Introduction to differential equations

Second-order linear differential equations.

Def.

A second order differential equation is an equation of the form:

\[ \frac{d^2 y}{dx^2} = f(x, y, \frac{dy}{dx}) \]

Example:

1) \[ m \cdot \frac{d^2 y}{dx^2} = F(x, y, \frac{dy}{dx}) \]

\[ \begin{array}{c}
\text{Initial value problem:} \\
\text{Find a function } y(x) \text{ such that}
\end{array} \]

\[ \begin{cases}
\frac{d^2 y}{dx^2} = f(x, y, \frac{dy}{dx}), \\
y(x_0) = y_0, \\
y'(x_0) = \gamma_0
\end{cases} \]
A linear differential equation of second order
\[ \frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t) y = g(t) \]

- non-homogeneous
\[ \frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t) y = 0 \]  \((*)\)
- homogeneous,

**Theorem (Existence and uniqueness)**
Suppose \( p(t) \) and \( q(t) \) are continuous on the interval \((a, b)\), \(-\infty \leq a < b \leq +\infty\),
to \((a, b)\). Then there exists one, and only one, function on \((a, b)\) such that
\[ y(t) = y_0, \ y'(t_0) = y_0', \] and \((*)\) holds.

**Corollary**
If \( y(t) \) is a solution of \((*)\),
\[ y(t) = y'(t_0) = 0 \] then \( y(t_1) = 0 \).

**Example**
\( y(t) \) is a solution of \((*)\), for \( t \in \mathbb{R} \),
\[ y(\frac{1}{2}) = 0 \] V and \( \forall t \in \mathbb{R} \), Find \( y(t) \).
**Solution:** \( y = 0 \), since \( y(0) = 0 \) (by cont.), \( y'(0) = 0 \).
Important remark

(*) can be rewritten as

\[ L(y) = 0, \] where

\[ L : C^2(\alpha, \beta) \to C^0(\alpha, \beta) \] - a linear operator,

\[ L(y) = y'' + p(t)y' + q(t)y, \]

\[ C^2(\alpha, \beta) = \{ f : (\alpha, \beta) \to \mathbb{R} \mid \exists f'' \text{ continuous on } (\alpha, \beta) \} \]

\[ C^0(\alpha, \beta) = \{ f : (\alpha, \beta) \to \mathbb{R} \mid f \text{ is continuous} \} \] since

\[ L(y_1 + y_2) = (y_1 + y_2)'' + p(t)(y_1 + y_2)' + q(t)(y_1 + y_2) = y_1'' + p(t)y_1' + q(t)y_1 + y_2'' + p(t)y_2' + q(t)y_2 = L(y_1) + L(y_2), \]

and \[ L(cy) = (cy)'' + p(t)(cy)' + q(t)(cy) = cL(y), \]

\[ L \text{ is a linear operator,} \]

Example

Consider an operator \[ L(y) = y'' + t^2 y', \]

Then \[ L(e^t) = e^t + t^2 e^t, \]

\[ L(e^{-t}) = e^{-t} - t^2 e^{-t}, \]

\[ L(t) = t^2, \]

\[ L(t^2) = 2 + 2t^3, \]

\[ L(1) = 0. \]
y is solution of (*) \( \Leftrightarrow \) \( \L(y) = 0 \),

**Remark**

Solutions of (*) = Ker \( \mathbf{L} \).

(Defi \( \text{Ker } \mathcal{L} = \{ y \in \mathcal{C}^2(\mathcal{P}) \mid \mathbf{L}y = 0 \} \))

**Lemma**

The space of solutions of (*) is a linear space.

**Proof**

\( \mathbf{L}(y_1) = 0, \mathbf{L}(y_2) = 0 \Rightarrow \mathbf{L}(y_1 + y_2) = \mathbf{L}(y_1) + \mathbf{L}(y_2) = 0 \),

so \( y_1 + y_2 \) is a solution too.

\( \mathbf{L}(c y_1) = c \mathbf{L}(y_1) = 0 \Rightarrow c y_1 \) is a solution too.

**Example**

\( y'' + y = 0 \)

\( y_1(t) = \cos t \) is a solution

\( y_2(t) = \sin t \) is a solution

\( y(t) = c_1 \cos t + c_2 \sin t \) is a solution \( \forall c_1, c_2 \in \mathbb{R} \).

**Proposition**

Any solution of \( y'' + y = 0 \) has the form

\( c_1 \cos t + c_2 \sin t \). \)
Proof

Assume that \( y(t) \) is a solution of the equation
\[ y'' + y = 0. \]
Let \( y(0) = y_0, \ y'(0) = y'_0 \), and take
\[ \tilde{y}(t) = y_0 \cos t + y'_0 \sin t. \]
Then \( \tilde{y}(0) = y_0 \)
\[ \tilde{y}'(0) = -y_0 \sin 0 + y'_0 \cos 0 = y'_0. \]
But there is only one solution such that
at \( 0 \) it is \( y_0 \) and its derivative is \( y'_0 \), so
\[ y(t) = \tilde{y}(t) \quad \Box. \]

Theorem

Let \( y_1(t) \) and \( y_2(t) \) be two solutions of \((\star)\) on \((\alpha, \beta)\), and
\[
\det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)
\]
does not vanish \( \forall t \in (\alpha, \beta) \).

