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Conservative Homoclinic Bifurcations and Some Applications

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Abstract—We study generic unfoldings of homoclinic tangencies of two-dimensional area-preserving diffeomorphisms (conservative Newhouse phenomena) and show that they give rise to invariant hyperbolic sets of arbitrarily large Hausdorff dimension. As applications, we discuss the size of the stochastic layer of a standard map and the Hausdorff dimension of invariant hyperbolic sets for certain restricted three-body problems. We avoid involved technical details and only concentrate on the ideas of the proof of the presented results.

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1. INTRODUCTION

In the case of dissipative dynamical systems, homoclinic bifurcations have been intensively investigated; some of the dynamical phenomena that appear after a bifurcation in this case are persistent tangencies and an infinite number of sinks (Newhouse phenomena [38, 39, 42]), strange attractors (Mora, Viana [35]), arbitrarily degenerate periodic points of arbitrarily high periods (Gonchenko, Shilnikov, Turaev [24]), and superexponential growth of periodic orbits (Kaloshin [27]).

The conservative (area-preserving) case is known to be more complicated. For example, it took over two decades to prove an analog of Newhouse results for area-preserving surface diffeomorphisms (Duarte [11, 12], Gonchenko, Shilnikov [23]). For the case of C^1 maps see also [40], but here we will be interested in the case of higher smoothness.

By $\mathrm{Diff}^r(M^2, \mathrm{Leb})$, $0 < r \le \infty$, we denote the space of C^r diffeomorphisms of a two-dimensional Riemannian manifold M^2 that preserve the natural Lebesgue measure on M^2. Let a diffeomorphism $f \in \mathrm{Diff}^\infty(M^2, \mathrm{Leb})$ have a quadratic homoclinic tangency associated with some hyperbolic fixed point P. Here is a zoo on known phenomena that appear as a result of a two-dimensional conservative homoclinic bifurcation:

Hénon map in the renormalization limit. An appropriately chosen and properly rescaled return map near a point of homoclinic tangency can be arbitrarily C^r-close to an area-preserving Hénon family $H_a(x, y) = (y, -x + a - y^2)$, where a can be arbitrary [34, 23].

Elliptic periodic points. A small perturbation of f may have an elliptic periodic point near a point of homoclinic tangency. One of the ways to prove this formally is to consider the aforementioned renormalization limit and to observe that the limit map H_a has an elliptic fixed point for some values of a.

Hyperbolic sets with persistent tangencies. Locally maximal hyperbolic sets exhibiting persistent tangencies of leaves of stable and unstable foliation can be born as a result of an unfolding of a homoclinic tangency [12]. A one-parameter version of this result is now also available [13].

Infinitely many coexisting elliptic periodic points (*conservative Newhouse phenomena*). Duarte [11, 13] and Gonchenko and Shilnikov [23] showed that near f there exists an open set $\mathcal{U} \subset$

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$\mathrm{Diff}^\infty(M^2, \mathrm{Leb})$ such that a generic diffeomorphism from $\mathcal{U}$ has infinitely many coexisting elliptic periodic points. Recently Duarte [13] also proved a one-parameter version of this result, therefore making it possible to apply it in many concrete finite-parameter families. Usually open sets of diffeomorphisms with persistent homoclinic tangency are called *Newhouse domains.*[1]

Elliptic periodic points of arbitrarily high order of degeneracy. Maps with infinitely many elliptic periodic orbits of every order of degeneracy are dense in the Newhouse regions in the space of two-dimensional area-preserving analytic maps [24].

Tangencies of arbitrarily high order. Area-preserving surface diffeomorphisms with homoclinic tangencies of arbitrarily high orders are dense in the Newhouse regions [24].

Universal maps. Gonchenko, Shilnikov, and Turaev [24] showed that near a homoclinic tangency one can approximate any given dynamics in the following sense: Every area-preserving diffeomorphism of a two-dimensional disc can be C^r-approximated by a diffeomorphism, which arises as an appropriately chosen and properly rescaled return map near a point of homoclinic tangency. In other words, every dynamical phenomenon which is generic for some open set of symplectic diffeomorphisms of a two-dimensional disc can be encountered arbitrarily close to any area-preserving two-dimensional map exhibiting a homoclinic tangency. It is hard to verify the appearance of this phenomenon in a concrete finite-parameter family. However, this serves as a great illustration of complexity of the dynamics of unfoldings of a homoclinic tangency.

The purpose of this paper is to add another animal to this zoo, namely,

Hyperbolic sets of large Hausdorff dimension. Locally maximal hyperbolic sets of Hausdorff dimension arbitrarily close to 2 appear after a generic one-parameter unfolding of a homoclinic tangency (see Section 2 for the formal statement).

In addition, we discuss two applications of this phenomenon. Namely, in Section 3 we prove that the stochastic layer (the set of orbits with nonzero Lyapunov exponents) of the Taylor–Chirikov standard map has full Hausdorff dimension for large generic values of the parameter. In Section 4 we construct invariant hyperbolic sets of Hausdorff dimension arbitrarily close to 2 in the Sitnikov problem and in the restricted planar circular three-body problem for many parameter values.

While the proofs in Section 3 are complete, Sections 2 and 4 present only outlines of the proofs. A complete account is fairly involved and will appear elsewhere.

2. HYPERBOLIC SETS OF LARGE HAUSDORFF DIMENSION

Several famous long-standing conjectures concern the measure of certain invariant sets of some dynamical system (see introduction to Sections 3 and 4). Any set of positive Lebesgue measure has Hausdorff dimension which is equal to the dimension of the ambient manifold. Therefore, it is reasonable to ask whether those invariant sets indeed have full Hausdorff dimension.

Downarowicz and Newhouse [9] proved that there is a residual subset $\mathcal{R}$ of the space of C^r diffeomorphisms of a compact two-dimensional manifold M such that if $f \in \mathcal{R}$ and f has a homoclinic tangency, then f has compact invariant topologically transitive sets of Hausdorff dimension 2. In their proof they used results by Gonchenko, Shilnikov, and Turaev [24] to create degenerate saddle–nodes. Therefore, this approach cannot be generalized to the conservative case, nor does it allow the result to be formulated for generic finite-parameter families of diffeomorphisms.

In the conservative setting Newhouse [41] proved that in $\mathrm{Diff}^1(M^2, \mathrm{Leb})$ there is a residual subset of maps such that every homoclinic class[2] for each of those maps has Hausdorff dimension 2.

[1] One of the results we establish is the existence of Newhouse domains for certain three-body problems (see Subsection 4.4).

[2] Let P be a hyperbolic saddle of a diffeomorphism f. A homoclinic class $H(P, f)$ is the closure of the union of all transversal homoclinic points of P.

Later Arnaud, Bonatti, and Crovisier [3, 2] significantly improved this result and showed that in the space of C^1 symplectic maps a residual subset consists of the transitive maps that have only one homoclinic class (the whole manifold). Notice that due to KAM theory this result cannot be extended to higher smoothness.

In this section we show that a generic one-parameter area-preserving homoclinic bifurcation always gives birth to a compact invariant topologically transitive set of Hausdorff dimension 2. This set is the closure of the union of a countable sequence of hyperbolic sets of Hausdorff dimension arbitrarily close to 2.

