\delta > 0$ and $\varepsilon > 0$ such that the following assertions hold.

(IH1) If $d(x,y) < \delta$, then there exists a unique point $w(x,y) = W^u_\varepsilon(x) \cap W^s_\varepsilon(y) \in \Lambda$. The mapping $(x,y) \mapsto w(x,y)$ is continuous.

(IH2) There exist constants C, ν, and ξ, where $C > 0$, $0 < \nu < 1$, and $0 < \xi < 1$, such that

$$y \in W^s_\varepsilon(x) \implies d(f^n(x), f^n(y)) \leqslant C\nu^n d(x,y), \quad n \in \mathbb{N},$$
$$y \in W^u_\varepsilon(x) \implies d(f^{-n}(x), f^{-n}(y)) \leqslant C\xi^n d(x,y), \quad n \in \mathbb{N}.$$

The simplest examples are provided by ordinary locally maximal hyperbolic sets and by the Bernoulli shifts.

Lemma 2. *Under the conditions of Theorem 3, the mapping $\mathfrak{f}\colon \mathfrak{L}^r \to \mathfrak{L}^r$ is intrinsically hyperbolic.*

4.5. Hölder continuity of conjugacies: intrinsically hyperbolic case.

Theorem 5. *Let Λ and Λ' be compact metric spaces, let Lipschitz-morphisms $f\colon \Lambda \to \Lambda$ and $f'\colon \Lambda' \to \Lambda'$ be intrinsically hyperbolic, and let $h\colon \Lambda \to \Lambda'$ be a topological conjugacy, $h = f'hf^{-1}$. Then both h and h^{-1} are Hölder continuous.*

The mapping $\mathfrak{f}\colon \mathfrak{L}^r \to \mathfrak{L}^r$ is intrinsically hyperbolic by Lemma 2. Take $\sigma\colon \Sigma^N \to \Sigma^N$ for $f\colon \Lambda \to \Lambda$ and $\mathfrak{f}\colon \mathfrak{L}^r \to \mathfrak{L}^r$ for $f'\colon \Lambda' \to \Lambda'$ in Theorem 5. Then the conjugacy $H\colon (\Sigma^N, \sigma) \to (\mathfrak{L}^r, \mathfrak{f})$ is Hölder continuous. This implies Theorem 3.

We must also be able to estimate the Hölder exponent of the conjugacy thus obtained. The following statement yields the desired estimate.

Theorem 6. *Let Λ and Λ' be compact metric spaces, let Lipschitz-morphisms $f\colon \Lambda \to \Lambda$ and $f'\colon \Lambda' \to \Lambda'$ be intrinsically hyperbolic with constants ν, ξ and ν', ξ', respectively, and let $h\colon \Lambda \to \Lambda'$ be a topological conjugacy, $h = f'hf^{-1}$.*

Let P be a Lipschitz constant for f and let Q be a Lipschitz constant for f^{-1}.

Assume that, for some $\alpha > 0$ and $\alpha' > 0$, the following inequalities hold:

$$\xi P^\alpha < 1, \quad \nu Q^\alpha < 1, \quad \xi' P^{\alpha'} < 1, \quad \nu' Q^{\alpha'} < 1.$$

Then the conjugacy h is Hölder continuous with Hölder exponent $\alpha\alpha'$.

Theorem 6 can be used to show that the Hölder exponent of the conjugacy $H\colon \Sigma^N \to \mathfrak{L}^r$ can be taken arbitrarily large by choosing a k-realization with a sufficiently small coefficient k.

Remark. Theorem 5 can be used to prove the Hölder continuity of the dependence of central leaves in the C^r-norm on the point of the base in a much more general context. It can have applications [13] that differ from those in the present paper. We are grateful to Andrei Török for this remark.

§5. Mild Random Dynamical Systems: The Idea of the Proof of Theorem A

The dynamics of the mapping $\mathfrak{G}$ in Theorem 2 restricted to some its locally maximal invariant set Δ can be described in terms of mild random dynamical systems. The properties of these systems allow us to generalize Theorem 1, and thus obtain Theorem A.