Then
\[ y(t) = C_1 y_1(t) + C_2 y_2(t) \]
is a general solution of \((\star)\).
An expression \[ \text{det} \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \]
is called the Wronskian of \( y_1 \) and \( y_2 \).

\textit{Notation:} \( W(t) = W[y_1, y_2](t) \).

\textbf{Proof}

Let \( y(t) \) be a solution of \( (r) \).

Take any \( t_0 \in (a, b) \), and let \( y_0, y_0' \) be the values of \( y(t_0) \) and \( y'(t_0) \).

\[
\begin{align*}
C_1 \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} = y_0 & \quad \Rightarrow \quad y_2'(t_0) \\
C_1 \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} = y_0' & \quad \Rightarrow \quad y_2(t_0)
\end{align*}
\]

\( C_1 \cdot W[y_1, y_2](t_0) = y_0 \cdot y_2'(t_0) - y_0' \cdot y_2(t_0) \),

\( C_1 = \frac{y_0 \cdot y_2'(t_0) - y_0' \cdot y_2(t_0)}{W[y_1, y_2](t_0)} \).

Similarly,

\( C_2 = \frac{y_0' \cdot y_1(t_0) - y_0 \cdot y_1'(t_0)}{W[y_0, y_2](t_0)} \).

Set \( \psi(t) = C_1 \cdot y_1(t) + C_2 \cdot y_2(t) \).

Then \( \psi(t_0) = y_0 \),

\( \psi'(t_0) = y_0' \), so \( \psi(t) \equiv y(t) \).
Theorem

Let \( p(t), q(t) \) be continuous on \((a, b)\), and \( y_1, y_2 \) be two solutions of \((*)\).

Then either \( W[y_1, y_2] \equiv 0 \) or 
\[
W[y_1, y_2](t) \neq 0 \quad \forall t \in (a, b).
\]

Proof

\[
W[y_1, y_2](t) = y_1(t)y_2'(t) - y_2(t)y_1'(t), \quad \text{so}
\]
\[
\frac{d}{dt} W(t) = y_1'(t)y_2(t) + y_1(t)y_2''(t) - y_2'(t)y_1(t) - y_2(t)y_1''(t) = \]
\[
= y_1(t)
\left(- p(t) y_2'(t) - q(t) y_2(t)\right) -
- y_2(t)
\left(- p(t) y_1'(t) - q(t) y_1(t)\right) =
\]
\[
= - p(t) W(t)
\]

Take any \( t_0 \in (a, b) \). Then
\[
W[y_1, y_2](t) = W[y_1, y_2](t_0) \cdot e^{- \int_{t_0}^{t} p(s) ds},
\]

therefore either \( W(t) \equiv 0 \) or 
\[
W(t) \neq 0 \quad \forall t \in (a, b).
\]
Remark

If \( y_1 = 0 \) then \( W[y_1, y_2] = 0 \),
If \( y_1 = Cy_2 \) then \( W[y_1, y_2] = 0 \).

Def

Functions \( y_1(t), y_2(t) \) are linearly dependent if \( y_1 = Cy_2 \) (or \( y_2 = Cy_1 \)).

Theorem

If \( y_1, y_2 \) are two solutions, and at some point \( t_0 \in \mathbb{R} \) \( W(t_0) = 0 \) then \( y_1 \) and \( y_2 \) are linearly dependent.

Proof

\[ W[y_1, y_2](t_0) = 0 \implies W(t) = 0, \text{ so} \]

\[ y_1(t), y_2(t) = y_1(t_0), y_2(t_0). \]

If \( y_1(t), y_2(t) \neq 0 \) for \( t < t_0 < t_1 \) then

\[ \frac{y_2(t)}{y_2(t_0)} = \frac{y_1(t)}{y_1(t_0)} \]

\[ (\mathcal{L} y_1(t))' = (\mathcal{L} y_1(t_1))' \]

\[ \mathcal{L} y_2(t) = \mathcal{L} (y_1(t) + \bar{c}) \]

\[ |y_2(t)| = e^{\bar{c}} |y_1(t)| \]

\[ y_2(t) = C y_1(t). \]
If \( \exists t^* \) s.t. \( y_1(t^*) = 0 \) then

\[ y_1'(t^*) = 0 \]

If \( y_1'(t^*) = 0 \) then \( y_1 \equiv 0 \).

If \( y_1'(t^*) \neq 0 \) then \( y_2(t^*) = 0 \),

and \( y_2'(t^*) = \left[ \frac{y_2'(t^*)}{y_1'(t^*)} \right] y_1'(t^*) \).

Consider \( \psi(t) = \left[ \frac{y_2'(t^*)}{y_1'(t^*)} \right] y_1(t) \) - solution of (1).

We have \( \psi(t^*) = 0 = y_2(t^*) \)

\( \psi'(t^*) = y_2'(t^*) \),

then \( \psi(t) = y_2(t) \), so

\( y_1 \) and \( y_2 \) are linearly independent \( \square \)

**Example**

1) \( e^{at}, e^{bt} \) are linearly independent if \( a \neq b \).

2) \( t, t^3 \)

3) \( \cos t, \sin t \)

4) \( t^2, t, |t| \) are linearly independent on \( \mathbb{R} \), but \( W(t, 1) \equiv 0 \). These functions are not solutions of (1) for any \( p(t), q(t) \).

\( \text{the same} \)