ELEMENTARY ANALYSIS 140A

Midterm Exam 2 Sample ANSWERS

Problem 1.

Find the limits of the following sequences (no proof is required):

$\lim_{n\to\infty} \frac{n^3+n^2+n+1}{3n^3+2n^2+n+1} =$

$\lim_{n\to\infty} (2n)^{\frac{1}{4n}} =$

$\lim_{n\to\infty} \left(n^{1/n} - \frac{1}{2}\right)^n =$

ANSWERS: $\frac{1}{3}, 1, 0$

Problem 2.

For each sequence, find the set of subsequential limits:

1) $s_n = (-1)^n + \frac{1}{2^n}$

2) $s_n = \cos\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)$

3) $s_n = (-1)^n + \frac{1}{2^n} + \cos\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)$

ANSWERS: 1) $S = \{-1, 1\}$;   2) $S = \{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$;   3) $S = \{-1 - \frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}, -1 + \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}\}$.

Problem 3.

Prove that the closure of an interior of any closed set $E \subset \mathbb{R}$ is a subset of E. Does it have to coincide with E?

ANSWER: Since $\text{int } E \subset E$, we have $\overline{\text{int } E} \subset \overline{E} = E$. The closure of the interior of E does not have to coincide with E. Consider, for example, the set $E = [0, 1] \cup \{3\}$. Then the closure of the interior of E is $[0, 1] \neq E$.

Problem 4.

For each of the following sets find its closure:

1) $E = \left\{\frac{1}{n} + \frac{1}{2^n} \mid n \in \mathbb{N}\right\}$

2) $E = \mathbb{Q} \cap (0, 1)$

3) $E = \mathbb{Q} \cap \{x \in \mathbb{R} \mid x^2 < 2\}$

ANSWERS: 1) $\overline{E} = E \cup \{0\}$, 2) $\overline{E} = [0, 1]$, 3) $\overline{E} = [-\sqrt{2}, \sqrt{2}]$.

Problem 5.

Consider a metric space (X, d), where $X = \{x, y, z, u\}$ and
$d(x, y) = d(x, z) = d(x, u) = d(y, x) = d(z, x) = d(u, x) = 2$,
$d(y, z) = d(y, u) = d(z, u) = d(z, y) = d(u, y) = d(u, z) = 1$,
$d(x, x) = d(y, y) = d(z, z) = d(u, u) = 0$.
Check that d is indeed a metric, and prove that (X, d) is a complete metric space.

ANSWER: Solution was presented on the lecture.