2.1. The area-preserving Hénon family. First of all, we consider the area-preserving Hénon family. For $a = -1$ this map has a degenerate fixed point at $(x, y) = (-1, 1)$. We construct invariant hyperbolic sets of large Hausdorff dimension for a slightly larger than -1 near this fixed point. Later we use the renormalization results to reduce the case of a generic unfolding of an area-preserving surface diffeomorphism with a homoclinic tangency to this construction.

Theorem 1. *Consider the family of area-preserving Hénon maps*

$$H_a\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -x + a - y^2 \end{pmatrix}. \qquad (1)$$

There is a (piecewise continuous) family of sets Λ_a, $a \in [-1, -1+\varepsilon]$ for some $\varepsilon > 0$, such that the following properties hold:

1. *The set Λ_a is a locally maximal hyperbolic set of the map H_a.*
2. *The set Λ_a contains a saddle fixed point of the map H_a.*
3. *The set Λ_a has an open and closed (in Λ_a) subset $\widetilde{\Lambda}_a$ such that the first return map for $\widetilde{\Lambda}_a$ is a two-component Smale horseshoe.*
4. *The Hausdorff dimension $\dim_{\mathrm{H}} \widetilde{\Lambda}_a \to 2$ as $a \to -1$.*

A similar statement also holds for any generic one-parameter unfolding of an extremal periodic point (see [11] for a formal definition) as soon as the form of the splitting of separatrices can be established (see [20, 21] for the relevant results on splitting of separatrices).

Theorem 1 can be considered as an improvement of Lemma A from [13], where Duarte proves that area-preserving Hénon maps have hyperbolic sets of large "left–right thickness" (see [13, 36] for a definition) for values of a slightly larger than -1.

Sketch of proof of Theorem 1. *Step* 1: *Change of coordinates and rescaling.* Up to the change of a parameter and coordinates, there exists only one one-parameter area-preserving quadratic family with some conditions on the fixed points (Hénon family) (see [25, 14]). In particular, we can consider the family

$$F_\varepsilon\colon\ (x, y) \mapsto (x + y - x^2 + \varepsilon,\ y - x^2 + \varepsilon) \qquad (2)$$

instead of (1). In this form it is a partial case of a so-called *generalized standard family*, and it was considered in [20].

An affine change of coordinates conjugates $\{F_\varepsilon\}$ with the family of maps

$$(u, v) \mapsto (u, v) + \delta(v, 2u - u^2) + \delta^2(2u - u^2, 0), \qquad (3)$$

where $\delta = \varepsilon^{1/4}$. This is a family of maps close to the identity. For each of these maps the origin is a saddle with eigenvalues

$$\lambda_1 = 1 + \delta^2 + \sqrt{\delta^4 + 2\delta^2} = 1 + \sqrt{2}\delta + O(\delta^2) > 1,$$

$$\lambda_2 = \lambda_1^{-1} = 1 + \delta^2 - \sqrt{\delta^4 + 2\delta^2} = 1 - \sqrt{2}\delta + O(\delta^2) < 1.$$

Set $h = \ln\lambda_1$. By definition $h = \sqrt{2}\delta + O(\delta^2)$, and δ can be expressed as a nice function of h. We parameterize the maps of the family (3) by h and denote the family (3) by $\mathfrak{F}_h$.

Step 2: *Gelfreich normal form and splitting of separatrices for the Hénon family.* The family $\mathfrak{F}_h$ is closely related to the conservative vector field

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 2x - x^2. \end{cases} \tag{4}$$

Namely, due to Theorems A and A′ from [16] (see also Proposition 5.1 from [15]), the separatrix phase curve of the vector field (4) (let us denote it by σ) gives a good approximation of some finite pieces of $W^{\mathrm{s}}(0,0)$ and $W^{\mathrm{u}}(0,0)$. Denote by $\widetilde{\sigma}$ a finite segment of the separatrix σ.

The restriction of the map $\mathfrak{F}_h$ to the local unstable separatrix $W^{\mathrm{u}}_h(0)$ is conjugate to a multiplication $\xi \mapsto \lambda\xi$, $\xi \in (\mathbb{R}, 0)$. Call a parameter t on $W^{\mathrm{u}}_h(0)$ *standard* if it is obtained by substituting e^t for ξ in the conjugating function. Such a parametrization is defined up to a substitution $t \mapsto t + \mathrm{const}$.

Theorem 4 in [18] states that there is an area-preserving real analytic change of coordinates Ψ_h that conjugates the map $\mathfrak{F}_h$ in a neighborhood of $\widetilde{\sigma}$ with the shift $(t, E) \mapsto (t + h, E)$, $\Psi_h^{-1}(W_h^{\mathrm{u}}) = \{E = 0\}$, and t gives a standard parametrization of the unstable manifold. Moreover, from [19, 20] it follows that in these normalizing coordinates the stable manifold $\Psi_h^{-1}(W_h^{\mathrm{s}})$ can be represented as the graph of a real analytic h-periodic function $\Theta(t)$,

$$\Theta(t) = 8\sqrt{2}|\Theta_1|h^{-6}e^{-2\pi^2/h}\sin\frac{2\pi t}{h} + O\big(h^{-5}e^{-2\pi^2/h}\big).$$

Gelfreich and Sauzin [22] proved that $|\Theta_1| \neq 0$ (see also [5], where some numerical results are described).

Step 3: *Birkhoff normal form and construction of a horseshoe.* Recall that a real analytic area-preserving diffeomorphism of a two-dimensional domain in a neighborhood of a saddle with eigenvalues (λ, λ^{-1}) can be reduced by an analytic change of coordinates to the Birkhoff normal form [57] (see also [53]):

$$N(x,y) = (\Delta(xy)x, \Delta^{-1}(xy)y), \tag{5}$$

where $\Delta(xy) = \lambda + a_1xy + a_2(xy)^2 + \dots$ is analytic. From [15, 12, 13] it follows that for an analytic one-parameter family of maps the change of coordinates and the function Δ depend analytically on the parameter. Together with the description of the splitting of separatrices, this allows one not only to construct the horseshoes for $\mathfrak{F}_h$ using the transversal homoclinic points, but also to estimate some quantitative characteristics of these horseshoes. Namely, the dynamics in a neighborhood of the saddle is controlled by the Birkhoff normal form, the dynamics and geometry in a neighborhood of $\widetilde{\sigma}$ are described by the Gelfreich normal form and the form of the splitting of separatrices, and all the transitions and changes of coordinates have uniformly bounded distortions.

Step 4: *Estimates of the left and right thicknesses for the constructed horseshoes.* Topologically, a constructed horseshoe K is a product of a "stable" and "unstable" Cantor sets K^{s} and K^{u}. Moreover, the Hausdorff dimension $\dim_{\mathrm{H}} K = \dim_{\mathrm{H}} K^{\mathrm{s}} + \dim_{\mathrm{H}} K^{\mathrm{u}}$ (see [32, 47]). Therefore, we can consider each of the Cantor sets separately. We will first estimate the left and right thicknesses of K^{s} and K^{u}.

