Recovering Short Generators of Principal Ideals in Cyclotomic Rings

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Joint work with
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[1] Slides revised on Sept. 7, 2015.

Recovering Short Generators for Cryptanalysis

A few cryptosystems (Fully Homomorphic Encryption [Smart and Vercauteren, 2010] and Multilinear Maps [Garg et al., 2013, Langlois et al., 2014]) share this KEYGEN:

- sk Choose a *short* g in some ring R as a private key
- pk Give a *bad* $\mathbb{Z}$-basis $\mathbf{B}$ of the ideal (g) as a public key (e.g. HNF).

Cryptanalysis in two steps (Key Recovery Attack)

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1. Principal Ideal Problem (PIP)
   - Given a $\mathbb{Z}$-basis $\mathbf{B}$ of a principal ideal $\mathfrak{I}$,
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2. Short Generator Problem
   - Given an arbitrary generator $h \in R$ of $\mathfrak{I}$
   - Recover g (or some g' equivalently short)

Cost of those two steps

1. Principal Ideal Problem (**PIP**)
   - sub-exponential time ($2^{\tilde{O}(n^{2/3})}$) classical algorithm [Biasse and Fieker, 2014, Biasse, 2014].
   - progress toward quantum polynomial time algorithm [Eisenträger et al., 2014, Biasse and Song, 2015b, Campbell et al., 2014, Biasse and Song, 2015a].
2. Short Generator Problem
   - equivalent to the **CVP** in the *log-unit* lattice
   - becomes a **BDD** problem in the crypto cases.
   - claimed to be easy [Campbell et al., 2014] in the cyclotomic case $m = 2^k$
     - confirmed by experiments [Schank, 2015]

This Work [Cramer et al., 2015]

We focus on step 2, and prove it can be solved in *classical polynomial time* for the aforementioned cryptanalytic instances, when the ring R is the ring of integers of the cyclotomic number field $K = \mathbb{Q}(\zeta_m)$ for $m = p^k$.

Overview

The Logarithmic Embedding

Let K be a number field of degree n, $\sigma_1 \dots \sigma_n : K \mapsto \mathbb{C}$ be its embeddings, and let R be its ring of integers. The logarithmic Embedding is defined as

$$\begin{aligned} \mathrm{Log} : K &\to \mathbb{R}^n \\ x &\mapsto (\log|\sigma_1(x)|, \dots, \log|\sigma_n(x)|) \end{aligned}$$

It induces

- a group morphism from $(K \setminus \{0\}, \cdot)$ to $(\mathbb{R}^n, +)$
- a monoid morphism from $(R \setminus \{0\}, \cdot)$ to $(\mathbb{R}^n, +)$

The Unit Group

Let $R^\times$ denotes the multiplicative group of units of R.
Let $\Lambda = \operatorname{Log} R^\times$. By Dirichlet Unit Theorem

- the kernel of Log is the cyclic group T of roots of unity of R
- $\Lambda \subset \mathbb{R}^n$ is an lattice of rank $r + c - 1$

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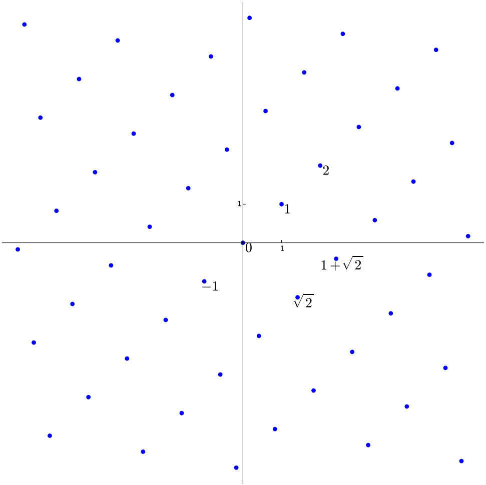
Reduction to CVP

Elements $g, h \in R$ generate the same ideal if and only if $h = g \cdot u$ for some unit $u \in R^\times$. In particular

$$\operatorname{Log} g \in \operatorname{Log} h + \Lambda.$$

and g is the "smallest" generator iff $\operatorname{Log} u \in \Lambda$ is a vector "closest" to $\operatorname{Log} h$.

