Second order linear differential equations.

The non-homogeneous equation,

\[ L(y) = y'' + p(t) y' + q(t) y = g(t), \quad (*) \]

\( p(t), q(t), g(t) \) are continuous on \((a, b)\).

Thus,

let \( y_1(t) \) and \( y_2(t) \) be two linearly independent solutions of

\[ y'' + p(t) y' + q(t) y = 0, \]

and \( y(t) \) be a solution of

\[ y'' + p(t) y' + q(t) y = g(t), \quad (*) \]

Then a general solution of \((*)\)

has the form

\[ C_1 y_1(t) + C_2 y_2(t) + y(t). \]

**Proof**

We need to show that any function of this form is a solution of \((*)\), and that any solution of \((*)\) has this form.
\[ L(L_y + L_y + L_{\psi}) = \]
\[ = C_1 L(y_1 + y_2 + y_3) + L(L_y) + \]
\[ = C_1 y_1 + C_2 y_2 + \psi (t) = \psi (t), \]
so
\[ C_1 y_1(t) + C_2 y_2(t) + \psi (t) \text{ is a solution of } (*) \].

Assume that \( y (t) \) is a solution of \((*)\),

then \( L(y - \psi) = L(y) - L(\psi) = y(t) - \psi(t) = 0 \),

so \( y - \psi \) is a solution of a homogeneous equation
\[ y'' + p(t)y' + q(t)y = 0, \]
so
\[ y' - \psi(t) = C_1 y_1(t) + C_2 y_2(t), \]
and
\[ y(t) = C_1 y_1(t) + C_2 y_2(t) + \psi (t) \quad \square \]

**Example**

Suppose that \( y_1(t), y_2(t), \) and \( y_3(t) \) are solutions of \((*)\), and \( W[y_1, y_2, y_3] \neq 0 \).

Find a general solution.

**Solution:** \( L(y_1 - y_3) = 0 = L(y_2 - y_3) \), so

\( y_1 - y_3 \) and \( y_2 - y_3 \) are (linearly indep.) solutions of \((*)\). Therefore a general solution has the form \( y(t) = C_1 (y_1 - y_3) + C_2 (y_2 - y_3) \).
The method of variation of parameters

\[ y'' + p(t)y' + q(t)y = g(t) \quad (*) \]

Assume that we know that \( y_1(t) \) and \( y_2(t) \) are linearly independent solutions of the homogeneous equation

\[ y'' + p(t)y' + q(t)y = 0 \quad (\text{hom}) \]

Then the general solution of (\text{hom}) is

\[ C_1 y_1(t) + C_2 y_2(t) \]

Let us try to find a solution of (*) in a form

\[ y(t) = C_1(t) y_1(t) + C_2(t) y_2(t), \quad \text{where} \quad C_1(t) \quad \text{and} \quad C_2(t) \quad \text{are unknown functions.} \]

\[
\begin{align*}
y'(t) &= C_1'(t)y_1(t) + C_2'(t)y_2(t) - \\
&\quad + C_1(t)y_1'(t) + C_2(t)y_2'(t) \\
y''(t) &= C_1''(t)y_1(t) + C_2''(t)y_2(t) + \\
&\quad 2(C_1'(t)y_1'(t) + C_2'(t)y_2'(t)) - \\
&\quad + C_1(t)y_1''(t) + C_2(t)y_2''(t)
\end{align*}
\]
Suppose that
\[ C_1'(t) y_1(t) + C_2'(t) y_2(t) = 0. \]
Then \( y''(t) = (C_1'(t) y_1(t) + C_2(t) y_2'(t))' = \]
\[ = C_1'' y_1 + C_2'' y_2' + C_1 y_1'' + C_2 y_2'', \text{ and} \]
\[ L(y) = (C_1 y_1' + C_2 y_2') + p(t) (C_1 y_1' + C_2 y_2) + q(t) (C_1 y_1 + C_2 y_2) = \]
\[ = C_1 y_1' + C_2 y_2' = g(t). \]

Therefore we want
\[ y_2' \times \int C_1'(t) y_1(t) + C_2'(t) y_2(t) = 0 \]
\[ y_2 \times \left( C_1 y_1' + C_2 y_2' = g(t) \right) \]
\[ C_1'(y_1 y_2' - y_1' y_2) = -y_2 \cdot g(t) \]
\[ \begin{cases} 
C_1' = -\frac{y_2 \cdot g}{W[y_1, y_2]}, \\
C_2' = \frac{y_1 \cdot g}{W[y_1, y_2]} 
\end{cases} \]

From here we find \( C_1(t), C_2(t), \) and a particular solution of \((*)\).
Example

\[ y'' + 4y' + 4y = te^{2t} \]

Homogeneous equation:

\[ y'' - 4y' + 4y = 0 \]

\[ r^2 - 4r + 4 = 0 \]

\[ r = 2, \quad C_1 e^{2t} + C_2 te^{2t} \text{ is a general solution.} \]

