Introduction to differential equations.

Linear differential equations of second order.

\[ y'' + p(t)y' + q(t)y = g(t) \] \hspace{1cm} \text{non-homogeneous}

\[ y'' + p(t)y' + q(t)y = 0 \] \hspace{1cm} \text{homogeneous}

\[ L: y \rightarrow y'' + p(t)y' + q(t)y \] \hspace{1cm} \text{linear operator on} \ C^2[a,b]

\[ Ly = 0 \]

Space of solutions (\(\equiv \ker L\)) is a linear space.

Then let \(y_1, y_2\) be two solutions of (1) on \((a,b)\), and \(p(t), q(t)\) be continuous on \((a,b)\). Then if

\[ \det \begin{pmatrix} y_1' & y_2' \\ y_1 & y_2 \end{pmatrix} = y_1 y_2' - y_2 y_1' \neq 0 \quad \forall t \in (a,b) \]

then \(c_1 y_1 (t) + c_2 y_2 (t)\) is a general solution of (1).

Thus (on Wronskian)

If \(p(t), q(t)\) are continuous on \((a,b)\), and \(y_1, y_2\) are solutions of (1) on \((a,b)\), then either

\[ W[y_1, y_2] \equiv 0 \]

or \(W[y_1, y_2](t) \neq 0 \quad \forall t \in (a,b)\).
**Def**

Two functions \( y_1(t) \) and \( y_2(t) \), \( t \in (\alpha, \beta) \), are linearly dependent if for some \( c_1, c_2 \), \( |c_1| |c_2| \neq 0 \),
\[
C_1 y_1(t) + C_2 y_2(t) = 0 \quad \forall t \in (\alpha, \beta)
\]

(\( \equiv \) one function is a constant multiple of another).

**Def**

Two functions \( y_1(t) \), \( y_2(t) \), \( t \in (\alpha, \beta) \), are linearly dependent if \( \exists C_1, \ldots, C_n \), \( |C_1| + \ldots + |C_n| \neq 0 \), s.t.
\[
C_1 y_1(t) + \ldots + C_n y_n(t) = 0 \quad \forall t \in (\alpha, \beta)
\]

**Proposition**

Two solutions of (1) are linearly dependent

\( \implies \) \( W[y_1, y_2](t) = 0 \).

**Proof**

(1) \( \implies \) \text{Easy: } \( y_1(t) = Cy_2(t) \),

\[
\det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det \begin{pmatrix} Cy_2 & y_2 \\ Cy_2' & y_2' \end{pmatrix} = 0
\]

(\( \Leftarrow \) \) \text{Assume}

\[
y_1(t) y_2'(t) - y_1'(t) y_2(t) = 0 \quad \forall t \in (\alpha, \beta)
\]

If \( y_1(t) y_2(t) \neq 0 \), \( \forall t \in (\alpha, \beta) \), then

\[
\frac{y_1'(t)}{y_1(t)} = \frac{y_2'(t)}{y_2(t)} \quad \forall t \in (\alpha, \beta), \text{ and}
\]
\[ \int \frac{y_1'}{y_1} \, dt = \int \frac{y_2'}{y_2} \, dt \]

\[ C_1 y_1 = C_2 y_2 + C \]

\[ y_1' = C e^{\int y_2} \quad \text{and since } y_1 \text{ and } y_2 \text{ do not have zero on } (y_1, y_2) \]

\[ y_1 = e^{C y_2} \quad \text{for some } C \in \mathbb{R}. \]

If \( \exists t_0 \in (y_1, y_2) \) s.t. \( y_1(t_0), y_2(t_0) = 0 \) then

\[ y_1(t_0), y_2(t_0) = 0 \quad \text{and} \quad y_1'(t_0) = y_2'(t_0) = 0. \]

Then

\[ y_1(t_0), y_2(t_0) = 0 \quad \text{or} \quad y_1'(t_0) = 0. \]

If \( y_1'(t_0) = 0 \) (and \( y_1(t_0) = 0 \)) then \( y_1(t) \equiv 0 \), and \( y_1(t) = 0, y_2. \)

If \( y_1'(t_0) \neq 0 \) then \( y_2(t_0) = 0. \)

Consider \( \varphi(t) = \left( \frac{y_2'(t_0)}{y_1'(t_0)} \right) y_1(t) \).

