Introduction to partial differential equations
- The heat equation & separation of variables.

Chapter 5, p.481ff.

1. Introduction
So far considered ordinary differential eqns. (ODE), i.e. unknown fun. depends on only 1 variable. Now want to consider partial differential eqns. (PDE), where unknown fun. depends on more than one variable. A partial diff. eqn. is a relation that involves this fun. and some of its partial derivatives.

Examples: "classical" PDEs

1. heat flux between two bodies: Consider two bodies at fixed temperatures \( T_A \) and \( T_B \), \( T_A < T_B \). Connect them by a thin metal slab.

\[ \text{heat exchange from } A \to B \]

\[ \text{want to know about temperature distribution along the slab} \]

\[ T(x,t) \ldots \text{temp. at position } x \text{ in the slab and time } t \]
$T(x,t)$ is governed by heat equation:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

(1)

$k$ : thermal conductivity ($k > 0$).

well posed problem:

\[
\begin{cases}
\text{solve (1) given} & (t=0) \\
\text{(i) initial \ temperature \ distrib.} & T(x,0) \\
\text{(ii) boundary condition:} & T(0,t) = T_A \\
& T(L,t) = T_B, \text{ all } t.
\end{cases}
\]

2) wave equation

Consider a string fixed at its endpoints (e.g. string of a guitar). Describe "profile" of string by at time $t$ by $u(x,t)$:

This is described by wave equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$$

(2)

$c$: speed of propagation

Solve (2) given that:

(i) initial profile $u(x,0)$

(ii) initial speed $u_t(x,0)$, where $u_t = \frac{\partial u}{\partial t}$
(iii) Boundary condition:
\[ u(0,t) = u(L,t) = 0 \] (string is fixed).

3. Laplace equation

\[ \Delta u = 0, \quad \text{where} \]

\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]  \hspace{1cm} \text{(in } \mathbb{R}^2 \text{)} \hspace{1cm} (3)

application in electromatics where
\[ u \] describes the electrostatic potential
in vacuum.

Want to focus on the heat equation (1).

Terminology: Similarly to ODE we define
the order of a PDE to be the order
of the highest partial derivative
involved in the eqn. The above named examples are all of order
2.

2. The heat equation - separation of variables

Want to solve the following boundary value
problem (BVP):

\[ \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \]
\[ u(0,t) = u(L,t) = 0 \]  \hspace{1cm} (4)
properties of solutions to (4):

Prop: Let \( u_1(x,t) \) and \( u_2(x,t) \) be two solutions to the BVP in (4), then so is any linear combination \( (c_1 u_1 + c_2 u_2) \) \( c_1, c_2 \in \mathbb{R} \).

Proof: Since both \( u_1, u_2 \) are solutions to (4), we have for \( c_1, c_2 \in \mathbb{R} \):

\[
\left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right) (c_1 u_1(x,t) + c_2 u_2(x,t)) = 0
\]

\[
= c_1 \left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right) u_1(x,t) + c_2 \left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right) u_2(x,t) = 0
\]

\[
= 0 ; \text{ thus, } (c_1 u_1(x,t) + c_2 u_2(x,t)) \text{ satisfies the heat equation, hence,}
\]

\[
c_1 u_1(0,t) + c_2 u_2(0,t) = 0
\]

\[
c_1 u_1(L,t) + c_2 u_2(L,t) = 0
\]

\( (c_1 u_1 + c_2 u_2) \) also fulfills the B.C. \( \square \)

Hence, the family of solutions to (4) ("general solution of the BVP") form a vector space. Additionally, we may impose an initial condition on the solutions to (4):

\[
u(0,x,0) = f(x) \quad (5)
\]
In our considerations (and in many applications) \( f(x) \) is a piecewise continuous function on \([0, L]\).

How to solve (4)?

• \textit{Ansatz ("separation of variables")}: \[ u(x, t) = X(x) \cdot T(t) \quad (6) \]

(6) is a sin to (4) iff

\[
\left[ \alpha^2 \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right] u(x, t) = \left[ \alpha^2 \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right] X(x) T(t) = \\
= \alpha^2 \frac{\partial^2}{\partial x^2} (X(x) T(t)) - \frac{\partial}{\partial t} (X(x) T(t)) = \\
= \alpha^2 T(t) X''(x) - X(x) T'(t) = 0
\]

(\Rightarrow) \quad \frac{X''(x)}{X(x)} = \frac{1}{\alpha^2} \frac{T'(t)}{T(t)} \quad \text{all } x \text{ and } t

(\Rightarrow) \quad \frac{X''(x)}{X(x)} = -\lambda \quad \text{and} \quad \frac{T'(t)}{T(t)} = -\frac{\lambda}{\alpha^2} \quad \text{for some constant } \lambda \in \mathbb{R}.