Let us try to find a solution of a non-homogeneous equation in a form

\[ y(t) = C_1 e^{2t} + C_2 t e^{2t} + \]

\[ y'(t) = C_1' e^{2t} + C_2' t e^{2t} + \]

\[ + C_1' 2e^{2t} + C_2' (e^{2t} + 2te^{2t}) \]

Suppose \( C_1' e^{2t} + C_2' t e^{2t} = 0 \). Then,

\[ y''(t) = C_1' 2e^{2t} + C_2' (e^{2t} + 2te^{2t}) + \]

\[ + 4 C_1 e^{2t} + C_2 (2e^{2t} + 2e^{2t} + 4te^{2t}) \]

so

\[ \begin{cases} C_1' e^{2t} + C_2' t e^{2t} = 0 \\ 2C_1 e^{2t} + C_2' (e^{2t} + 2te^{2t}) + te^{2t} \end{cases} \]
\[ \left\{ \begin{array}{l}
C_1' + C_2' \cdot t = 0 \\
2C_1' + C_2' (1 + 2t) = t 
\end{array} \right. \]

\[ \left\{ \begin{array}{l}
C_2' = t \\
C_1' = -t^2 
\end{array} \right. \]

So
\[ \left\{ \begin{array}{l}
C_1 (t) = -\frac{t^3}{3} + \tilde{C}_1 \\
C_2 (t) = \frac{t^2}{2} + \tilde{C}_2 
\end{array} \right. \]

So
\[ y (t) = \left( \tilde{C}_1 - \frac{t^3}{3} \right) e^{2t} + \left( \tilde{C}_2 + \frac{t^2}{2} \right) t e^{2t} \]

is a general solution

(i.e., \[ -\frac{t^3}{3} e^{2t} + \frac{t^2}{2} t e^{2t} \]

is a partial solution

\[ = \frac{t^3}{6} e^{2t}, \quad \text{i.e.,} \]

\[ y (t) = C_1 e^{2t} + C_2 t e^{2t} + \frac{t^3}{6} e^{2t} \]

is a general solution.)
Example

\[ y'' - y = f(t) , \quad y(0) = y'(0) = 0 \]

\[ y'' - y = 0 \]

\[ e_1 e^t + e_2 e^{-t} \]

\[ y(t) = e_1 e^t + e_2 e^{-t} \]

\[ y'(t) = [e_1 e^t + e_2 e^{-t}]' + e_1 e^t - e_2 e^{-t} \]

\[ y''(t) = [e_1 e^t - e_2 e^{-t}] + e_1 e^t + e_2 e^{-t} \]

\[ \left\{ \begin{array}{l}
     e_1 e^t - e_2 e^{-t} = f(t) \\
     e_1 e^t + e_2 e^{-t} = 0
\end{array} \right. \]

\[ \left\{ \begin{array}{l}
     e_1(t) = \frac{1}{2} f(t) e^t \\
     e_2(t) = -\frac{1}{2} e^t f(t)
\end{array} \right. \]

\[ \left\{ \begin{array}{l}
     e_1(t) = \frac{1}{2} \int_0^t f(\tau) e^{\tau} d\tau + \bar{e}_1 \\
     e_2(t) = -\frac{1}{2} \int_0^t f(\tau) e^{\tau} d\tau + \bar{e}_2
\end{array} \right. \]

\[ y(t) = e^t \left( \bar{e}_1 + \frac{1}{2} \int_0^t f(\tau) e^{\tau} d\tau \right) + e^{-t} \left( \bar{e}_2 - \frac{1}{2} \int_0^t f(\tau) e^{\tau} d\tau \right) \]

\[ y(0) = 0 = \bar{e}_1 + \bar{e}_2 \]

\[ y'(0) = e^t \left( \bar{e}_1 + \frac{1}{2} \int_0^t f(\tau) e^{\tau} d\tau \right) - e^{-t} \left( \bar{e}_2 - \frac{1}{2} \int_0^t f(\tau) e^{\tau} d\tau \right) + \frac{1}{2} e^{-t} f(t) e^t = \bar{e}_1 - \bar{e}_2 = 0 \]
Therefore we need a solution

\[
y(t) = \frac{1}{2} e^t \int_0^t f(\tau) e^{-\tau} d\tau - \frac{1}{2} e^{-t} \int_0^t f(\tau) e^\tau d\tau =
\]

\[
= \frac{1}{2} \int_0^t f(\tau) (e^{t-\tau} - e^{-t-\tau}) d\tau =
\]

\[
y(t) = \int_0^t f(\tau) \sinh(t-\tau) d\tau
\]

Indeed,

\[
y' = f(t) + \int_0^t f(\tau) \cosh(t-\tau) d\tau
\]

\[
y'' = f(t) + \int_0^t f(\tau) \sinh(t-\tau) d\tau
\]