ELEMENTARY ANALYSIS 140A

Final Exam SAMPLE (with answers and hints)

Problem 1.

Prove that for every $n \geq 1$, $2^{2^n} - 1$ is divisible by at least n distinct primes.

Hint: Use the equality $2^{2^{n+1}} - 1 = (2^{2^n} + 1)(2^{2^n} - 1)$ and induction.

Problem 2.

Determine the limit of the sequence $\{5 - \frac{1}{n} + \frac{1}{n^2}\}_{n \in \mathbb{N}}$. Prove that the sequence converges to that limit using the definition of sequence convergence.

Answer: Set $s_n = 5 - \frac{1}{n} + \frac{1}{n^2}$. Let us prove that $\lim_{n\to\infty} s_n = 5$. We have

$$|s_n - 5| = \left|\frac{1}{n} - \frac{1}{n^2}\right| = \left|\frac{n-1}{n^2}\right| \leq \left|\frac{n}{n^2}\right| = \frac{1}{n}.$$

For a given $\varepsilon > 0$ choose $N > \frac{1}{\varepsilon}$, then for any $n > N$ we have $|s_n - 5| \leq \frac{1}{n} < \frac{1}{N} < \varepsilon$.

Problem 3.

If possible, give an example of each of the following. Write "not possible" when appropriate.

a) A sequence $\{s_n\}$ with $\limsup s_n = +\infty$ and $\liminf s_n = 0$.

b) A bounded sequence which diverges.

c) A series $\sum a_n$ which diverges, but for which the series $\sum a_n^2$ converges.

d) A continuous but not uniformly continuous function $f : [-2000, 2000] \to \mathbb{R}$.

e) A function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at exactly one point (and discontinuous at every other point).

Answers:

a) $s_n = (1 + (-1)^n)n$;

b) $s_n = (-1)^n$;

c) $\sum \frac{1}{n}$;

d) not possible;

e) $f(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$

Problem 4.

Prove that if the series $\sum a_n$ converges absolutely then the series $\sum(-1)^n a_n^2$ also converges.

Answer: Notice that if $\sum a_n$ converges, then for some $N \in \mathbb{N}$ for all $n > N$ we have $|a_n| < 1$. Therefore for $n > N$ we also have $|(-1)^n (a_n)^2| < |a_n|$. Since $\sum a_n$ converges absolutely, by Comparison test the series $\sum(-1)^n a_n$ also converges.

Problem 5.

Give an example of a metric space (X, d) for which there exists a continuous unbounded function $f : X \to \mathbb{R}$. Is it possible to take the standard Cantor set as such an example?

Answer: Consider $X = \mathbb{R}$ with the standard metric, $f(x) = x$ is continuous and unbounded. Since the standard Cantor set is compact, every continuous function on the Cantor set is bounded.

Problem 6.

Prove that $e^{-x} = x$ for some $x > 0$.

Answer: Consider $f(x) = e^{-x} - x$. Then $f(0) = 1 > 0$, and $f(1) = e^{-1} - 1 < 0$, so by the Intermediate Value Theorem, there exists $x_0 \in (0, 1)$ such that $f(x_0) = 0$, and therefore $e^{-x_0} = x_0$.

Problem 7.

Prove or Disprove: $f(x) = x^2 \sin \frac{1}{x^2}$ is uniformly continuous on $(0, 5)$.

Hint: Show that the function $f(x)$ can be extended to $[0, 5]$ as a continuous function.