Exact equations

Def

\[ M(t, y) + N(t, y) \frac{dy}{dt} = 0 \] is exact if

\[ \exists \phi(t, y) \text{ s.t.} \]

\[ M(t, y) = \frac{\partial \phi}{\partial t} \quad \text{and} \quad N(t, y) = \frac{\partial \phi}{\partial y}. \]

In this case the equation has the form

\[ \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial y} \cdot \frac{dy}{dt} = 0 \iff \frac{d}{dt}(\phi(t, y)) = 0, \]

so \( \phi(t, y) = C \) is a general solution.

Thus

\[ M(t, y) + N(t, y) \frac{dy}{dt} = 0 \]

is exact if, and only if

\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}. \]

Example

Any separable equation is exact:

\[ \frac{dy}{dt} = \frac{g(t)}{f(y)} \]

\[ g(t) - f(y) \frac{dy}{dt} = 0 \]

\[ M(t, y) = g(t), \quad N(t, y) = -f(y), \quad \frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial t}. \]
\[ \psi(t,y) = \int g(t)\,dt - \int f(y)\,dy \]

\[ \int g(t)\,dt - \int f(y)\,dy = C \quad \text{general solution of a separable equation.} \]

\[ M(t,y) + N(t,y) \frac{dy}{dt} = 0 \quad (\star) \]

**Def.** A function \( \mu(t,y) \) is called an integrating factor for \( (\star) \) if the equation

\[ \mu(t,y) \cdot M(t,y) + \mu(t,y) \cdot N(t,y) \frac{dy}{dt} = 0 \]

is exact.

**Remark 1**

\( \mu(t,y) \) is an integrating factor if

\[ \frac{\partial}{\partial y} \left( \mu(t,y) M(t,y) \right) = \frac{\partial}{\partial t} \left( \mu(t,y) N(t,y) \right) \]

**Remark 2**

Locally in a neighborhood of a non-degenerate point an integrating factor exists.
Example

\[ \frac{dy}{dt} + \alpha(t)y = b(t) \]

Let us try to find an integrating factor

\[ \alpha(t)y - b(t) + \frac{dy}{dt} = 0 \]

\[ N(t,y) = \alpha(t)y - b(t) \]

\[ N(t,y) = 1 \]

Let us try to find an integrating factor

\[ \mu(t) (\alpha(t)y - b(t)) + \mu(t) \frac{dy}{dt} = 0 \]

\[ \mu_1 \]

\[ \frac{\partial \mu_1}{\partial y} = \frac{\partial N_1}{\partial t} \]

\[ \mu(t) \alpha(t) = \frac{\partial \mu}{\partial t}, \quad \mu(t) = \int \alpha(t) dt \]

\[ \left\{ \begin{array}{l} \mu(t) (\alpha(t)y - b(t)) = \frac{\partial y}{\partial t} \\ \mu(t) = \frac{\partial y}{\partial y} \Rightarrow \phi(t,y) = \mu(t) \cdot y + h(t) \end{array} \right\} \]
\[ \frac{dy}{dt} = \frac{dm}{dt} \cdot y + b'(t) = \]

\[ \mu(t) q(t) y + b'(t) = \]

\[ \mu(t) (q(t)y - b(t)) , \]

so \[ b'(t) = -\mu(t) b(t) \], so

\[ b(t) = - \int \mu(t) b(t) dt + \mathcal{C} , \]

and

\[ \psi(t, y) = \mu(t) y - \int \mu(t) b(t) dt \]

**General solution:**

\[ \psi(t, y) = C , \]

so

\[ \mu(t) y - \int \mu(t) b(t) dt = C , \]

or

\[ y(t) = \left( \mu(t) \right)^{-1} \left( C + \int \mu(t) b(t) dt \right) , \]

where \[ \mu(t) = \int a(t) dt \].
Homogeneous Equations

Def. An equation

\[ \frac{dy}{dt} = f\left(\frac{y}{t}\right) \]

is called homogeneous differential equation.

Solution

Consider a function \( u(t) = \frac{y}{t} \),

then \( y(t) = u(t) \cdot t \), and

\[ y' = u(t) + u' \cdot t \],

so

\[ u(t) + u'(t) \cdot t = f(u) \],

and

\[ u' = \frac{f(u) - u}{t} \] separable equation,

\[ \int \frac{du}{f(u) - u} = \int \frac{dt}{t} \],

\[ \ln|t| + 1 = \int \frac{dy}{f(u) - u} + C \]
Example

\[ y' = \frac{y - t}{y + t} \]

\[ y' = \frac{y/t - 1}{y/t + 1} \] homogenous!

\[ u = \frac{y}{t}, \quad y = u \cdot t, \quad y' = u' \cdot t + u \]

\[ u' \cdot t + u = \frac{u - 1}{u + 1} \]

\[ u' \cdot t = \frac{u - 1}{u + 1} - u = \frac{u - 1 - u^2 - u}{u + 1} = -\frac{1 + u^2}{u + 1} \]

\[ \int \frac{(1 + u) \, du}{1 + u^2} = -\int \frac{dt}{t} \]

\[ \arctan u + \frac{1}{2} \ln (1 + u^2) = -\ln |t| + C/2 \]

\[ 2 \arctan u + \ln (1 + u^2) + \ln (t^2) = C \]

\[ 2 \arctan \frac{y}{t} + \ln (t^2 + u^2) = C \] general solution

\[ u = \frac{y}{t} \]
Geometrical interpretation

\[
\frac{dy}{dt} = \frac{y}{t}
\]

\[
\int \frac{dy}{y} = \int \frac{dt}{t}
\]

ln |y| = ln |t| + c

\[
y = c t, \quad c \in \mathbb{R}
\]

\[
\frac{dy}{dt} = -\frac{t}{y}
\]

\[
\int y \, dy = -\int t \, dt
\]

\[
\frac{y^2}{2} = -\frac{t^2}{2} + c
\]

\[
y^2 + t^2 = 2c
\]