5.1. Reduction to the properties of mild random dynamical systems. Let $\mathfrak{G}$ be the same as in Theorem 3. Then $\mathfrak{G}$ has an invariant subset Δ homeomorphic to $\Sigma^N \times M$. Let $\pi\colon \Sigma^N \times M \to M$ be the projection to the fiber along the base. A homeomorphism $\mathbf{H}\colon \Sigma^N \times M \to \Delta$, $\Delta \subset D \times M$, can be taken so that the coordinate φ is preserved, and hence the restriction of this homeomorphism to a single fiber is a C^r-diffeomorphism. One can consider the induced mapping

$$G = \mathbf{H}^{-1} \circ \mathfrak{G} \circ \mathbf{H}\colon \Sigma^N \times M \to \Sigma^N \times M.$$

Let us denote by f_ω the mapping $\pi \circ \mathbf{H}^{-1} \circ \mathfrak{G} \circ \mathbf{H}(\omega, \cdot)\colon M \to M$ depending on ω. The mappings $\mathbf{H}$, $\mathbf{H}^{-1}$, and $\mathfrak{G}$ are of class C^r, and hence $f_\omega \in C^r$. Then the mapping G becomes

$$G\colon \Sigma^N \times M \to \Sigma^N \times M, \qquad (\omega, \varphi) \mapsto (\sigma\omega, f_\omega(\varphi)). \tag{5.1}$$

Thus, G is a mild random dynamical system (1.4). Since the central leaves $\Phi^{-1}(\omega)$ of the set Δ depend Hölder continuously on the point $\omega \in \Sigma^N$ in the C^r-metric, it follows that the diffeomorphisms f_ω are Hölder continuous with respect to $\omega \in \Sigma^N$ in the C^r-metric. Since the mapping $\mathfrak{G}$ is C^r-close to the mapping $\mathfrak{F}$, it follows that the central leaves of the mapping $\mathfrak{G}$ are C^r-close to the central leaves of the mapping $\mathfrak{F}$ [11], and hence, for any $\omega \in \Sigma^N$, the diffeomorphism f_ω is C^r-close to the mapping f_{ω_0}.

The homeomorphism $\mathbf{H}\colon \Sigma^N \times M \to \Delta$ conjugates the mappings G and $\mathfrak{G}|_\Delta$, and its restriction to a single fiber is C^r-smooth. Hence, to investigate the properties of the mapping $\mathfrak{G}$, one must investigate a mild random dynamical system G with the corresponding properties (f_ω are smooth, close to f_{ω_0}, and depend Hölder continuously on ω).

5.2. Estimates for the φ-coordinate. It is easy to work with a step random dynamical system because one can think only about the φ-coordinate of the orbit and, at each step, trace its image under one of the N diffeomorphisms f_j. In the case of a mild random dynamical system, we do not know what mapping must be applied to the fiber over ω until we know the entire sequence ω. To overcome this difficulty, we need the following lemma.

Write

$$\gamma = \max_{\{\omega, \omega' \mid \omega_0 = \omega'_0\}} d_{C^0}(f_\omega, f_{\omega'}), \tag{5.2}$$

$$L = \max_{\omega \in \Sigma^N} \max_{\varphi \in S^1} (\|Df_\omega(\varphi)\|, \|Df_\omega^{-1}(\varphi)\|). \tag{5.3}$$

Lemma 3. *Let the mapping G given by formula* (5.1) *have the following properties:*

(1) $d_{C^0}(f_\omega, f_{\omega'}) \leqslant C(d_{\Sigma^N}(\omega, \omega'))^\alpha$ *for some* $C > 0$ *and* $\alpha > 0$ *and for any* f_ω *and* $f_{\omega'}$;

(2) $L2^{-\alpha} < 1$.