Let I be a finite closed interval and $\psi_1, \psi_2 \colon I \to I$ be strictly monotonous contracting maps, $\psi_1(I) \cap \psi_2(I) = \varnothing$. Denote $I_1 = \psi(I) \cup \psi_2(I)$ and set $I_{n+1} = \psi_1(I_n) \cup \psi_2(I_n)$. Then $C = \bigcap_{n\in\mathbb{N}} I_n$ is a Cantor set.

To define the left and right thickness, we consider the gaps in the Cantor set C. A *gap* of C is a bounded component of the complement $\mathbb{R} \setminus C$. The gaps of C are ordered in the following way. A bounded component U of $\mathbb{R} \setminus I_1$ is a gap of order zero (see the figure). A bounded component U' of $\mathbb{R} \setminus I_{n+1}$ which is not a gap of order less than or equal to $n - 1$ is a gap of order n. For example,

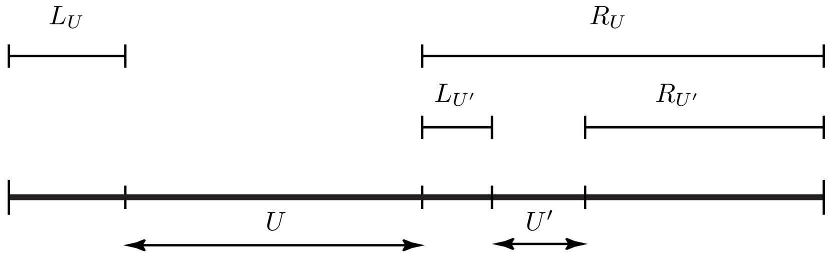

Figure.

U' in the figure is a gap of order 1. It is straightforward to check that every gap of C is a gap of some finite order.

Given a gap U of C of order n, we denote by L_U (respectively, R_U) the component of I_{n+1} that is left (respectively, right) adjacent to U. The greatest lower bounds

$$\tau_{\mathrm{L}}(C) = \inf\left\{\frac{|L_U|}{|U|}\colon U \text{ is a gap of } C\right\} \qquad \text{and} \qquad \tau_{\mathrm{R}}(C) = \inf\left\{\frac{|R_U|}{|U|}\colon U \text{ is a gap of } C\right\}$$

are called the *left* and *right thickness* of C, respectively. For more details on the left and right thickness, see [12] and [36]. See also [46] for a more standard definition and properties of the thickness of a Cantor set.

Fix any small constant $\nu > 0$. Using the Birkhoff normal form and the description of the splitting of separatrices, one can carry out the construction of the Cantor set K in such a way that $\tau_{\mathrm{L}}(K^{\mathrm{s}}) \sim h^{-1}$ and $\tau_{\mathrm{R}}(K^{\mathrm{s}}) \sim h^{\nu}$ as $h \to 0$.

Step 5: *Relation between the one-sided thicknesses and the Hausdorff dimension of a Cantor set.* We will use the following

Proposition 1. *Denote by τ_{L} and τ_{R} the left and right thicknesses of a Cantor set $C \subset \mathbb{R}$. Then the Hausdorff dimension*

$$\dim_{\mathrm{H}} C > \max\left(\frac{\ln\left(1 + \frac{\tau_{\mathrm{R}}}{1+\tau_{\mathrm{L}}}\right)}{\ln\left(1 + \frac{1+\tau_{\mathrm{R}}}{\tau_{\mathrm{L}}}\right)}, \frac{\ln\left(1 + \frac{\tau_{\mathrm{L}}}{1+\tau_{\mathrm{R}}}\right)}{\ln\left(1 + \frac{1+\tau_{\mathrm{L}}}{\tau_{\mathrm{R}}}\right)}\right).$$

In our case this implies that the Hausdorff dimension $\dim_{\mathrm{H}} K^{\mathrm{s}} > \frac{1}{1+\nu}$ if h is small enough. Therefore, $\dim_{\mathrm{H}} K > \frac{2}{1+\nu}$. Since ν could be chosen arbitrary small, this proves Theorem 1. ☐

2.2. Conservative homoclinic bifurcations and hyperbolic sets of large Hausdorff dimension. In order to construct transitive invariant sets of full Hausdorff dimension, we use the notion of a *homoclinic class.*

Definition 1. Let P be a hyperbolic saddle of a diffeomorphism f. A homoclinic class $H(P, f)$ is the closure of the union of all transversal homoclinic points of P.

It is known that $H(P, f)$ is a transitive invariant set of f (see [43]). Moreover, consider all basic sets (locally maximal transitive hyperbolic sets) that contain the saddle P. The homoclinic class $H(P, f)$ is the smallest closed invariant set that contains all of them.

Theorem 2. *Let $f_0 \in \mathrm{Diff}^{\infty}(M^2, \mathrm{Leb})$ have an orbit $\mathcal{O}$ of quadratic homoclinic tangencies associated with some hyperbolic fixed point P_0 and $\{f_\mu\}$ be a generic unfolding of f_0 in* $\mathrm{Diff}^{\infty}(M^2, \mathrm{Leb})$. *Then for any $\delta > 0$ there is an open set $\mathcal{U} \subseteq \mathbb{R}^1$, $0 \in \overline{\mathcal{U}}$, such that the following holds*:

(1) *for every $\mu \in \mathcal{U}$ the map f_μ has a basic set Δ_μ that contains the unique fixed point P_μ near P_0, exhibits persistent homoclinic tangencies, and whose Hausdorff dimension*

$$\dim_{\mathrm{H}} \Delta_\mu > 2 - \delta;$$

(2) *there is a dense subset* $\mathcal{D} \subseteq \mathcal{U}$ *such that for every* $\mu \in \mathcal{D}$ *the map* f_μ *has a homoclinic tangency of the fixed point* P_μ;

(3) *there is a residual subset* $\mathcal{R} \subseteq \mathcal{U}$ *such that for every* $\mu \in \mathcal{R}$

(3.1) *every point of the homoclinic class* $H(P_\mu, f_\mu)$ *is an accumulation point of generic elliptic points of* f_μ,

(3.2) *the homoclinic class* $H(P_\mu, f_\mu)$ *contains hyperbolic sets of Hausdorff dimension arbitrarily close to* 2; *in particular,* $\dim_{\mathrm{H}} H(P_\mu, f_\mu) = 2$.

Sketch of proof of Theorem 2. *Step* 1: *A sequence of bifurcation values* $\mu_n \to 0$ *with quadratic homoclinic tangencies.* A generic one-parameter family of diffeomorphisms unfolding a quadratic homoclinic tangency does not have isolated bifurcation values of the parameter (see, e.g., [46]). Therefore, we can choose a sequence of parameters $\{\mu_n\}$, $\mu_n \to 0$, such that f_{μ_n} has a quadratic homoclinic tangency and transversal homoclinic points.