Example: Embedding $\mathbb{Z}[\sqrt{2}] \hookrightarrow \mathbb{R}^2$



- x-axis: $a + b\sqrt{2} \mapsto a + b\sqrt{2}$
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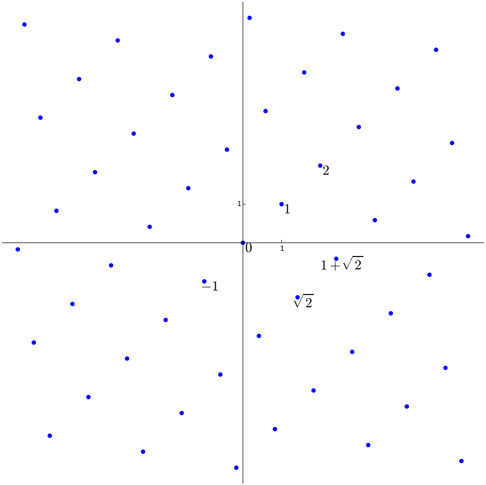

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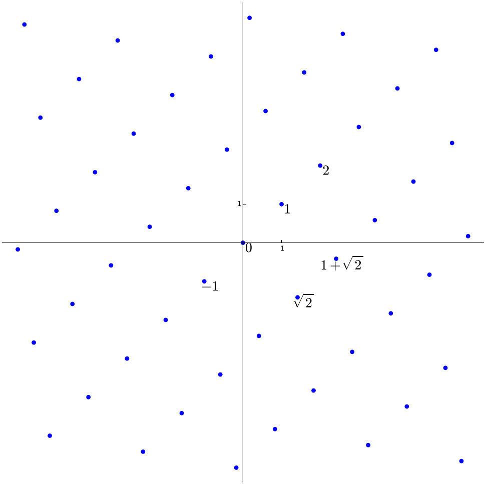

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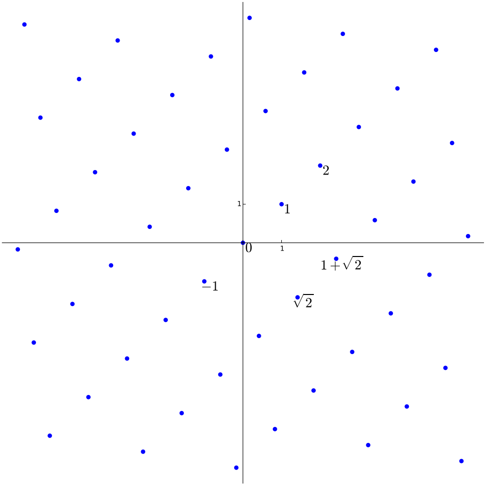

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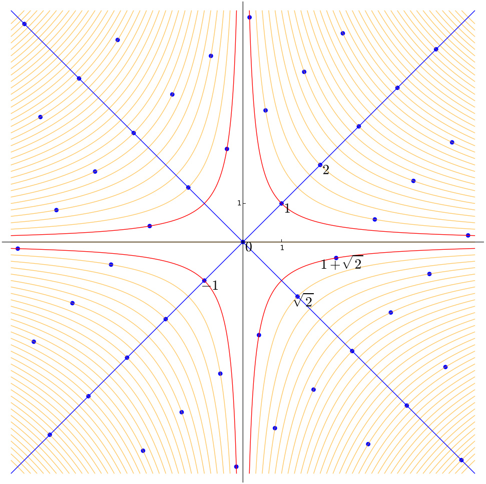