Then there exists $K > 0$, $K = K(L, C, \alpha)$, that does not depend on γ and for which

$$d_{\Sigma^N}(\omega, \omega') \leqslant 2^{-m} \implies d_M(\pi \circ G^m(\omega, \varphi), \pi \circ G^m(\omega', \varphi)) \leqslant K\gamma^b,$$

for any $m \in \mathbb{N}$, $\varphi \in M$, and $\omega, \omega' \in \Sigma^N$, where $b = 1 - \ln L / \ln 2^\alpha$.

Lemma 3 allows us to estimate the φ-coordinate of the points obtained by iterating the mapping G given by (5.1) even if we know the ω-coordinate of the initial point only approximately.

5.3. Final part of the proof of Theorem A. The previous lemma allows one to prove the following generalization of Theorem 1 for mild random dynamical systems.

Theorem 7. *For any N, there exist an open finite interval $I \subset \mathbb{R}$, $0 \in I$, and N open sets $U_0, \dots, U_{N-1} \subset$ $\mathrm{Diff}^2(S^1)$ such that, if the mapping G given by* (5.1) *has properties* (1)–(3) *below:*

(1) $f_\omega \in U_{\omega_0}$ *for any* $\omega \in \Sigma^N$,

(2) $d_{C^1}(f_\omega, f_{\omega'}) \leqslant C(d_{\Sigma^N}(\omega, \omega'))^\beta$ *for some* $C > 0$ *and* $\beta > 0$ *and for any* f_ω *and* $f_{\omega'}$,

(3) $L2^{-\beta} < 1$,

then G has properties (i)–(iii) *stated for the mapping F in Theorem* 1.

The proof of Theorem A is now performed as follows.

Let us take neighborhoods $U_0, \dots, U_{n-1}$ as in Theorem 7, take diffeomorphisms $f_j \in U_j$, and consider the corresponding step system (1.2). Then we construct a k-realization $\mathfrak{F}$ of the mapping F with sufficiently small k. Let $L/2$ be the maximum of the Lipschitz constants of the mappings $f_0, \dots, f_{N-1}, f_0^{-1}, \dots, f_{N-1}^{-1}$.

If $k(L/2)^2 < 1$, then Theorem 2 is applicable. Hence, any mapping $\mathfrak{G}$ from a small neighborhood $\mathfrak{U}$ of $\mathfrak{F}$ in C^2 has a partially hyperbolic invariant set Δ with C^2-smooth central leaves. The restriction of $\mathfrak{G}$ to Δ is a mild system G of the form (5.1). For sufficiently small k and $\mathfrak{U}$, G satisfies the conditions of Theorem 7.

Indeed, (1) holds for a small $\mathfrak{U}$. For any $\beta > 0$, the constant k may be taken so small that (2) holds for the chosen β. This can be derived from Theorem 6. Hence, relation (3) is ensured by a suitable choice of k. The application of Theorem 7 proves Theorem A.

§6. Partially Hyperbolic Attractors: Addendum to Theorem A

The set Δ in Theorem A can be an attractor. Below we prove an analog of assertion (i) in Theorem 1. The system we construct here is an "almost step" system on the attractor.

6.1. Skew product over a solenoid. Consider a Smale–Williams solenoid mapping $\mathbf{S}\colon T \to T$, $T = S^1 \times B$, $B = \{z \in \mathbb{C} \mid |z| < 2\}$, $S^1 = \mathbb{R}/\mathbb{Z}$, and $\mathbf{S}(s, z) = (2s, \frac{1}{3}z + \exp 2\pi i s)$. Consider a skew product

$$\mathscr{H}\colon T \times S^1 \to T \times S^1, \qquad (x, \varphi) \mapsto (\mathbf{S}(x), f_x(\varphi)), \quad x \in T,\ \varphi \in S^1. \tag{6.1}$$

For any choice of f_x, the mapping (6.1) has a maximal attractor Δ. Denote the projection to the base T along the fibers S^1 by Λ. The set Λ is a solenoid indeed, and does not depend on f_x. Let $\mathscr{H}|_\Delta = G$.

Choose f_x to be "almost piecewise constant." The family f_x cannot be exactly piecewise constant because the set Λ is connected.