Step 2: *Appearance of invariant hyperbolic sets of large Hausdorff dimension.* Using the renormalization techniques by Mora–Romero [34], one can make an appropriately chosen and rescaled map near a homoclinic tangency C^r-close to a Hénon map H_a for any given a. By Theorem 1, for a slightly larger than -1, the map H_a has an invariant hyperbolic set $\widetilde{\Lambda}_a$ of Hausdorff dimension close to 2 with persistent hyperbolic tangencies. By continuous dependence of the Hausdorff dimension of an invariant hyperbolic set on a diffeomorphism [32, 47], near each μ_n there is an open interval of parameters U_n such that for $\mu \in U_n$ the map f_μ has an invariant locally maximal transitive hyperbolic set Δ^*_μ which also has persistent homoclinic tangencies and whose Hausdorff dimension is greater than $2 - \delta$.

Step 3: *Connecting the invariant set* Λ^*_μ *with* P_μ. The hyperbolic saddle P_μ and the set Δ^*_μ are homoclinically related (see Lemma 2 from [11]). Therefore, for every $\mu \in U_n$ there exists a basic set Δ_μ such that $P_\mu \in \Delta_\mu$ and $\Delta^*_\mu \subset \Delta_\mu$. Since Δ^*_μ has persistent homoclinic tangencies, so does Δ_μ. Also, $\dim_{\mathrm{H}} \Delta_\mu \geq \dim_{\mathrm{H}} \Delta^*_\mu > 2 - \delta$. This proves statement (1).

Step 4: *Completion of the proof of statement* (2). Since Λ^*_μ has persistent homoclinic tangencies, standard arguments (see, e.g., [46]) show that for each $\mu \in D_n$ in a dense subset of parameters $D_n \subset U_n$, f_μ has a homoclinic tangency for the fixed point P_μ. This step completes the proof of statement (2).

Step 5: *Construction of elliptic periodic points.* Take any $\mu \in U_n$. If Q_μ is a transversal homoclinic point of the saddle P_μ, then it can be continued for some intervals of parameters $I_Q \subseteq U_n$. Assume that $I_Q \subseteq U_n$ is a maximal subinterval of U_n where such a continuation is possible. All homoclinic points of P_μ for all values $\mu \in U_n$ generate a countable number of such subintervals $\{I_s\}_{s\in\mathbb{N}}$ in U_n.

From [34] it follows that for each I_s there exists a residual set $R_s \subseteq I_s$ of parameters such that for $\mu \in R_s$ the corresponding homoclinic point Q_μ is an accumulation point of elliptic periodic points of f_μ. Denote $\widetilde{R}_s = (U_n \setminus \overline{I}_s) \cup R_s$, a residual subset of U_n. Now set $\mathcal{R}_1 = \bigcap_{s\in\mathbb{N}} \widetilde{R}_s$, which is also a residual subset in U_n. For $\mu \in \mathcal{R}_1$ every transversal homoclinic point of the saddle P_μ is an accumulation point of elliptic periodic points of f_μ, and this proves (3.1).

Step 6: *Construction of a homoclinic class of full Hausdorff dimension.* From Theorem 1 and [34] it follows that for every $m \in \mathbb{N}$ there exists an open and dense subset $A_m \subset U_n$ such that for every $\mu \in A_m$ there exists a hyperbolic set Δ^m_μ such that $\dim_{\mathrm{H}} \Delta^m_\mu > 2 - \frac{1}{m}$. From [11, Lemma 2] it follows that P_μ and Δ^m_μ are homoclinically related. Therefore, there exists a basic set $\widetilde{\Delta}_\mu$ such that $P_\mu \in \widetilde{\Delta}^m_\mu$ and $\Delta^m_\mu \subset \widetilde{\Delta}^m_\mu$. In particular, for $\mu \in \mathcal{R}_2 = \bigcap_{m\geq 1} A_m$ we have $\dim_{\mathrm{H}} H(P_\mu, f_\mu) = 2$. Set $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$. This proves (3.2).

This completes the sketch of the proof of Theorem 2. □

3. STANDARD MAP

The KAM theorem on the conservation of quasiperiodic motions in near-integrable Hamiltonian systems gave rise to the question on dynamical behavior in the regions where invariant tori are destroyed. In a more general form this question can be stated in the following way: Can an analytic symplectic map have a chaotic component of positive measure coexisting with the Kolmogorov–Arnold–Moser (KAM) tori? Katok [28] gave a construction of a C^∞-smooth Bernoulli diffeomorphism on the two-dimensional disc which is equal to the identity on the boundary. One can perform a "smooth surgery" to combine this transformation with any other type of transformations. It turns out that in principle quasiperiodic motions and the Bernoulli (chaotic) component can coexist in a smooth area-preserving dynamical system. Since then several more or less artificial examples of coexistence of regular and chaotic component were suggested [4, 8, 29, 50, 58]. Nevertheless, for "natural" examples (including those that appear in applications) the rigorous proof of positivity of the metric entropy (due to Pesin's theory [49] this is equivalent to the existence of a positive measure set of orbits with nonzero Lyapunov exponents) is still missing. The simplest and most famous system where one would expect *mixed behavior* (KAM tori and orbits with nonzero Lyapunov exponents both have positive measure) is the Taylor–Chirikov standard map of the two-dimensional torus $\mathbb{T}^2$, given by

$$f_k(x,y) = \big(x + y + k\sin(2\pi x), y + k\sin(2\pi x)\big) \mod \mathbb{Z}^2. \tag{6}$$

This family is also a model for numerous physical problems (see, e.g., [6, 26, 56]).

Problem (Sinai [54])**.** Is the metric entropy of f_k positive for some values of k? for positive measure of values of k? for all nonzero values of k?

Currently, even the existence of at least one value of k with this property is not known. In the study of the standard family in the current context Duarte [10] proved the following important result:

Theorem A (Duarte [10])**.** *There is a family of basic sets Λ_k of f_k such that*

(i) *Λ_k is dynamically increasing, meaning that for small $\varepsilon > 0$, $\Lambda_{k+\varepsilon}$ contains the continuation of Λ_k at parameter $k+\varepsilon$;*

(ii) *the Hausdorff dimension of Λ_k increases up to 2. For large k,*

$$\dim_{\mathrm{H}}(\Lambda_k) \ge 2\frac{\ln 2}{\ln\big(2 + \frac{9}{k^{1/3}}\big)};$$

(iii) *Λ_k fills in $\mathbb{T}^2 \ni (x,y)$, meaning that as k goes to ∞, the maximum distance from any point in $\mathbb{T}^2$ to Λ_k tends to 0. For large k, the set Λ_k is δ_k-dense on $\mathbb{T}^2$ for $\delta_k = \frac{4}{k^{1/3}}$.*

Theorem B (Duarte [10])**.** *There exists $k_0 > 0$ and a residual set $R \subseteq [k_0, \infty)$ such that for $k \in R$ the closure of the elliptic points of f_k contains Λ_k.*

Here we provide an improvement of Theorems A and B that states, roughly speaking, that the stochastic layer of the standard map has full Hausdorff dimension for large parameters from a residual set in the space of parameters.

