First-order linear differential equations

**Definition**

\[ y' + a(t)y = b(t), \quad y(t) \]

the first order linear (non-homogeneous) differential equation

\[ y' + a(t)y = 0 \]

the first order linear homogeneous diff. equation

**Example**

\[ y' + ty + 3y = 0 \quad \times \]

\[ y' + ty + t^2 + 1 = 0 \quad \checkmark \]

**How to solve**

\[ y' + a(t)y = 0 \quad \times \]

\[
\frac{y'}{y} = -a(t)
\]

\[
\frac{d}{dt} \ln |y| = -a(t)
\]

\[
\ln |y| = -\int a(t) \, dt + C
\]

\[
|y(t)| = e^{-\int a(t) \, dt + C}
\]

\[
y(t) = Ce^{-\int a(t) \, dt}
\]

general solution of \((*)\)
Example
\[ y' = ty \]
\[ \frac{y'}{y} = t \]
\[ y(t) = ce^{\frac{t^2}{2}} \]

How to solve \( y' + a(t)y = b(t) \)? \( \star \star \star \)

Solve first \( y' + a(t)y = 0 \)
\[ y(t) = c \cdot e^{-\int a(t) dt} \]

Let us try to find a solution of \( \star \star \star \) in a form
\[ y(t) = \varphi(t) \cdot e^{-\int a(t) dt} \]
\[ \varphi'(t) e^{-\int a(t) dt} + \varphi(t) \cdot e^{-\int a(t) dt} \cdot (-a(t)) + \]
\[ + a(t) \varphi(t) e^{-\int a(t) dt} = b(t) \]
\[ \varphi'(t) = b(t) \cdot e^{\int a(t) dt} \]
\[ \varphi(t) = C + \int b(t) e^{\int a(t) dt} dt \]
\[ \varphi(t) = C + G(t) \]
\[ y(t) = e^{-\int a(t) dt} \left( C + G(t) \right) \]

("Method of "variation of parameters")
Example

\[ y' + y + e^{-t} = 0 \]
\[ y' = -y \]
\[ y = c e^{-t} \]

Let us try to find the solution of the (non-homogeneous) equation in a form

\[ y(t) = \varphi(t) e^{-t} \]
\[ \varphi'(t) e^{-t} - (\varphi(t)) e^{-t} + \varphi(t) e^{-t} + t = 0 \]
\[ \varphi'(t) = -t e^{-t} \]

\[ y(t) = -S + e^{t} dt + c = -(e^{-t} e^{t}) + c = c - (e^{-t} e^{t}) \]
\[ y(t) = (c - e^{-t} e^{t}) e^{-t} = c e^{-t} - t + 1 \]

- general solution

Remark

Solutions of a homogeneous linear shift equation from a linear space:

\[ y' + a(t) y = 0 \quad y = y(t) \]
\[ z' + a(t) z = 0 \quad z = z(t) \]

\[ \frac{d}{dt} \left( y + z + a(t) (y + z) \right) = 0 \]
\[ (\lambda y)' + a(t) \cdot (\lambda y) = 0 \quad \forall \lambda \in \mathbb{R} \]
Moreover, if

\( z, y \) are solutions of a non-homogeneous equation then

\((z - y)\) is a solution of a homogeneous equation.

In other words, if

\( w(t) \) is a solution of a non-homogeneous equation,

then any its solution \( z(t) \)

can be represented as \( z(t) = w(t) + u(t) \),

where \( u(t) \) is a solution of a homogeneous equation.

\( C'(\mathbb{R}) \)

solutions of a homogeneous equation

\( y' = \frac{dy}{dx} \)

partial solution of a non-homogeneous equation

solutions of a non-homogeneous equation

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**Example**

\[
\begin{align*}
X(t) &= 100 + t \\
\frac{dX}{dt} &= 1 + \frac{4}{100+t} \\
Y(0) &= 0
\end{align*}
\]

Can the bug bite within 2 hours?
Homogeneous equation:

\[ y'(t) = \frac{y}{100+t} \]

\[ \frac{y'}{y} = \frac{1}{100+t} \]

\[ y = C \cdot (100+t) \]

Non-homogeneous equation:

\[ y' = 1 + \frac{y}{100+t} \]

\[ y(t) = y(t) \cdot \frac{1}{100+t} \]

\[ y'(t) + (100+t) + y(t) = 1 + y(t) \]

\[ y'(t) = \frac{1}{100+t} \]

\[ u(t) = \ln(100+t) + \epsilon \]

\[ u(t) = (100+t) \cdot (C + \ln(100+t)) \]

\[ y(0) = 0 \]

\[ C = \ln 100 \]

\[ y(t) = (100+t) \cdot \ln \left(1 + \frac{t}{100}\right) \]

\[ y(t) = x \cdot 14 = 100+t \]

\[ \ln \left(1 + \frac{t}{100}\right) = 1 \Rightarrow 1 + \frac{t}{100} = e \]

\[ t = 100 \cdot (e - 1) \approx 1.7 \cdot 10^3 \text{ min} \]

\[ 170 \text{ min} > 2 \text{ hours} \]
On linear shift equations of first order with periodic coefficients.

\[ y'(t) + a(t) y = 0, \quad a(t+T) = a(t), \quad T > 0, \]

\[
\begin{align*}
  y(t) &= C e^{-\int_{0}^{t} a(t) \, dt} \\
  y(T) &= y(0) e^{-\int_{0}^{T} a(t) \, dt} = \lambda
\end{align*}
\]

If \[ \lambda < 1 \] then \[ y(t) \to 0 \text{ as } t \to +\infty \]

If \[ \lambda > 1 \] then \[ y(t) \to \infty \text{ as } t \to +\infty \]

If \[ \lambda = 1 \] then \[ y(t) \text{ is periodic} \]

\[ y' + a(t) y = b(t) = a(t), \quad b(t) \text{ are periodic} \]

If \[ \lambda < 0 \] then \[ \exists \] periodic solution,

if \[ \lambda > 0 \] then all solutions \[ \to \text{ periodic} \]

if \[ \lambda > 1 \] then all other solutions \[ \to \infty \text{ as } t \to +\infty \]