Let g_0 and g_1 be the same as in Sec. 2.2 with $a \in (1, 2^{1/4})$. Set

$$f_x = g_j \quad \text{for } x = (s, z),\ s \in [j/2 + 1/2^{11}, (j+1)/2],\ j = 0, 1. \tag{6.2}$$

For all $x \in T$, let f_x be C^2-smooth with respect to x and φ and let

$$\left|\frac{\partial f_x}{\partial \varphi}\right| \leqslant a, \qquad \left|\frac{\partial (f_x^{-1})}{\partial \varphi}\right| \leqslant a, \qquad a \in (1, 2^{1/4}). \tag{6.3}$$

6.2. Dense sets of periodic orbits.

Proposition 2. *The mapping $\mathscr{H}$ that satisfies conditions* (6.1)–(6.3) *has two sets of periodic points each of which is dense in the attractor Δ. The first set consists of the periodic orbits with one-dimensional unstable manifolds and the other of those with two-dimensional unstable manifolds.*

Proof. We will prove the existence of the second set only. The existence of the first set is proved in a similar way, replacing the mapping $\mathscr{H}$ by its inverse on the attractor.

For any point $x \in T$, define the *fate* of x as a sequence of 0s and 1s that is infinite to the right as follows:

$$\omega(x) = \{\omega_n(x)\}, \quad \omega_n(x) = j \quad \text{for } \mathbf{S}^n(x) \in D_j,$$

where $D_j = \{(s, x) \mid s \in [j/2, j/2 + 1/2)\}$.

Important Remark. If the segment $\omega_1(x) \dots \omega_{10}(x)$ contains at least one symbol 1, then $f_x = g_{\omega_0(x)}$ by (6.2).

Take an arbitrary open set of $T \times S^1$. There exists a segment $\alpha_{-n} \dots \alpha_{n-1}$ that contains at least one symbol 0 and an arc $I \subset S^1$ such that $U \times I$ belongs to the above open set, where

$$U = \{x \in T \mid \omega_k(x) = \alpha_k \text{ for } -n \leqslant k \leqslant n-1\}.$$

Take

$$U' = \{x \in U \mid \omega_k(x) = 1 \text{ for } -(n+m) \leqslant k < -n,\ n \leqslant k \leqslant n+m-1\},$$

where the positive integer m will be chosen later. Let us find a periodic point with $\dim W^u = 2$ in $U' \times I$. Let $\pi\colon T \times S^1 \to S^1$ be the projection along the first factor.

Let us take $z \in U'$ so that $\omega_{n+m}(z) = 1$. For any $x \in U'$ we set $I_x = \{x\} \times I$ and $I^{\pm}(x) = \pi \circ G^{\pm(n+m)} I_x$. Let $J^+(x)$ be the "middle third" of the arc $I^+(x)$, and let $J^-(x)$ be the arc with the same center as $I^-(x)$ and of doubled length.

Apply Lemma 1 to the arcs $J^-(z)$ and $J^+(z)$ instead of J^- and J^+ and to $K > a^{n+m}$. Let the composition g of mappings g_0 and g_1 given by this lemma consist of q factors and correspond to a segment of 0 and 1, which we denote by $\bar{\beta}$.

Without loss of generality we may assume that the segment $\bar{\beta}$ contains at most nine zeros in succession. Indeed, the composition of ten rotations g_0 is the identity mapping, which permits us to use the cancellation.

Denote by $\bar{\alpha}$ the string $1 \dots 1 \alpha_{-n} \dots \alpha_{n-1} 1 \dots 1$ with m symbols 1 before α_{-n} and after α_{n-1}. Take a point $y \in \Lambda$ with the periodic fate $\omega = (\bar{\alpha}\bar{\beta})$.

Let us prove that the arc I_y contains a periodic point of G for which the absolute value of the multiplier along the circle is greater than one.