Theorem 3. *There exists $k_0 > 0$ and a residual set $\mathcal{R} \subset [k_0, +\infty)$ such that for every $k \in \mathcal{R}$ there exists an infinite sequence of transitive locally maximal hyperbolic sets of the map f_k*

$$\Lambda_k^{(0)} \subseteq \Lambda_k^{(1)} \subseteq \Lambda_k^{(2)} \subseteq \ldots \subseteq \Lambda_k^{(n)} \subseteq \ldots \tag{7}$$

that has the following properties:

(i) *the set $\Lambda_k^{(0)} = \Lambda_k$, where the family of sets $\{\Lambda_k\}$ is described in Theorem* A;

(ii) *the Hausdorff dimension* $\dim_{\mathrm H}\Lambda_k^{(n)} \to 2$ *as* $n\to\infty$;

(iii) $\Omega_k = \overline{\bigcup_{n\in\mathbb N}\Lambda_k^{(n)}}$ *is a transitive invariant set of the map* f_k, *and* $\dim_{\mathrm H}\Omega_k = 2$;

(iv) *for any* $x\in\Omega_k$, $k\in\mathcal R$, *and any* $\varepsilon>0$, *the Hausdorff dimension*

$$\dim_{\mathrm H} B_\varepsilon(x)\cap\Omega_k = \dim_{\mathrm H}\Omega_k = 2,$$

where $B_\varepsilon(x)$ *is an open ball of radius* ε *centered at* x;

(v) *each point of* Ω_k *is an accumulation point of elliptic islands of the map* f_k.

For an open set of parameters our construction provides invariant hyperbolic sets of Hausdorff dimension arbitrarily close to 2.

Theorem 4. *There exists* $k_0>0$ *such that for any* $\delta>0$ *there exists an open and dense subset* $U\subset[k_0,+\infty)$ *such that for every* $k\in U$ *the map* f_k *has an invariant hyperbolic set of Hausdorff dimension greater than* $2-\delta$.

Notice that these results give a partial explanation of the difficulties that we encounter studying the standard family. Indeed, one of the possible approaches is to consider an invariant hyperbolic set in the stochastic layer and to try to extend the hyperbolic behavior to a larger part of the phase space through homoclinic bifurcations. Unavoidably Newhouse domains (see [42, 51] for the dissipative case and [11–13] for the conservative case) associated with the absence of hyperbolicity appear after small change of the parameter. If the Hausdorff dimension of the initial hyperbolic set is less than 1, then the measure of the set of parameters that correspond to Newhouse domains is small and has zero density at the critical value (see [44, 45]). For the case when the Hausdorff dimension of the hyperbolic set is slightly greater than 1, a similar result was recently obtained by Palis and Yoccoz [48], and the proof is astonishingly involved. They also conjectured that an analogous property holds for an initial hyperbolic set of any Hausdorff dimension, but the proof would require even more technical and complicated considerations. Here is what Palis and Yoccoz [48] wrote:

> Of course, we expect the same to be true for all cases $0<\dim_{\mathrm H}(\Lambda)<2$. For that, it seems to us that our methods need to be considerably sharpened: we have to study deeper the dynamical recurrence of points near tangencies of higher order (cubic, quartic, …) between stable and unstable curves. We also hope that the ideas introduced in the present paper might be useful in broader contexts. In the horizon lies the famous question whether for the standard family of area preserving maps one can find sets of positive Lebesgue probability in parameter space such that the corresponding maps display nonzero Lyapunov exponents in sets of positive Lebesgue probability in phase space.

Theorems 3 and 4 show that in order to understand the dynamics of the stochastic layer of the standard map, one has to face these difficulties.

Proof of Theorems 3 and 4. First of all, we reduce Theorem 3 to the following proposition. Denote by $\mathcal N(N)=(n_1,\dots,n_N)$ an N-tuple with $n_i\in\mathbb N$.

Proposition 2. *There exists* $k_0>0$ *such that for each* $N\in\mathbb N$ *there is a family of finite open intervals* $\mathcal U_{\mathcal N(N)}\subseteq[k_0,+\infty)$ *indexed by* N*-tuples* $\mathcal N(N)=(n_1,\dots,n_N)$ *satisfying the following properties*:

(U1) *For any pair of tuples* $\mathcal N(N)\ne\mathcal N'(N)$, *the intervals* $\mathcal U_{\mathcal N(N)}$ *and* $\mathcal U_{\mathcal N'(N)}$ *are disjoint.*

(U2) *For any tuple* $\mathcal N(N+1)=(\mathcal N(N),n_{N+1})$, *we have* $\mathcal U_{\mathcal N(N+1)}\subseteq\mathcal U_{\mathcal N(N)}$.

(U3) *The union* $\bigcup_{n_1\in\mathbb N}\mathcal U_{n_1}$ *is dense in* $[k_0,+\infty)$, *and for each* $N\in\mathbb N$ *the union* $\bigcup_{j\in\mathbb N}\mathcal U_{(\mathcal N(N),j)}$ *is dense in* $\mathcal U_{\mathcal N(N)}$.

(U4) *Every diffeomorphism* f_k, $k \in \mathcal{U}_{\mathcal{N}(N)}$, *has a sequence of invariant basic sets*

$$\Lambda_k^{(n_1)} \subseteq \Lambda_k^{(n_1,n_2)} \subseteq \ldots \subseteq \Lambda_k^{\mathcal{N}(N)},$$

and $\Lambda_k^{\mathcal{N}(N)}$ *depends continuously on* $k \in \mathcal{U}_{\mathcal{N}(N)}$.

(U5) $\Lambda_k \subseteq \Lambda_k^{(n_1)}$ *for each* $n_1 \in \mathbb{N}$ *and* $k \in \mathcal{U}_{n_1}$, *where* Λ_k *is a hyperbolic set from Theorem* A.

(U6) $\dim_{\mathrm{H}} \Lambda_k^{\mathcal{N}(N)} > 2 - 1/N$.

(U7) *For any point* $x \in \Lambda_k^{\mathcal{N}(N)}$, *there exists an elliptic periodic point* p_x *of* f_k *such that* $\mathrm{dist}(p_x, x) < 1/N$.

Theorems 3 and 4 follow from Proposition 2. Indeed, set $\mathbf{U}_N = \bigcup_{\mathcal{N}(N)} \mathcal{U}_{\mathcal{N}(N)}$. Due to (U3) the set $\mathbf{U}_N$ is dense in $[k_0, +\infty)$. Therefore, $\mathcal{R} = \bigcap_{N\in\mathbb{N}} \mathbf{U}_N$ is a residual subset of $[k_0, +\infty)$. Properties (U1) and (U2) imply that for each $k \in \mathcal{R}$ the value k belongs to each element of the uniquely defined nested sequence of intervals

$$\mathcal{U}_{n_1} \supseteq \mathcal{U}_{n_1,n_2} \supseteq \ldots \supseteq \mathcal{U}_{\mathcal{N}(N)} \supseteq \ldots.$$

Therefore, for $k \in \mathcal{R}$ the sequence of basic sets

$$\Lambda_k \subseteq \Lambda_k^{(n_1)} \subseteq \Lambda_k^{(n_1,n_2)} \subseteq \ldots \subseteq \Lambda_k^{\mathcal{N}(N)} \subseteq \ldots$$

is defined such that the Hausdorff dimension $\dim_{\mathrm{H}} \Lambda_k^{\mathcal{N}(N)} > 2 - 1/N$. Since k is fixed now, redenote $\Lambda_k^N = \Lambda_k^{\mathcal{N}(N)}$. Statements (i) and (ii) of Theorem 3 follow from (U5) and (U6).