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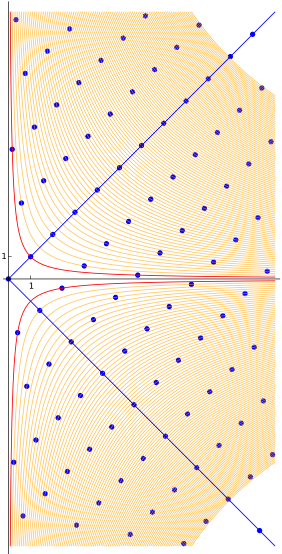
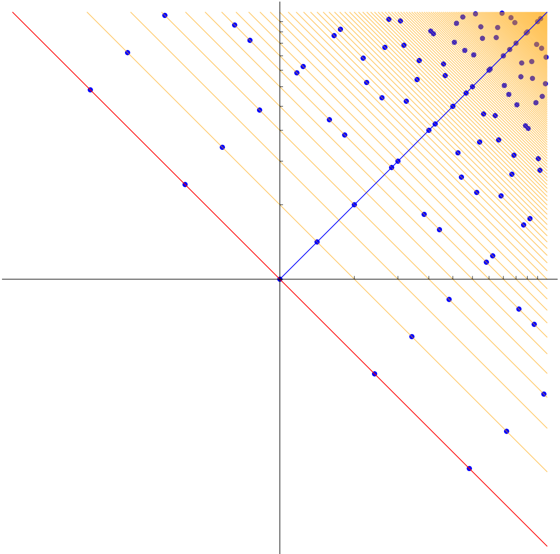
■ "Orthogonal" elements
■ Units (algebraic norm 1)
■ "Isonorms" curves

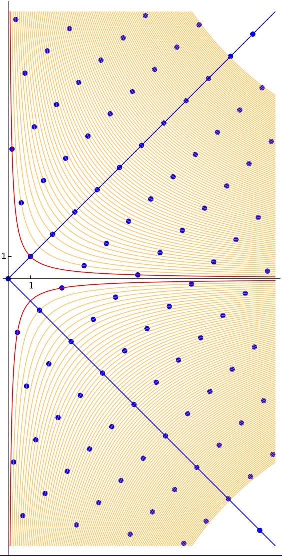
$(\{\bullet\}, +)$ is a sub-monoid of $\mathbb{R}^2$

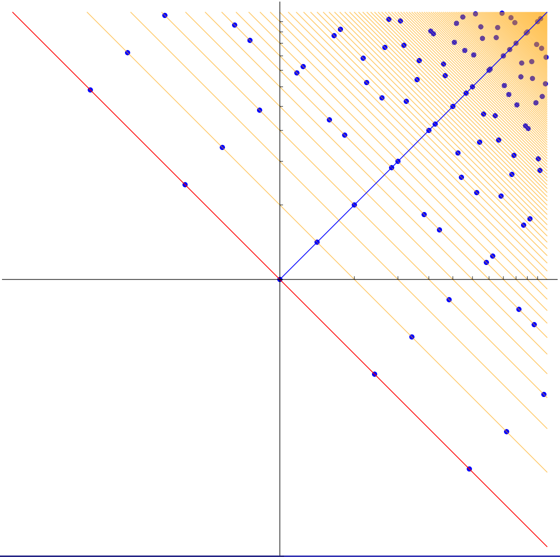




Example: Logarithmic Embedding $\text{Log}\,\mathbb{Z}[\sqrt{2}]$

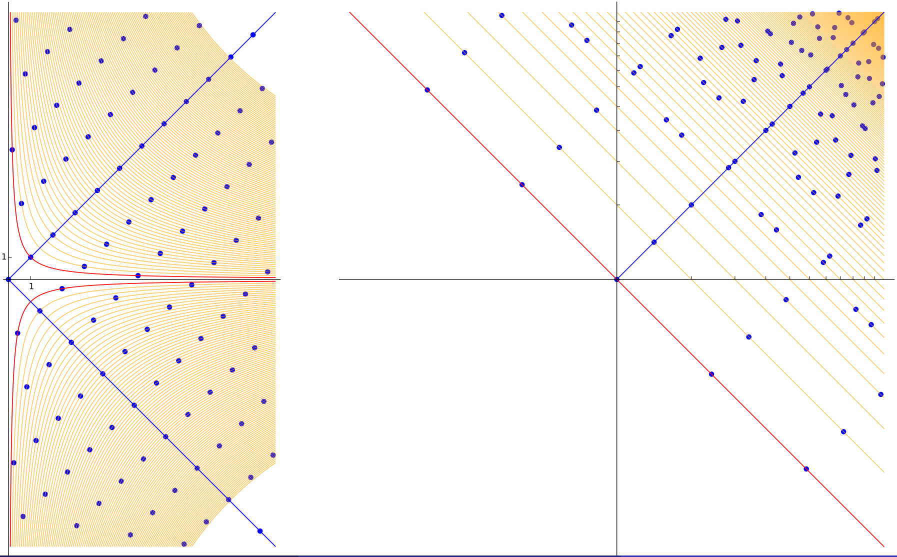
$\Lambda = (\{\bullet\}, +) \cap \diagdown$ is a lattice of $\mathbb{R}^2$, orthogonal to $(1, 1)$