Set $y_0 = y$ and $y_l = \mathbf{S}^l y$. For $m+n < l \leqslant m+n+q$, the map f_{y_l} is equal to either g_0 or g_1 and is determined by the corresponding element of the sequence β. Indeed, for such l, the fate of the point y_l does not contain ten zeros in succession in the segment $\omega_1 \dots \omega_{10}$. Hence, $s(\mathbf{S}(y_l)) \notin [0, 1/2^{11}] \cup [1/2, 1/2 + 1/2^{11}]$ by the above important remark. Therefore, $\pi \circ G^{m+n+q} I_y = g I^+(y)$. In Sec. 6.3 we will prove the following lemma.

Lemma 4. *For m large enough and for I small enough, and for y and z chosen above, we have $I^-(y) \subset J^-(z)$, $J^+(z) \subset I^+(y)$, and $|J^-(z)| < 0.1$.*

Lemma 4 implies $I^-(y) \subset J^-(z) \subset g(J^+(z)) \subset g(I^+(y))$. Moreover, $g'|_{I^+(y)} \geqslant K$ by Lemma 1. Hence, the arc I_y contains a periodic point of the mapping G as desired. This proves Proposition 2 up to Lemma 4.

6.3. Divergence of orbits: sketch of the proof of Lemma 4. By (6.3),

$$a^{-(n+m)}|I| \leqslant |I^{\pm}(z)| \leqslant a^{n+m}|I|.$$

Let

$$\mathbb{L} = \max_{T \times S^1} \left| \frac{\partial f_x(\varphi)}{\partial x} \right|.$$

We prove below that, for any $a \in (1, 2^{1/4})$ and $\varphi \in S^1$ and for the points $y, z \in U'$ chosen above,

$$|\pi \circ G^{\pm(n+m)}(z, \varphi) - \pi \circ G^{\pm(n+m)}(y, \varphi)| \leqslant C\mathbb{L}\, a^{n+m} 2^{-m}. \tag{6.4}$$

Let us derive Lemma 4 from inequality (6.4). Let $|I| = 0.1 a^{-(n+m)}$. Then $|I^-(z)| \leqslant 0.1$ and $|I^+(z)| \geqslant 0.1 a^{-2(n+m)}$. Let us prove that the distance between the endpoints of the arcs $I^+(z)$ and $I^+(y)$ corresponding to the same endpoint of the arc I is smaller than $|I^+(z)|/3$ for m sufficiently large. Indeed,

$$C\mathbb{L} a^{n+m} 2^{-m} \leqslant 0.1 a^{-2(n+m)}/3$$

for $a \in (1, 2^{1/4})$ and $m > 3n + C_1$, where C_1 depends on C and $\mathbb{L}$. The same holds for the endpoints of $I^-(z)$ and $I^-(y)$. This proves the lemma up to inequality (6.4).

Let us prove (6.4) for G^{n+m} (the negative exponent is treated in the same way). We have

$$|f_z(\varphi) - f_y(\psi)| \leqslant \rho(z, y)\mathbb{L} + a|\varphi - \psi|. \tag{6.5}$$

Let y_l be as above, $z_{l+1} = \mathbf{S}(z_l)$, and $z_0 = z$. By the definition of U' and $\mathbf{S}$, for $0 \leqslant l \leqslant n$ we have $\rho(z_l, y_l) \leqslant 2^{-m}$. Hence, by (6.5)

$$|\pi \circ G^n(z, \varphi) - \pi \circ G^n(y, \varphi)| \leqslant C\mathbb{L}\, a^n 2^{-m}. \tag{6.6}$$

For $n < l \leqslant n+m$ we obtain $f_{z_l} = f_{y_l} = g_1$. This can be derived from (6.2) as follows. For $n < l \leqslant n+m-1$, by the construction of z and y we have $\omega_0(z_l) = \omega_1(z_l) = \omega_0(y_l) = \omega_1(y_l) = 1$. For $l = n+m$, we still have $\omega_0(z_l) = \omega_1(z_l) = \omega_0(y_l) = 1$. Moreover, the first 10 elements of the sequence $\omega_1(y_l) \dots \omega_{10}(y_l) \dots$ cannot vanish simultaneously. Therefore, $f_{y_l} = g_{\omega_0(y_l)} = g_1$ by (6.2).