The closure of the union of a nested sequence of transitive sets is transitive, so property (iii) follows.

For a locally maximal transitive invariant set of a surface diffeomorphism, the Hausdorff dimension of the set is equal to the Hausdorff dimension of any open subset of this set (see [32]). This implies property (iv) for the sets Ω_k, $k \in \mathcal{R}$.

Finally, property (v) follows directly from (U7).

In order to prove Theorem 4, one just needs to consider the family of basic sets $\Lambda_k^{\mathcal{N}(N)}$ defined for $k \in \mathbf{U}_N$ for large enough N. Then Proposition 2 itself can be reduced to the following

Lemma 1. *Given* $k^* \in (k_0, +\infty)$, $\varepsilon > 0$, *and* $\delta > 0$, *there exists a finite open interval* $V \subset (k^* - \varepsilon, k^*)$ *such that for all* $k \in V$ *the map* f_k *has a basic set* Λ_k^* *such that*

(1) Λ_k^* *depends continuously on* $k \in V$;

(2) $\Lambda_k^* \supseteq \Lambda_k$, *where* Λ_k *is a basic set from Theorem* A;

(3) *the Hausdorff dimension* $\dim_{\mathrm{H}} \Lambda_k^* > 2 - \delta$;

(4) *for any point* $x \in \Lambda_k^*$, *there exists an elliptic periodic point* p_x *of* f_k *such that* $\mathrm{dist}(p_x, x) < \delta$.

Indeed, let us show how to construct the intervals $\mathcal{U}_{n_1}$ and the sets $\Lambda_k^{(n_1)}$. Let $\{k_l\}_{l\in\mathbb{N}}$ be a dense set of points in $(k_0, +\infty)$. Apply Lemma 1 to each $k^* = k_l$, $l \in \mathbb{N}$, for $\delta = \delta_1$ and $\varepsilon = \varepsilon_l < \frac{1}{l}$. This gives a sequence of open intervals $\{V_l\}_{l\in\mathbb{N}}$. Since the sequence $\{k_l\}_{l\in\mathbb{N}}$ is dense in $(k_0, +\infty)$ and $\varepsilon_l \to 0$, the intervals $\{V_l\}$ are dense in $(k_0, +\infty)$.

Take $\mathcal{U}_1 = V_1$. If $\mathcal{U}_1, \ldots, \mathcal{U}_t$ are constructed, take V_s, the first interval in the sequence $\{V_l\}_{l\in\mathbb{N}}$ that is not contained in $\overline{\bigcup_{n_1=1}^t \mathcal{U}_{n_1}}$. Then $V_s \setminus \overline{\bigcup_{n_1=1}^t \mathcal{U}_{n_1}}$ is a finite union of K open intervals. Take these intervals as $\mathcal{U}_{t+1}, \ldots, \mathcal{U}_{t+K}$ and continue in the same way. This gives a sequence of disjoint intervals $\{\mathcal{U}_{n_1}\}_{n_1\in\mathbb{N}}$ with the desired properties.

Now, assume that intervals $\{\mathcal{U}_{\mathcal{N}(N)}\}$ are constructed. Take one of the intervals $\mathcal{U}_{\mathcal{N}(N)}$. The set $\Lambda_k^{\mathcal{N}(N)}$ exhibits persistent tangencies, as the following result by Duarte states:

Theorem C (Duarte [10]). *There exists $k_0 > 0$ such that for any $k \geq k_0$ and any periodic point $P \in \Lambda_k$, the set of parameters $k' \geq k$ at which the invariant manifolds $W^{\mathrm{s}}(P(k'))$ and $W^{\mathrm{u}}(P(k'))$ generically unfold a quadratic tangency is dense in $[k, +\infty)$.*

Recall that $P(k')$ denotes the continuation of the periodic saddle P at parameter k'.

Therefore, the application of Theorem 2 gives a dense sequence of intervals $\{V_{\mathcal{N}(N),l}\}_{l\in\mathbb{N}}$ in $\mathcal{U}_{\mathcal{N}(N)}$ such that for each $k \in V_{\mathcal{N}(N),l}$ the map f_k has a basic set Δ_k such that the Hausdorff dimension $\dim_{\mathrm{H}} \Delta_k > 2 - \frac{1}{N+1}$ and $\Delta_k \cap \Lambda_k^{\mathcal{N}(N)} \neq \varnothing$.

The following lemma is a standard statement from hyperbolic dynamics.

Lemma 2. *Let Δ_1 and Δ_2 be two basic sets of a diffeomorphism $f\colon M^2 \to M^2$ of a surface M^2. Suppose that $\Delta_1 \cap \Delta_2 \neq \varnothing$. Then there is a basic set $\Delta_3 \subseteq M^2$ such that $\Delta_1 \cup \Delta_2 \subseteq \Delta_3$.*

Apply Lemma 2 to Δ_k and $\Lambda_k^{\mathcal{N}(N)}$, and denote by $\widetilde{\Lambda}_k^{\mathcal{N}(N)} \supset \Delta_k \cup \Lambda_k^{\mathcal{N}(N)}$ the corresponding basic set. The set $\widetilde{\Lambda}_k^{\mathcal{N}(N)}$ also has persistent tangencies. The unfolding of a homoclinic tangency creates elliptic periodic orbits that shadow the orbit of homoclinic tangencies. The creation of these generic elliptic points can be seen from the renormalization at conservative homoclinic tangencies (see [34]). Shrinking $V_{(\mathcal{N}(N),l)}$ if necessary, we can guarantee that every point of $\widetilde{\Lambda}_k^{\mathcal{N}(N)}$ can be δ_{N+1}-approximated by an elliptic periodic point. Now the same procedure that we applied above to the intervals $\{V_l\}$ gives a collection of disjoint intervals $\{\mathcal{U}_{(\mathcal{N}(N),n_{N+1})}\}_{n_{N+1}\in\mathbb{N}}$ in $\mathcal{U}_{\mathcal{N}(N)}$. For any $k \in \mathcal{U}_{\mathcal{N}(N),n_{N+1}} \subset V_{(\mathcal{N}(N),l)}$ we take $\Lambda_k^{(\mathcal{N}(N),n_{N+1})} = \widetilde{\Lambda}_k^{\mathcal{N}(N)}$. Now all the properties in Proposition 2 are satisfied.

