1) (5 pts) Find the general solution to the differential equation \( y'' + 2y' + y = 2e^{-x} \)

First, the complementary solution: \( y'' + 2y' + y = 0 \) corresponds to the auxiliary equation \( m^2 + 2m + 1 = (m+1)^2 = 0 \), so the complementary solution is \( y_c(x) = c_1e^{-x} + c_2xe^{-x} \).

Now for the particular solution: The differential operator is \( L = D^2 + 2D + 1 \). We would normally expect it to be of the form of \( y_p(x) = Ae^{-x} \), but there is a homogeneous solution of this form. We'd next try \( y_p(x) = Axe^{-x} \), but again, there is a homogeneous solution of this form. Finally, we try \( y_p(x) = Ax^2e^{-x} \), which is not a homogeneous solution, so we proceed with this.

\[
y_p' = 2Axe^{-x} - Ax^2e^{-x} \rightarrow y_p'' = 2Ae^{-x} - 2Axe^{-x} - \left(2Axe^{-x} - Ax^2e^{-x}\right) = 2Ae^{-x} - 4Axe^{-x} + Ax^2e^{-x}
\]

\[
Ly_p = 2Ae^{-x} - 4Axe^{-x} + Ax^2e^{-x} + 4Axe^{-x} - 2Ax^2e^{-x} + Ax^2e^{-x} = 2Ae^{-x}
\]

This must be equal to \( 2e^{-x} \), so \( A = 1 \). Thus \( y_p(x) = x^2e^{-x} \).

As \( y(x) = y_c(x) + y_p(x) \), we finally have \( y(x) = c_1e^{-x} + c_2xe^{-x} + x^2e^{-x} \)

2) (5 pts) Find the general solution to the differential equation \( 2x^2y'' + 2xy' + y = 0 \)

Memorization way: As we saw in the handout, \( a_x^2y'' + a_xy' + a_0y = g(x) \rightarrow a_x\tilde{y}'' + \left(a_1 - a_x\right)\tilde{y}' + a_0\tilde{y} = g(e^x) \) so this ODE corresponds to \( 2\tilde{y}'' + 0\tilde{y}' + \tilde{y} = 0 \).

The "derive it" way:

Let \( x(t) = e^t \), thus \( t(x) = \ln x \). Let \( \tilde{y}(t) = y(x(t)) = y(\ln x) \), thus

\[
y(x) = \tilde{y}(t(x)) = \tilde{y}(\ln x).
\]

\[
\frac{dy}{dx} = \frac{d}{dx} \left[ y(x) \right] = \frac{d}{dx} \left[ \tilde{y}(t(x)) \right] = \frac{d\tilde{y}}{dt} \frac{dt}{dx} = \frac{d\tilde{y}}{dt} \frac{1}{x} = \tilde{y}'e^{-t}
\]

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{dy}{dx} \right] = \frac{d}{dx} \left[ \frac{d\tilde{y}}{dt} \frac{1}{x} \right] = -\frac{1}{x^2} \frac{d\tilde{y}}{dt} + \frac{1}{x} \frac{d}{dx} \left[ \frac{d\tilde{y}}{dt} (t(x)) \right] = -\frac{1}{x^2} \frac{d\tilde{y}}{dt} + \frac{1}{x} \frac{d^2\tilde{y}}{dt^2} = e^{2t} \left( \tilde{y}'' - \tilde{y}' \right)
\]

Substituting in, we have \( 2e^{2t} \left[ e^{2t} \left( \tilde{y}'' - \tilde{y}' \right) \right] + 2e^t \left[ \tilde{y}'e^{-t} \right] + \tilde{y} = 0 \), or rearranging, \( 2\tilde{y}'' + 0\tilde{y}' + \tilde{y} = 0 \).

In either case, this corresponds to the auxiliary equation \( 2m^2 + 1 = 0 \), so \( m = \pm \frac{i}{\sqrt{2}} \). Thus

\[
\tilde{y}_c(t) = c_1 \cos \left( \frac{t}{\sqrt{2}} \right) + c_2 \sin \left( \frac{t}{\sqrt{2}} \right).
\]

Reversing the transformation, we find

\[
y_c(x) = c_1 \cos \left( \frac{\ln x}{\sqrt{2}} \right) + c_2 \sin \left( \frac{\ln x}{\sqrt{2}} \right).
\]