Hence, by (6.3) we obtain $|f_{z_l}(\varphi) - f_{y_l}(\psi)| = |g_1(\varphi) - g_1(\psi)| \leqslant a|\varphi - \psi|$. This, together with inequality (6.6), implies (6.4), which completes the proof of Lemma 4, and hence of Proposition 2.

§7. Random Dynamical Systems on the Boundary of the Morse–Smale Set

Partially hyperbolic sets with random dynamical systems on them occur under nonlocal bifurcations. This effect was described implicitly in [14]; partially hyperbolic sets were described, but random dynamical systems were not mentioned. Below we reproduce and continue this description, which finally leads to Theorem B.

7.1. Several homoclinic surfaces of a saddlenode cycle. Consider a vector field in $\mathbb{R}^n$, $n \geqslant 4$, that has a saddlenode cycle and satisfies some genericity assumptions. A saddlenode cycle is a periodic orbit one of whose multipliers is equal to 1 while the others are outside of the unit circle. We do not reproduce the genericity assumptions; they can be found in [14]. The vector fields with a saddlenode cycle form a set of codimension one in $\mathfrak{X}^r$, $r \geqslant 1$. Without decreasing the codimension of the degeneracy, one may assume that the vector fields with a saddlenode cycle have N smooth invariant surfaces diffeomorphic to a two-dimensional torus T^2 or to a Klein bottle K^2 and satisfy the following property. Each of these surfaces contains a saddlenode cycle L that is simultaneously the α- and ω-limit set of all other orbits on the surface. Such a surface is called a homoclinic surface of the cycle L.

7.2. The description of bifurcation.

Theorem 8. *Let a generic one-parameter family of vector fields X_ε contain a vector field X_0 that belongs to the boundary of the Morse–Smale set and has a saddlenode cycle* L *with N smooth homoclinic surfaces diffeomorphic to either T^2 or K^2. Let U be a small neighborhood of the union of these homoclinic surfaces. Then, possibly after a reparametrization, each of the fields X_ε, for small $\varepsilon > 0$, has an invariant set Ω_ε that contains all nonwandering points of X_ε in U. There is a global cross-section Γ transversal to X_ε in U, and a global Poincaré mapping between open subsets V and V' of Γ, $\mathfrak{G}_\varepsilon\colon V \to V'$, $V \subset \Gamma$, $V' \subset \Gamma$, along the orbits of X_ε. The intersection $\Delta_\varepsilon = \Omega_\varepsilon \cap \Gamma$ is invariant under $\mathfrak{G}_\varepsilon$ and admits a surjective mapping $\Phi_\varepsilon\colon \Delta_\varepsilon \to \Sigma_N$ onto the space of all bi-infinite sequences of N symbols such that the diagram*

$$\begin{array}{ccc} \Delta_\varepsilon & \xrightarrow{\mathfrak{G}_\varepsilon} & \Delta_\varepsilon \\ {\scriptstyle \Phi_\varepsilon}\downarrow & & \downarrow{\scriptstyle \Phi_\varepsilon} \\ \Sigma^N & \xrightarrow{\sigma} & \Sigma^N \end{array}$$

is commutative. Here σ is the Bernoulli shift. Moreover, for any $\omega \in \Sigma^N$, the fiber $\Phi_\varepsilon^{-1}(\omega)$ is an embedded circle. The embedding $i_{\omega,\varepsilon}\colon S^1 \to \Phi_\varepsilon^{-1}\omega \subset \Gamma$ is a Lipschitzian mapping and depends continuously on ω.

This theorem is proved in [14, §5.7] in a slightly different but equivalent form.

Remark. The set Δ_ε in Theorem 8 is a maximal invariant set for $\mathfrak{G}_\varepsilon$ in $U \cup \Gamma$.