Finally, Lemma 1 follows directly from Theorem 2 and Theorem C. □

4. HYPERBOLIC SETS OF LARGE HAUSDORFF DIMENSION IN THREE-BODY PROBLEMS

Initially our interest in the conservative Newhouse phenomena was motivated by the fact that it appears in the three-body problem. The classical three-body problem consists in studying the dynamics of three point masses in the plane or in the three-dimensional space that attract each other under the Newton gravitation. The three-body problem is called *restricted* if one of the bodies has mass zero and the other two masses are strictly positive. In the pioneering work [1] Alexeev found an important application of hyperbolic dynamics to the three-body problem. He proved the existence of the so-called *oscillatory motions*. A motion in the three-body problem is called *oscillatory* if the limsup of the mutual distances is infinite and the liminf is finite. The existence of such motions was a long-standing open problem. The first rigorous example of the existence of such motions is due to Sitnikov [55] for the restricted spatial three-body problem. Alexeev extended the Sitnikov example to the spatial three-body problem. Later, interpreting homoclinic intersections, Moser [37] gave a conceptually transparent proof of the existence of oscillatory motions for the Sitnikov example. This paved the way for a variety of applications of hyperbolic dynamics to the three-body problem.

In this section we discuss the size of compact invariant hyperbolic sets in the Sitnikov example and the restricted planar circular three-body problem and show that these sets often have almost full Hausdorff dimension.

4.1. Sitnikov's example.

Consider two point masses q_1 and q_2 of equal mass $m_1 = m_2 = 1/2$. Suppose they move on the plane so that the center of mass is at the origin. Assume that their orbits are elliptic of eccentricity $e > 0$ and period 2π. We will treat e as a parameter. Consider a third massless point q_3 moving along the z-axis. Due to symmetry, if the initial condition and velocity belong to the z-axis, then the whole orbit of q_3 also belongs to the z-axis. Denote by $(t, z(t), \dot{z}(t))$ the orbit of q_3, where the time t (mod 2π) determines the position of the primaries. Denote by $r(t) = r_e(t)$ the distance from the primaries to the origin. Then the equation of motion of the massless body has the form

$$\ddot{z} = -\frac{z}{\sqrt{z^2 + r^2(t)}}, \tag{8}$$

and the corresponding Hamiltonian is time-periodic,

$$H(z, Z, t) = \frac{Z^2}{2} - \frac{1}{\sqrt{z^2 + r^2(t)}},$$

where Z is the variable conjugate to z, which coincides with the velocity of z.

Theorem 5. *There is an open set $\mathcal{N} \subset (0,1)$ of values of the eccentricity e and a residual subset $\mathcal{R} \subset \mathcal{N}$ such that for $e \in \mathcal{R}$ there are compact invariant hyperbolic sets of Hausdorff dimension arbitrarily close to* 3.

4.2. The restricted planar circular three-body problem. Consider the restricted planar circular three-body problem (RPC3BP). Namely, consider two massive bodies, called *primaries*, performing uniform circular motion about their center of mass. Normalizing the masses of the primaries so that their sum is equal to one, we obtain primaries of mass μ and $1-\mu$, where $0 < \mu < 1$ is called the *mass ratio*. In addition, we chose coordinates so that the center of mass of the system is located at the origin, and we normalize the period of the circular motion to 2π. By considering a frame that rotates with the primaries, we can choose rectangular coordinates (x, y) so that the primaries are fixed at $(1-\mu, 0)$ and $(-\mu, 0)$, respectively. Finally, we introduce a third massless body P into the system, so that it does not effect the primaries. In RPC3BP one investigates how P moves.

The distance from P to the primaries is given by $d_1(x,y) = [(x-(1-\mu))^2+y^2]^{1/2}$ and $d_2(x,y) = [(x-\mu)^2+y^2]^{1/2}$. The standard formula for the *Jacobi constant* C, the only integral for RPC3BP, is given by

$$C_\mu(x, y, \dot{x}, \dot{y}) = x^2 + y^2 + \frac{2\mu}{d_1} + \frac{2(1-\mu)}{d_2} - (\dot{x}^2 + \dot{y}^2). \tag{9}$$

Denote by RPC3BP(μ, C) the RPC3BP with mass ratio μ restricted to the energy surface $\Pi_C = \{C_\mu(x, y, \dot{x}, \dot{y}) = C\}$. We will treat both μ and C as parameters. Below we will consider $C > 2\sqrt{2}$. Consider the set

$$\left\{(x,y)\colon\ (x^2+y^2) + \frac{2\mu}{d_1} + \frac{2(1-\mu)}{d_2} \geq C\right\}.$$

Notice that this set defines the set of possible positions of P provided that its initial condition is on the energy surface Π_C. One could show that for $C > 2\sqrt{2}$ this set consists of three disjoint regions, called *Hill regions*: one surrounds the primary with mass $1-\mu$, another, which is smaller, surrounds the other primary, and the last one occupies the complement of an open set that covers both primaries. The first one is called the *inner Hill region*, the second is the *lunar Hill region*, and the last one is the *outer Hill region*. Below we will study only the outer Hill region.

Here is the main result for RPC3BP.

Theorem 6. A. *For any $C > 2\sqrt{2}$ there is an open set of mass ratios $\mathcal{N}_C \subset (0,1)$ such that for a residual subset $\mathcal{R} \subset \mathcal{N}_C$ and for any $(\mu, C) \in \mathcal{R}$ in the three-dimensional energy surface Π_C there are compact invariant hyperbolic sets of* RPC3BP(μ, C) *of Hausdorff dimension arbitrarily close to* 3.

B. *For any $\mu \in (0,1)$ there is an open set $\mathcal{N}_\mu \subset (2\sqrt{2}, \infty)$ such that for a Baire generic $C \in \mathcal{N}_\mu$ in the three-dimensional energy surface Π_C there are compact invariant hyperbolic sets of* RPC3BP(μ, C) *of Hausdorff dimension arbitrarily close to* 3.

Remark 1. The minimal distance to the origin for a bounded orbit of the two-body problem in terms of the Jacobi constant can be arbitrarily close to $C^2/8$. Therefore, for $C = 2\sqrt{2}$ such an orbit might pass nearly at unit distance from the origin. This might lead to a near collision with the primary of mass μ. We want to avoid that.

Our technique could also be applied to the three-body problem on the line [31, 52], but we do not elaborate on it here.

4.3. Reduction to area-preserving maps. A natural way to reduce the Sitnikov example to a two-dimensional Poincaré map is as follows. Define

$$f_e\colon\ (z,\dot z)\mapsto (z',\dot z'),\qquad (z,\dot z)\in\mathbb{R}^2, \tag{10}$$

where a trajectory of (8) with the initial condition $(0,z,\dot z)$ at time 2π is located at $(2\pi,z',\dot z')$. Since the equations of motion are Hamiltonian, this map is *area-preserving*.

There are many ways to define a Poincaré map for RPC3BP(μ,C) with $C\ge 2\sqrt{2}$. Let us pick one. Consider the polar coordinates (r,φ) on the (x,y)-plane, and let (P_r,P_φ) be their symplectic conjugate. Write the Hamiltonian of RPC3BP in these coordinates:

$$H(r,P_r,\varphi,P_\varphi)=\frac{P_r^2}{2}+\frac{P_\varphi^2}{2r^2}-\frac{1}{r}-P_\varphi+\left(\frac{1}{r}-\frac{\mu}{d_1}-\frac{1-\mu}{d_2}\right)=:H_0+\Delta H,$$

where d_1 and d_2 are the distances to the primaries as above (9) and P_r (respectively, P_φ) is the variable conjugate to r (respectively, φ). In other words, $P_r=\dot r$ and P_φ is the angular momentum. One can rewrite the Jacobi constant in the polar coordinates.

