Introduction to differential equations

Lecture 1 (Sept. 26, 2008)

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textbooks: M. Braun - required
Arnold's ODE - recommended

Differential equations

Ordinary differential equations (ODE)

Find a function \( y(t) \)

such that

\[ f(t, y, y', \ldots, y^{(n)}) = 0 \]

Example:

\[
\begin{align*}
y' &= 2y \\
y'' + ty + \cos t &= 0 \\
y''' &= y + t, \quad y = y(t)
\end{align*}
\]

Definition

The order of a differential equation

is the order of the highest derivative of the function \( y \) that appears in the equation

Partial differential equations (PDE)

Find a function \( u(t_1, t_2, \ldots, t_n) \)

such that

\[ f(t_1, u, \frac{\partial u}{\partial t_1}, \ldots, \frac{\partial u}{\partial t_n}, \ldots, \frac{\partial^n u}{\partial t_1^a_1 \cdots \partial t_n^a_n}) = 0\]

Example:

\[
\begin{align*}
\frac{\partial^2 u(t, x)}{\partial t^2} &= \frac{\partial^2 y}{\partial x^2}, \quad u = u(t, x) \\
\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} &= 0 \\
\end{align*}
\]

\[ u = u(x_1, x_2, x_3) \]
Solution of a differential equation:

If \( y(t) \) is a solution of an ODE if it satisfies the equation.

Example

\[
\begin{align*}
(\star) \quad & y' = y, \quad y(t) = e^t \quad \text{solution} \quad (e^t)' = e^t \\
& y(t) = 2e^t \quad \text{solution} \quad (2e^t)' = 2e^t \\
& y(t) = C e^t \quad \text{solution of (\star)} \quad \text{for each} \quad C \in \mathbb{R}.
\end{align*}
\]

Thus, \( y(t) = C e^t \) is a general solution of (\star).

To find a general solution of a differential equation is to find all solutions.

Initial value problem:

To find a solution \( y(t) \) of a differential equation such that

\[
y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \ldots, \quad y^{(n-1)}(t_0) = y_{n-1},
\]

where \( n \) is an order of the differential equation.

Example

Find a solution of an equation

\[
y' = y
\]

such that \( y(0) = 1 \).

Solution:

\[
y(t) = C e^t
\]

If \( y(0) = 1 \), then \( C \cdot e^0 = C = 1 \), so we are looking for a solution \( y(t) = e^t \).
Simplest differential equation:
\[ y'(t) = f(t) \quad (**) \]

General solution:
\[ y(t) = \int_{t_0}^{t} f(t) \, dt + C \]

Initial value problem:
\[ y(t_0) = \int_{t_0}^{t} f(t) \, dt + y_0 \quad \text{is a solution of (**), such that } y(t_0) = y_0. \]

First order ODE:
\[ y' = f(t, y) \quad , \quad y = y(t) \]

Geometrical interpretation.
Def. Assume that at each point of the plane \( \mathbb{R}^2 \) a straight line passing through this point has been chosen. In this case we say that a direction field has been defined.

Remark

A differential equation \( y' = f(t, y) \) defines a direction field:

\[ y' = \frac{dy}{dt} = f(t, y) \]

If \( y(t) \) is a solution, then the graph of \( y(t) \) is tangent to the chosen line in every point.

Def. A closed curve which is (at each of its points) tangent to a direction field is called an integral curve of the direction field.

Geometrical problem

Given a direction field, find integral curves (or integral curve passing through a given point).
Example

\[ y' = y e^t / e^t \]

\[ y(t) = C e^t \]

How to solve \( y' = y \)?

Notice that \( \frac{d}{dt} \ln |y| = \frac{y'}{y} \), so

\[ \frac{d}{dt} (\ln |y|) = 1 \]

exp; \( \ln |y| = t + C \),

\[ |y| = e^{t+C} = e^C e^t \]

so \( y = \frac{e^C e^t}{C} e^t \)

\( C \in \mathbb{R} \)

\[ y = C e^t \] - general solution.
Equation of normal reproduction
\[ y' = ky, \quad k > 0 \]
\[ \frac{d}{dt}(kt y) = k \]
\[ kt y = k t + c, \]
\[ y = ce^{kt} \quad \text{general solution.} \]

Equation of radioactive decay
\[ y' = -ky, \quad k > 0. \]
\[ y = ce^{-kt} \]

Explosion Equation
\[ y' = ky^2 \]
\[ \frac{y'}{y^2} = \frac{d}{dt}(\frac{1}{y}) \]
\[ -\frac{1}{y} = kt + c \]
\[ y = -\frac{1}{c + kt} = \frac{1}{kt - (-c)} \]
First-order linear differential equations

**Definition**

(1) \[ y' + a(t)y = b(t) \]

is the first order linear differential equation.

(2) \[ y' + a(t)y = 0 \] - homogeneous first order linear differential equation

\[ b(t) \neq 0 \] - non-homogeneous.

**How to solve**

\[ y' + a(t)y = 0 \]

\[ \frac{y'}{y} = -a(t) \]

\[ \frac{d}{dt} \ln |y| = -a(t) \]

\[ \ln |y| = -\int a(t) \, dt + C \]

\[ |y| = ce^{-\int a(t) \, dt} \]

- general solution of (2)

**Example**

\[ y' = ty \]

\[ y' = t \]

\[ y(t) = ce^{\frac{t^2}{2}} \]

\[ (ce^{\frac{t^2}{2}})' = ce^{\frac{t^2}{2}} \cdot t \]

\[ y' = y \cdot t \]
How to solve \( y' + a(t) y = b(t) \)?

**Method of variation of constant**

\( y' + a(t) y = b(t) \)  \( (\text{nh}) \)

Solve first \( y' + a(t) y = 0 \)

\[
\begin{align*}
y(t) &= C e^{-\int a(t) \, dt} \\

\text{Let us try to find a solution of (nh) in a form}
\end{align*}
\]

\[
\begin{align*}
y(t) &= \varphi(t) e^{-\int a(t) \, dt} \\

\varphi'(t) &= -\int a(t) \, dt + \varphi(t) \left(-\alpha(t) e^{-\int a(t) \, dt}\right) + \alpha(t) \left(\varphi(t) e^{-\int a(t) \, dt}\right)
\end{align*}
\]

\[
\begin{align*}
\varphi(t) &= \frac{b(t)}{\alpha(t)} e^{-\int a(t) \, dt} \\

\text{Find } \varphi(t) &= G(t) + C,
\end{align*}
\]

\[
\begin{align*}
y(t) &= (G(t) + C) e^{-\int a(t) \, dt} - \text{general solution}
\end{align*}
\]
Example

\[ y' + y + t = 0 \]

Homog. equation:
\[ y' = -y \]
\[ y = ce^{-t} \]

Let us try to find the solutions of the initial (non-homogeneous) equation in a form
\[ y(t) = \varphi(t)e^{-t} \]

\[ \varphi'(t)e^{-t} - \varphi(t)e^{-t} + \varphi(t)e^{-t} + t = 0 \]

\[ \varphi'(t) = -te^{t} \]

\[ \varphi(t) = -\int te^{t}dt + C = \]

\[ = -(te^{t} - e^{t}) + C = C - te^{t} + e^{t} \]

\[ y(t) = (C - te^{t} + e^{t})e^{-t} \quad \text{general solution} \]