The boundary conditions for \( u(x, t) \) imply:

\[ 0 = u(0, t) = X(0) T(t) \]
\[ 0 = u(L, t) = X(L) T(t), \text{ all } t \]

\[ \Rightarrow \quad X(0) = X(L) = 0 \quad \text{ (for non-trivial } T(t)) \]
Solving (4) is thus equivalent to solving two ordinary differential equations, one of which is a BVP:

\[
\begin{align*}
X''(x) + \lambda X(x) &= 0 \\
X(0) &= X(L) = 0
\end{align*}
\]

and

\[
T''(t) + \alpha^2 \pi^2 T(t) = 0
\]

We want to find a non-trivial soln. to (9) and (10) (i.e. a soln. which is not zero everywhere).

1. Start with the soln. of (9):

a) 2nd order ode (homog.):

differential polynomial: \( r^2 + \lambda = 0 \)

3 cases:

i) \( \lambda < 0 \) \( \Rightarrow \) \( r^2 = -\lambda \geq 0 \)

\[ r_{1/2} = \pm \sqrt{\lambda} \]

= gen. soln. to (9): \( X(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} \)

boundary condition: \( X(0) = X(L) = 0 \)

(11) only satisfies these bc. iff \( c_1 = c_2 = 0 \) (trivial soln.)

\[ \Rightarrow \] No non-trivial soln. for \( \lambda < 0 \).
\[ X''(x) = 0 \]

\[ = \text{general sol.: } X(x) = c_1 + c_2 x \quad (12) \]

(12) satisfies the bc. iff \( c_1 = c_2 = 0 \).

\[ = \text{No non-trivial sol. for } X = 0. \]

(3) \( \lambda > 0 \): roots of characteristic polynomial are complex, \( r_{1,2} = \pm i \sqrt{\lambda} \).

\[ = \text{general sol.: } X(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x) \quad (13) \]

Boundary cond.:

\[ 0 = X(0) = c_1 \]

\[ 0 = X(L) = c_1 \cos(\sqrt{\lambda} L) + c_2 \sin(\sqrt{\lambda} L) = c_2 \sin(\sqrt{\lambda} L) \]

\[ \implies c_2 = 0 \]

\[ \implies X(x) \text{ non-trivial for } \pm i \sqrt{\lambda} \neq 0 \]

The only non-trivial solns. to the BVP (3) are obtained for \( \lambda > 0 \) and satisfy:

\[ \sin(\sqrt{\lambda} L) = 0 \]

\[ \iff \sqrt{\lambda} L = n\pi, \quad n \in \mathbb{Z} \setminus \{0\} \]

\[ \iff \sqrt{\lambda} = \frac{n\pi}{L}, \quad n = \pm 1, \pm 2, \pm 3, \ldots \]

Therefore, the family of funs.

\[ \left\{ \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots \right\} \]

forms a system of linearly independent solutions to the BVP in (9).
Note that the bc. restriction restricts the permissible values for the parameter $A$ in the eqn. (9) and hence in the heat eqn. (4).

Now we can solve 

\[ T(t) = a^2 x \frac{\partial^2 T}{\partial x^2} = 0, \]

where

\[ \lambda_n = \frac{n^2 \pi^2}{L^2}, \quad n = 1, 2, 3, \ldots \]

General solution to (10):

\[ T(t) = c_n e^{-\frac{a^2 n^2 \pi^2}{L^2} t} \]

Let us summarize: The family of fns.

\[ \left\{ e^{-\frac{a^2 n^2 \pi^2}{L^2} t} \sin \left( \frac{n \pi x}{L} \right) : n = 1, 2, 3, \ldots \right\} \]

forms a system of linearly independent solns. of (4). By above theorem in particular, by above Prop. any finite linear combination of these fns.

\[ \sum_{n=1}^{N} c_n \sin \left( \frac{n \pi x}{L} \right) e^{-\frac{a^2 n^2 \pi^2}{L^2} t} \]

some $N \in \mathbb{N}$ and $c_n \in \mathbb{R}$, is a soln. to (4).