The cross-section Γ is a hypersurface in a tubular neighborhood of the saddlenode cycle L. It is diffeomorphic to a product of a circle S^1 and a ball B. The coordinate φ on the circle S^1 is thus lifted to Γ. This coordinate depends on the diffeomorphism $S^1 \times B \to \Gamma$ and on the parametrization of S^1. This choice may be made in a special way so that Theorem 10 holds (see below). Moreover, we can choose the embedding $i_{\omega,\varepsilon}$ in Theorem 8 so that it preserves the coordinate φ; hence, the image $i_{\omega,\varepsilon}(S^1)$ is a graph of a mapping $S^1 \to B$ that depends on ω and ε.

Consider the mapping

$$\mathbf{H}_\varepsilon\colon \Sigma^N \times S^1 \to \Delta_\varepsilon, \qquad \mathbf{H}_\varepsilon^{-1}(p) = (\Phi_\varepsilon(p), \varphi). \tag{7.1}$$

Then $\mathbf{H}_\varepsilon^{-1}$ conjugates the map $\mathfrak{G}_\varepsilon|_{\Delta_\varepsilon}$ with a mild random dynamical system

$$G_\varepsilon\colon \Sigma^N \times S^1 \to \Sigma^N \times S^1, \qquad (\omega, \varphi) \mapsto (\sigma\omega, f_\omega(\varphi, \varepsilon)). \tag{7.2}$$

By Theorem 8, for a chosen ε, the mapping f_ω is Lipschitzian with respect to φ and continuous with respect to ω. In order to extend the properties of step random dynamical systems to the mild ones, we need better analytic properties of the mapping f_ω with respect to both φ and ω. The following theorem is an analog of Theorem 3 for the sets Δ_ε.

Theorem 9. *In Theorem 8, the embedding $i_{\omega,\varepsilon}$ of the circle into Γ is of class C^r for arbitrary r provided that $\varepsilon > 0$ is sufficiently small. Moreover, this embedding is Hölder continuous with respect to ω as an element of $C^r(S^1, B)$, and the Hölder exponent tends to infinity as $\varepsilon \to +0$.*

Corollary. *The mild system* (7.2) *is conjugated by the mapping* (7.1) *to the mapping $\mathfrak{G}_\varepsilon$ restricted to its maximal invariant set Δ_ε. The mapping $f_\omega(\,\cdot\,, \varepsilon)\colon S^1 \to S^1$ in* (7.2) *is a C^r-diffeomorphism* (*not necessarily orientation preserving*) *and depends Hölder continuously on ω as an element of* $\mathrm{Diff}^r(S^1, S^1)$; *the Hölder exponent tends to infinity as $\varepsilon \to +0$.*

7.3. Rescaling: return to the critical value of the parameter. As the parameter ε tends to 0 from the right, the domain of the mapping $\mathfrak{G}_\varepsilon$ becomes smaller. It turns out that the set Δ_ε tends to the union of N circles that form the intersection of Γ with the homoclinic surfaces. There is no well-defined limit map on this union for the family $\mathfrak{G}_\varepsilon$. On the other hand, the mappings G_ε tend to a well-defined limit as $\varepsilon \to 0$ over a specially chosen subsequence.

Theorem 10. *The critical vector field X_0 in Theorem 8 determines N diffeomorphisms of the circle, $f_j\colon S^1 \to S^1$, $j \in \{0, \dots, N-1\}$. There exists a continuous function $T(\varepsilon)$ that is monotone and tends to $+\infty$ as $\varepsilon \to +0$ and for which in (7.2) we have*

$$\lim_{\varepsilon \to +0} f_\omega(\varphi + T(\varepsilon), \varepsilon) = f_{\omega_0}(\varphi). \tag{7.3}$$

Moreover, let $\tau(\varepsilon) = \{T(\varepsilon)\}$ (thus, τ is the fractional part of the number T) and let $\mathbb{I}$ be any open interval. Then the set $\tau^{-1}(\mathbb{I})$ has positive density at the origin.