Then

\[ \varphi'(t_0) = \frac{y_2'(t_0)}{y_1'(t_0)} y_1(t_0) = 0 = y_2(t_0) \]

\[ \varphi'(t_0) = \frac{y_2'(t_0)}{y_1'(t_0)} y_1'(t_0) = y_2'(t_0) \]

so \( y_2(t) = \left( \frac{y_2'(t_0)}{y_1'(t_0)} \right) y_1(t) \).

\[ \text{Therefore if } y_1 \text{ and } y_2 \text{ are solutions of } (*) \text{ then } \]

\[ C_1 y_1 + C_2 y_2 \quad \text{and } \quad W[y_1, y_2] \neq 0 \quad \text{so } y_1 \text{ and } y_2 \text{ are linearly independent.} \]

in a general solution
Example

If $a \neq b$ then $e^{at}$ and $e^{bt}$ are linearly independent functions.

Indeed, $c_1 e^{at} + c_2 e^{bt} = e^{at} (c_1 e^{(b-a)t}) \neq 0$, for all $c_1, c_2$.

\[
\text{W} \left[ e^{at}, e^{bt} \right] = \\
= \det \begin{pmatrix} e^{at} & e^{bt} \\ e^{at} & e^{bt} \end{pmatrix} \\
= e^{(a+b)t} - e^{(a-b)t} \neq 0.
\]

Example

$y_1(t) = t$, $y_2(t) = e^t$,

\[
\text{W} \left[ y_1, y_2 \right] = \det \begin{pmatrix} t & e^t \\ 1 & e^t \end{pmatrix} = e^t - e^t = 0.
\]

Not a contradiction since $p(t), q(t)$ such that $e^t$ and $t$ are solutions of $(*)$!
Linear homogeneous differential equations of second order with constant coefficients.

Consider an equation

\[(**+) \quad ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}^1\]

Let us try to find a solution of \((**+)\)
in a form \(y(t) = e^{rt}, \quad r \in \mathbb{R}^1\).

\[ar^2e^{rt} + br e^{rt} + ce^{rt} = 0\]

\[ar^2 + br + c = 0\]

Def. An equation \(ar^2 + br + c = 0\) is a characteristic equation of the diff. equ. \((**+)\).

\(y(t) = e^{rt}\) is a solution of \((**+)\) \(\iff\) \(r\) is a root of \(ar^2 + br + c = 0\).

If \(r_1, r_2\) are two different real roots of \(ar^2 + br + c = 0\) then

\(y_1(t) = e^{r_1t}\) is a solution,

\(y_2(t) = e^{r_2t}\) is a solution, and \(y_1\) and \(y_2\) are linearly independent, so

\(C_1 e^{r_1t} + C_2 e^{r_2t}\) is a general solution of \((**+)\).
Example

1) \( y'' - 3y' + 2 = 0 \)

Characteristic equation:
\( r^2 - 3r + 2 = 0 \)

\( D = 9 - 8 = 1 \),

\( r_{1,2} = \frac{3 \pm 1}{2} = \{2, 1\}, \) so

\( e^t \) and \( e^{2t} \) are solutions, so

\( C_1 e^t + C_2 e^{2t} \) is a general solution.

2) \( y'' - 5y = 0 \)

Characteristic equation:
\( r^2 - 5 = 0 \)

\( r_{1,2} = \{-\sqrt{5}, \sqrt{5}\}, \) so

\( e^{-\sqrt{5}t} \) and \( e^{\sqrt{5}t} \) are solutions.

\( C_1 e^{\sqrt{5}t} + C_2 e^{-\sqrt{5}t} \) is a general solution.
3) Find a solution of
\[ y'' - 3y' + 2y = 0 \]
such that \( y(0) = 2 \)
\[ y'(0) = 3 \]

**Solution:**

We know that a general solution has the form \( y(t) = C_1 e^t + C_2 e^{2t} \).

\[ y(0) = C_1 + C_2 = 2 \]
\[ y'(0) = C_1 + 2C_2 = 3 \]

\[ \Rightarrow C_2 = 1 = C_1 \]

\[ y(t) = e^t + e^{2t} \]

**Example**

Euler's equation:

\[ y'' + \frac{\alpha}{t} y' + \frac{B}{t^2} y = 0 \], \( t > 0 \).

**Solution:**

Try to find \( y(t) = t^\nu \):

\[ \nu (\nu - 1) t^{\nu - 2} + \nu + \frac{B}{t^2} = 0 \]

\[ \nu (\nu - 1) + \nu B + B = 0 \]

\[ \nu^2 + (\nu - 1) \nu + B = 0 \]

Find \( \nu_1, \nu_2 \) (roots of):

\[ C_1 t^{\nu_1} + C_2 t^{\nu_2} \] - general solution.
\[ y'' - \frac{2}{t} y' + \frac{2}{t^2} y = 0, \ t > 0 \]

\[ r^2 - 3r + 2 = 0, \ r_{1,2} = 1, 2, \]

\[ y(t) = C_1 t + C_2 t^2 \quad \text{general solution}. \]