Since the Jacobi constant is a first integral of this problem, there is a three-dimensional "energy" surface $\Pi_C=\{C=C_\mu(r,\varphi,P_r,P_\varphi)\}$. It turns out that for $C>2\sqrt{2}$ in the outer Hill region by the implicit function theorem one can express $P_\varphi=P_\varphi(r,\varphi,P_r,C)$ on Π_C and consider a three-dimensional differential equation for (r,φ,P_r). On a "large" open set $\dot\varphi=1-P_\varphi/r^2>0$ and $\varphi(t)$ is strictly monotone. Choose a two-dimensional surface $S=\{\varphi=0\}\subset\Pi_C$ and a Poincaré return map

$$f_{\mu,C}\colon\ (r,P_r)\mapsto (r',P_r'), \tag{11}$$

where a trajectory of RPC3BP with the initial condition $(r,0,P_r,P_\varphi(r,0,P_r,C))$ passes through $(r',2\pi,P_r',P_\varphi(r',2\pi,P_r',C))$. This gives rise to an *area-preserving map* $f_{\mu,C}\colon U\to\mathbb{R}^2$ defined on an open set $U\subset\mathbb{R}^2$.

4.4. Newhouse domains in three-body problems. Recall that a saddle periodic point p of an area-preserving map f exhibits a homoclinic tangency if the stable and unstable manifolds $W^{\mathrm s}(p)$ and $W^{\mathrm u}(p)$ of p, respectively, have a point of tangency. We say that f has a homoclinic tangency if some of its saddle points has a homoclinic tangency. Call an open set with a dense subset of maps with a homoclinic tangency a *Newhouse domain*.

Theorem 7. *Let* $\{f_e\}_{0<e<1}$ *be the family of maps* (10). *Then there is a Newhouse domain* $\mathcal{N}\subset(0,1)$; *i.e., for a dense set of* e *in* $\mathcal{N}$ *the Poincaré map* f_e *has a homoclinic tangency.*

Theorem 8. *Let* $\{f_{\mu,C}\}$ *be the family of maps* (11). *Then there exists* $C_0\ge 2\sqrt{2}$ *such that*

(A) *for any* $C>C_0$ *there is a Newhouse domain* $\mathcal{N}_C\subset(0,1)$; *i.e., for a dense set of* μ *in* $\mathcal{N}_C$ *the Poincaré map* $f_{\mu,C}$ *has a homoclinic tangency*;

(B) *for any* $\mu\in(0,1)$ *there is a Newhouse domain* $\mathcal{N}_\mu\subset(C_0,+\infty)$; *i.e., for a dense set of* C *in* $\mathcal{N}_\mu$ *the Poincaré map* $f_{\mu,C}$ *has a homoclinic tangency.*

Using the ideas of Newhouse, Robinson [51] showed that for a generic one-parameter unfolding of a homoclinic tangency there are Newhouse domains on the parameter line. In a sense we prove a similar statement for the two concrete conservative systems. Namely, we show that the above one-parameter families are nondegenerate and Newhouse domains occur on the parameter line, not in an infinite-dimensional space of mappings. The proofs of these two theorems are based on similar results for area-preserving Hénon maps (see [13] or Theorem 1 above).

4.5. Plan of proof of Theorems 5 and 7. The proofs of both parts of Theorem 6 and Theorem 8 follow very similar strategies burdened by more involved technical details.

In what follows the following motions play a special role.

Definition 2. A motion of the massless body is called *future* (respectively, *past*) *parabolic* if the body escapes to infinity with vanishing speed as time tends to $+\infty$ (respectively, $-\infty$).

Plan of proof of Theorems 5 and 7. Recall that $f_e\colon (z,\dot z)\mapsto (z',\dot z')$ is the Poincaré map (10).

The change of coordinates $(z,\dot z)\to (u=z^{-1/2},\ v=\dot z)$ sends $(z=\infty,\ \dot z=0)$ to the origin and the origin $(u,v)=0$ becomes a degenerate saddle fixed point. McGehee [33] showed that its separatrices are smooth manifolds, denoted $W^{\mathrm s}_e(0)$ and $W^{\mathrm u}_e(0)$, which correspond to parabolic motions. For $e=0$ the manifolds coincide $W^{\mathrm s}_0(0)=W^{\mathrm u}_0(0)$ and form a separatrix loop given by

$$\frac{v^2}{2}-\frac{2u^2}{\sqrt{4+u^2}}=0.$$

It turns out that for small positive e the separatrices $W^{\mathrm s}_e(0)$ and $W^{\mathrm u}_e(0)$ intersect transversally. This follows from the nondegeneracy of the Melnikov function proved by Moser [37]. An explicit form was calculated in [17] (see a related paper by Dankowicz and Holmes [7]). A similar statement for RPC3BP was proved by Llibre and Simó [30, 31] and later, with a different method, by Xia [59, Section 3].

Step 1: *Invariant cone field near degenerate saddle.* Similarly to the results in [36], one can show that there exists an invariant cone filed in a neighborhood of the degenerate saddle. Moreover, the differential of the transition from a point near $W^{\mathrm s}_e(0)$ to a point near $W^{\mathrm u}_e(0)$ through that neighborhood expands vectors from the invariant cones.

Step 2: *Hyperbolic periodic saddles near parabolic motions.* Similarly to the classical Poincaré–Birkhoff theorem, the cone condition in a neighborhood of the degenerate saddle and the existence of transversal homoclinic points imply the existence of hyperbolic saddle periodic points $\{p^m_e\}$ that accumulate to a homoclinic point. The compact parts of the stable and unstable manifolds of $\{p^m_e\}$ are C^1-close to the corresponding pieces of $W^{\mathrm s}_e(0)$ and $W^{\mathrm u}_e(0)$.

Step 3: *Appearance of a homoclinic tangency.* Using the splitting of separatrices, one can show that there is a sequence of e_k monotonically decreasing to zero such that there is a quadratic tangency of $W^{\mathrm s}_{e_k}(0)$ and $W^{\mathrm u}_{e_k}(0)$. Moreover, the unfolding of this tangency in e is nondegenerate. Since $W^{\mathrm s}_e(p^m_{e_k})$ and $W^{\mathrm u}_e(p^m_{e_k})$ are close to $W^{\mathrm s}_{e_k}(0)$ and $W^{\mathrm u}_{e_k}(0)$, there is also a sequence e'_k such that $W^{\mathrm s}_e(p^m_{e'_k})$ and $W^{\mathrm u}_e(p^m_{e'_k})$ have a quadratic tangency, and the unfolding of this tangency in e is also nondegenerate.

Step 4: *Generic unfolding of a homoclinic tangency and appearance of hyperbolic sets of large Hausdorff dimension.* The application of Theorem 2 to the nondegenerate unfolding of a quadratic tangency between $W^{\mathrm s}_e(p^m_{e'_k})$ and $W^{\mathrm u}_e(p^m_{e'_k})$ immediately proves both Theorem 5 and Theorem 7. $\square$

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