Remarks. 1. Property (7.3) allows us to do a rescaling and to pass to the limit as $\varepsilon \to 0$ in the family G_ε. Namely, define $\varepsilon(k, \mu)$ by the relation $T(\varepsilon(k, \mu)) = k + \mu$, $k \in \mathbb{Z}^+$, $\mu \in [0, 1)$. Since the function f_ω is 1-periodic in φ, it follows from (7.3) that

$$f_\omega(\varphi, \varepsilon(k, \mu)) \to f_{\omega_0}(\varphi - \mu) \quad \text{as } k \to \infty. \tag{7.4}$$

Let $\varepsilon_k = \varepsilon(k, 0)$. Then the sequence G_{ε_k} tends to a limit F of the form (1.2).

2. Consider a sequence h_k of Smale horseshoe maps with rectangular domains that contract to a segment as $k \to \infty$. The sequence of maps h_k has no reasonable limit. However, any mapping h_k is conjugate to the same Bernoulli shift $\sigma\colon \Sigma^2 \to \Sigma^2$, which can be regarded as the limit of the sequence of horseshoe mappings that is obtained after a rescaling. In the same sense, the mappings $\mathfrak{G}_{\varepsilon_k}|_{\Delta_{\varepsilon_k}}$, after the rescaling, tend to a limit of the form (1.2), which has rich dynamical properties.

7.4. Sketch of the proof of Theorem B. Theorem B follows from the results of this section and from Theorem 7. Namely, let $N \in \mathbb{Z}$ be arbitrary with $N > 1$. Let $U_0, \dots, U_{N-1}$ be as in Theorem 7. Choose any $f_j \in U_j$. One can take a field X_0 in Theorem 8 such that the corresponding mappings on the right-hand side of (7.3) are equal to f_j. The conditions on the vector field X_0 in Theorem 8 imply assertion (1) of Theorem B.

Let X_ε be a generic one-parameter unfolding of the field X_0, let $\mathfrak{G}_\varepsilon$ be the corresponding family of the Poincaré mappings in Theorem 8, let G_ε be the derived family (7.2), and let G_{ε_k} be a subsequence of this family given by Remark 1 in Sec. 7.3. It follows from (7.4) that assertion (1) of Theorem 7 holds for the mappings G_{ε_k} for large k.

Conditions (2) and (3) of Theorem 7 hold by the corollary to Theorem 9. This corollary implies assertion (2) of Theorem B.

Let us prove assertion (3) of Theorem B. Let $\mathbb{I} \subset [0, 1]$ be an open interval such that a mapping $\varphi \mapsto f_j(\varphi - \mu)$ still belongs to U_j for $\mu \in \mathbb{I}$. Then Theorem 7 can be applied to the mappings $G_{\varepsilon(k,\mu)}$ for $\mu \in \mathbb{I}$ and a sufficiently large k. The corresponding values $\varepsilon(k, \mu)$ form the intersection of the set $\tau^{-1}(\mathbb{I})$ with small neighborhood of the origin. The last statement of Theorem 10 now implies assertion (3) of Theorem B.

§8. Conjectures

8.1. Stable ergodicity and robust transitivity. A volume-preserving C^2-diffeomorphism is said to be *stably ergodic* if each of its sufficiently C^1-small, volume-preserving perturbations is ergodic. Anosov diffeomorphisms are examples. In [15] it was shown that there exist partially hyperbolic stably ergodic diffeomorphisms.

Conjecture 1. *A stably ergodic diffeomorphism is robustly transitive, that is, each of its sufficiently C^1-small, not necessarily volume-preserving perturbations is transitive.*

8.2. Skew extensions and robust transitivity. It is proved that most skew extensions over a transitive hyperbolic set by compact Lie groups are stably ergodic in the class of all skew extensions of this type [16]. A skew extension has a smooth realization if there exists a smooth diffeomorphism such that its restriction to a certain partially hyperbolic invariant set is conjugate to this skew extension.

Conjecture 2. *If a skew extension over a transitive hyperbolic set by a compact Lie group is stably ergodic (in the class of skew extensions) and has a smooth realization, then this smooth realization has a partially hyperbolic invariant set that is robustly transitive in the class of all C^1-perturbations.*

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