Suppose, we are also given an initial condition for $u(x,t)$,

\[ u(x,0) = f(x). \]
If \( \phi \) the sol. is of the form
\[
u(x,t) = \sum_{n=1}^{N} c_n \sin \left( \frac{n \pi x}{L} \right) e^{-\frac{a^2 n^2 \pi^2 t}{L^2}}
\]
then
\[
f(x) = u(x,0) = \sum_{n=1}^{N} c_n \sin \left( \frac{n \pi x}{L} \right)
\] (14)

We want the fun. \( f \) to be piecewise cont., but not every piecewise cont. on \([0,L]\) will be a finite \( \{n \} \) linear comb. of
\[
\sin \left( \frac{n \pi x}{L} \right) \quad (\text{in fact only very few, of them will})
\]

Could try to generalize (14) by writing
\[
f(x) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n \pi x}{L} \right)
\] (15)

**Questions:**

(i) Does this formal expression make sense

(convergence? in what sense?)

(ii) Can every piecewise cont. fun. on \([0,1]\)

be written \( \approx \) in the form (15)?

This question was first raised by

Joseph Fourier for exactly the purpose

of solving the heat eqn.

Let us generalize (15) \( \approx \) to
\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \left( \frac{n \pi x}{L} \right) + b_n \sin \left( \frac{n \pi x}{L} \right) \right\}
\] (16)

and ask questions (i) and (ii).
Let us suppose we want to approximate $f$ by a finite nb. of terms in (16):

$$S_n = \frac{a_0}{2} + \sum_{k=1}^{N} \left( a_k \cos\left( \frac{k\pi x}{L} \right) + b_k \sin\left( \frac{k\pi x}{L} \right) \right).$$

We could quantify the quality of this approximation by calculating the mean square deviation of $S_n$ from $f$:

$$\frac{1}{L} \int_0^L \left| f(x) - S_n(x) \right|^2 \, dx.$$

Averaging over the interval $[0, L]$, i.e.

$$\frac{1}{L} \int_0^L \left| f(x) - S_n(x) \right|^2 \, dx.$$

for this we require the fun. $f$ to be square integrable, i.e. $\int |f(x)|^2 \, dx < \infty$. We denote the class of fun. with that property by $L^2([0, L])$. Every piecewise cont. fun. will be in $L^2([0, L])$. We could then understand convergence of an approximation $S_n$ to $f$ as $n \to \infty$:

$$\frac{1}{L} \int_0^L \left| f(x) - S_n(x) \right|^2 \, dx \to 0$$

(mean square error goes to zero as $n \to \infty$)

It turns out that this is the mathematically correct way to understand the convergence of the formal expression (16). One can...
prove the following to completely answer (i) and (ii):

then (Fourier-expansion):

Let \( f \) be a square integrable function on \([0, L]\) (i.e. \( f \in L^2[0, L] \)). Then, the sequence of \( \Phi \) approximate fns.

\[
S_n = \frac{a_0}{2} + \sum_{k=1}^{n} \left[ a_k \cos\left(\frac{k \pi x}{L}\right) + b_k \sin\left(\frac{k \pi x}{L}\right) \right]
\]

(17)

converges to \( f \) in the sense

\[
\int_0^L |S_n(x) - f(x)|^2 \, dx \to 0 \quad \text{as} \quad n \to \infty
\]

if and only if the coefficients in (17) are given by

\[
a_k = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{k \pi x}{L}\right) \, dx, \quad k = 0, 1, 2, \ldots
\]

\[
b_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k \pi x}{L}\right) \, dx, \quad k = 1, 2, 3, \ldots
\]

We then write

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n \pi x}{L}\right) + b_n \sin\left(\frac{n \pi x}{L}\right) \right]
\]

(18)

and call this the Fourier expansion series for the function \( f \). The coefficients \( a_n \) and \( b_n \) are called Fourier coefficients of \( f \).
Therefore, with this in mind we can write down the general solution to the BVP in (4) subject to the initial cond.:

\[ u(x,0) = f(x) \], with freewall cond. \( f \) on \([0,L] \), as

\[ u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2 \pi^2 a^2 t}{L^2}} \sin \left( \frac{n \pi x}{L} \right) \]

where \( c_n \) are the Fourier coefficients of \( f \):

\[ f(x) = \sum_{n=1}^{\infty} c_n \sin \left( \frac{n \pi x}{L} \right) \]

\[ c_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{n \pi x}{L} \right) \, dx \].