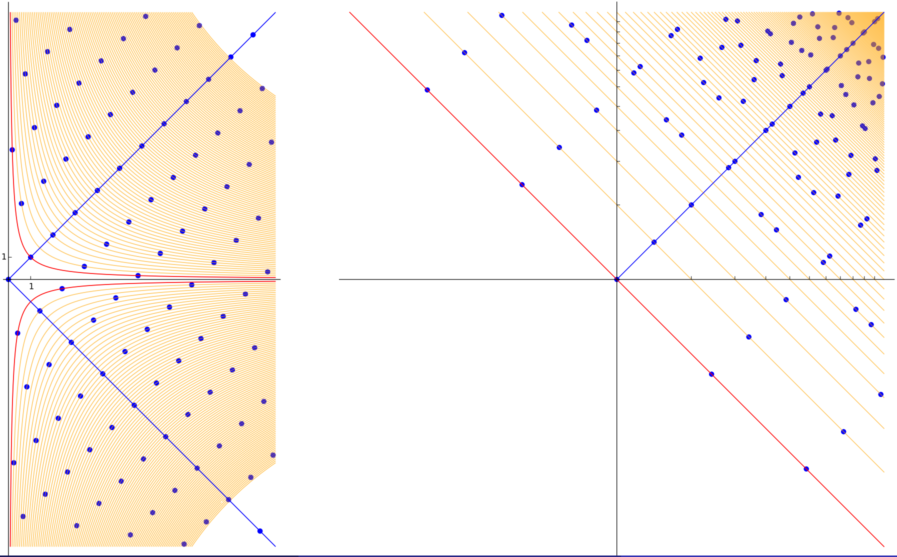
Example: Logarithmic Embedding $\operatorname{Log}\mathbb{Z}[\sqrt{2}]$

$\{\bullet\} \cap$ ╲ are shifted finite copies of Λ



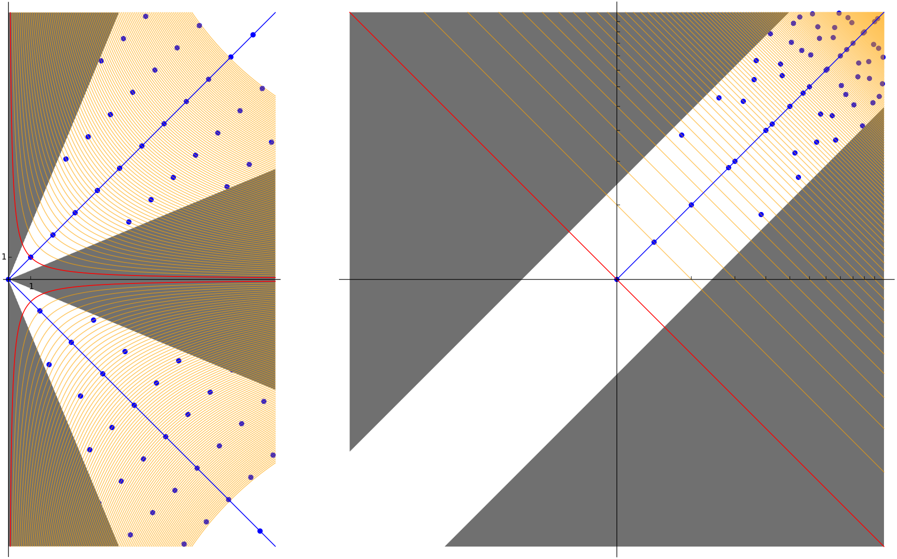
Example: Logarithmic Embedding Log $\mathbb{Z}[\sqrt{2}]$

Some $\{\bullet\} \cap$ ╲ may be empty (e.g. no elements of Norm 3 in $\mathbb{Z}[\sqrt{2}]$)



Reduction modulo $\Lambda = \operatorname{Log} \mathbb{Z}[\sqrt{2}]^\times$

The reduction $\bmod \Lambda$ for various fundamental domains.



