Final exam (sample)

In all the problems $D$ is the unit disc, $D = \{z \in \mathbb{C} \mid |z| < 1\}$.

**Problem 1.**
Let $L \subset \mathbb{C}$ be the line $L = \{z = x + iy \mid x = 3\}$. Assume that $f: \mathbb{C} \to \mathbb{C}$ is an entire function such that for any $z \in L$ we have $f(z) \in L$. Assume that $f(0) = 1 + 2i$. Find $f(6)$.

**Problem 2.**
Suppose a function $f: \overline{D} \to \mathbb{C}$ is continuous and holomorphic in $D$. Suppose also that for any $z \in \partial D$ we have $\text{Re} f(z) = 5$. Prove that $f$ is a constant.

**Problem 3.**
Let $f: \{z = x + iy \mid y > 0\} \to \mathbb{C}$ be a holomorphic function such that $f\left(\frac{i}{\sqrt{n}}\right) = 0$ for every $n \in \mathbb{N}$. Prove that $f$ is unbounded.

**Problem 4.**
\(a\) TRUE OR FALSE: A continuous function $v: \mathbb{R} \to \mathbb{R}$ is linear if and only if for every $x, h \in \mathbb{R}$ one has
\[\frac{1}{2} (v(x - h) + v(x + h)) = v(x) .\]

\(b\) TRUE OR FALSE: A continuous function $u: \mathbb{C} \to \mathbb{R}$ is harmonic if and only if for every $z, h \in \mathbb{C}$ one has:
\[\frac{1}{4} (u(x - h) + u(x + h) + u(x - ih) + u(x + ih)) = u(x) .\]

**Problem 5.**
Let $\{f_\alpha\}_{\alpha \in A}$ be a family of holomorphic functions on $D$ such that
\[\forall x \in D \ \forall f \in \{f_\alpha\} \ \text{Re} f(z) \neq \text{Im} f(z) .\]
Prove that $\{f_\alpha\}_{\alpha \in A}$ is a normal family (use the definition of normal family that allows a sequence of function to converge to $\infty$, uniformly on compact sets).