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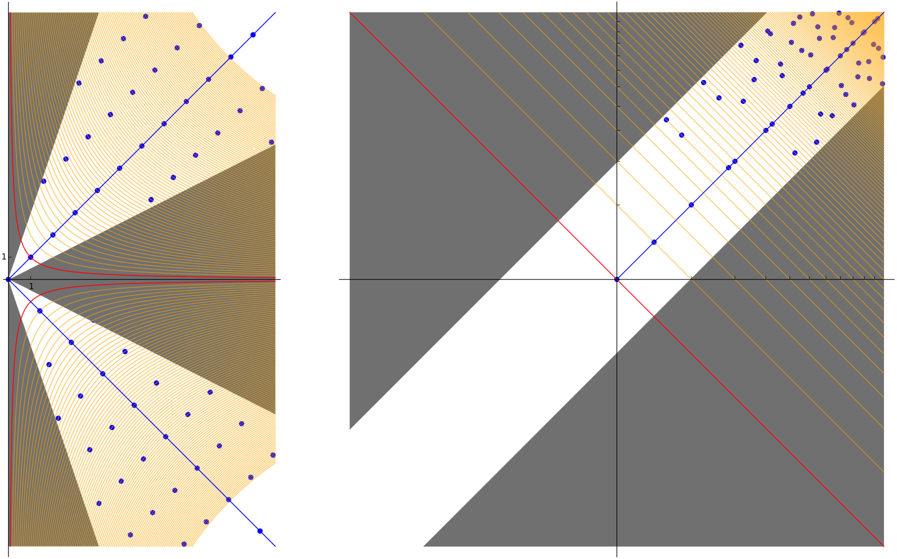

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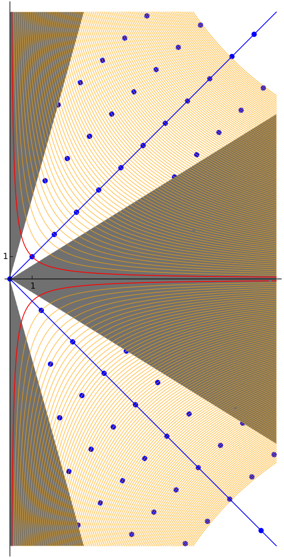

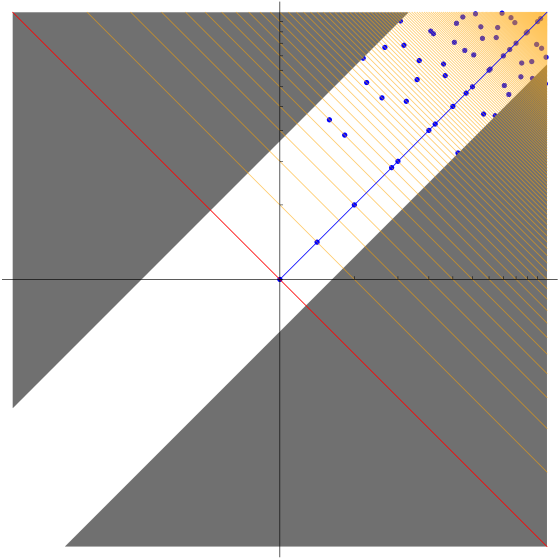

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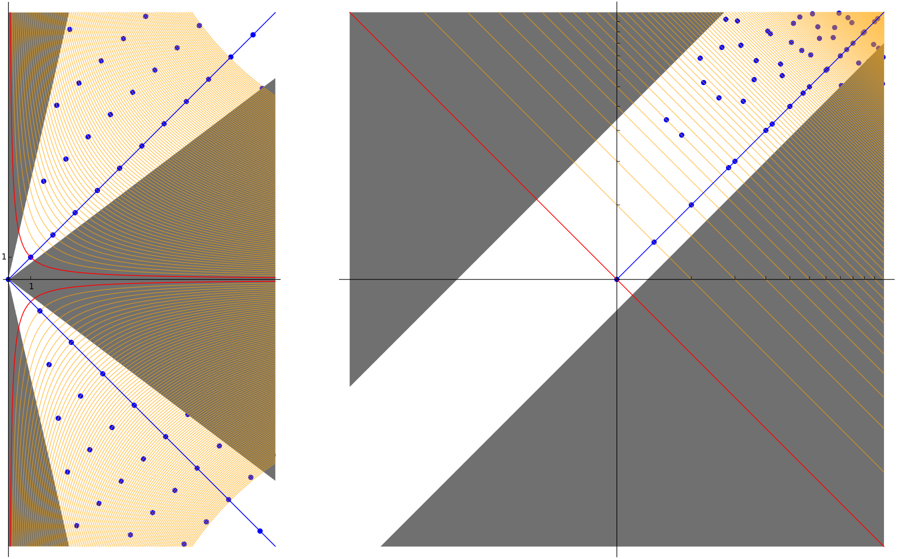


Decoding with the RoundOff algorithm

The simplest algorithm [Babai, 1986] to reduce modulo a lattice

RoundOff$(\mathbf{B}, \mathbf{t})$, $\mathbf{B}$ a $\mathbb{Z}$-basis of Λ

$\mathbf{v} = \mathbf{B} \cdot \lfloor (\mathbf{B}^\vee)^\top \cdot \mathbf{t} \rceil$
$\mathbf{e} = \mathbf{t} - \mathbf{v}$
return $(\mathbf{t}, \mathbf{e})$ where $\mathbf{t} \in \mathbf{B}$

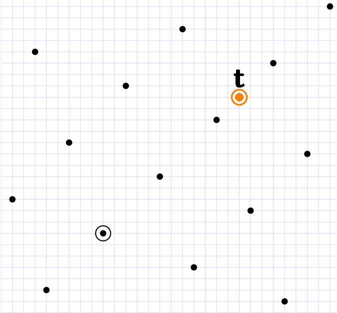
Used as a *d*ecoding algorithm, its correctness is characterized by the error $\mathbf{e}$ and the *dual basis* $\mathbf{B}^\vee$.

Fact(Correctness of RoundOff)

let $\mathbf{t} = \mathbf{v} + \mathbf{e}$ for some $\mathbf{v} \in \Lambda$. If $\langle \mathbf{b}_j^\vee, \mathbf{e} \rangle \in [-\frac{1}{2}, \frac{1}{2})$ for all j, then

$$\text{RoundOff}(\mathbf{B}, \mathbf{t}) = (\mathbf{v}, \mathbf{e}).$$

RoundOff in pictures



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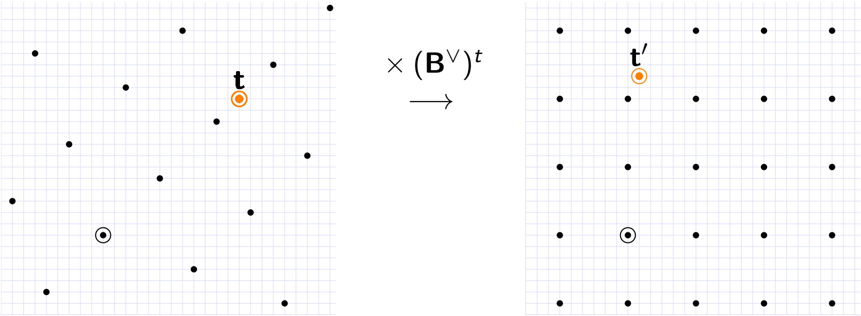

RoundOff algorithm:

1. use basis **B** to switch to the lattice $\mathbb{Z}^n$ $(\times(\mathbf{B}^\vee)^t)$

$$\mathbf{t}' = (\mathbf{B}^\vee)^t \cdot \mathbf{t};$$

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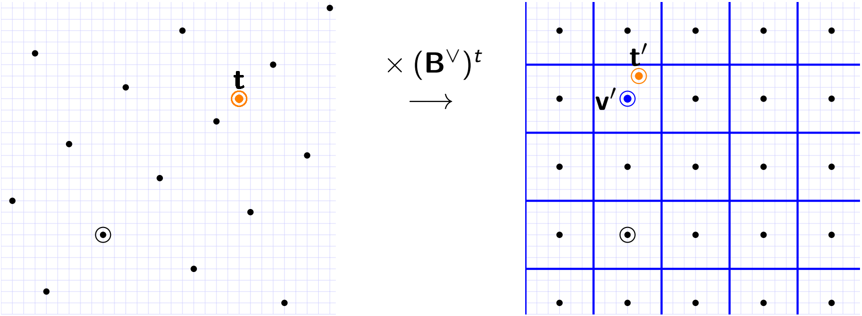
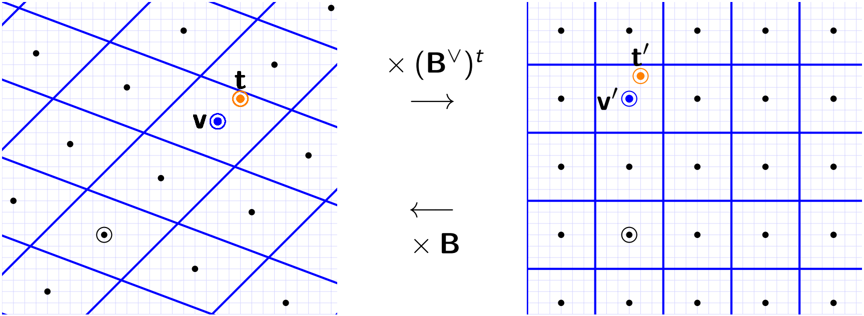


RoundOff algorithm:

1. use basis $\mathbf{B}$ to switch to the lattice $\mathbb{Z}^n$ $(\times(\mathbf{B}^\vee)^t)$
2. Round each coordinate

$$\mathbf{t}' = (\mathbf{B}^\vee)^t \cdot \mathbf{t}; \quad \mathbf{v}' = \lfloor \mathbf{t}' \rceil;$$



RoundOff algorithm:

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3. Switch back to the lattice L $(\times\mathbf{B})$

$$\mathbf{t}' = (\mathbf{B}^\vee)^t \cdot \mathbf{t}; \quad \mathbf{v}' = \lfloor \mathbf{t}' \rceil; \quad \mathbf{v} = \mathbf{B} \cdot \mathbf{v}'$$

Recovering Short Generator: Proof Plan

Folklore strategy [Bernstein, 2014, Campbell et al., 2014] to recover a short generator g

1. Construct a basis $\mathbf{B}$ of the unit-log lattice $\operatorname{Log} R^\times$
   - For $K = \mathbb{Q}(\zeta_m)$, $m = p^k$, an (almost[2]) canonical basis is given by

$$\mathbf{b}_j = \operatorname{Log} \frac{1-\zeta^j}{1-\zeta}, \quad j \in \{2, \ldots, m/2\}, j \text{ co-prime with } m$$

2. Prove that the basis is "good", that is $\|\mathbf{b}_j^\vee\|$ are all small
3. Prove that $\mathbf{e} = \operatorname{Log} g$ is small enough

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Technical contributions [CDPR15]

2. Estimate $\|\mathbf{b}_j^\vee\|$ precisely using analytic tools [Washington, 1997, Littlewood, 1924]
3. Bound $\mathbf{e}$ using theory of sub-exponential random variables [Vershynin, 2012]

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Overview

Cyclotomic units

We fix the number field $K = \mathbb{Q}(\zeta_m)$ where $m = p^k$ for some prime p. Set

$$z_j = 1 - \zeta^j \quad \text{and } b_j = z_j / z_1 \text{ for all } j \text{ coprimes with } m.$$

The b_j are units, and the group C generated by

$$\zeta, \quad b_j \quad \text{for } j = 2, \ldots m/2, j \text{ coprime with } m$$

is known as the group of *cyclotomic units*.

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Simplification 2 (for this talk)

We study the dual matrix $\mathbf{Z}^\vee$, where $\mathbf{z}_j = \operatorname{Log} z_j$.
It can be proved to close to $\mathbf{B}^\vee$ where $\mathbf{b}_j = \mathbf{z}_j - \mathbf{z}_1$.

[3]One just need the index $[R^\times : C] = h^+(m)$ to be small.

The matrix **Z**

The field K admits exactly $\varphi(m)/2$ pairs of conjugate complex embeddings

$$\sigma_i = \overline{\sigma_{-i}}, \text{ where } \sigma_i : \zeta \mapsto \omega^i \text{ is defined for all } i \in \mathbb{Z}_m^\times.$$

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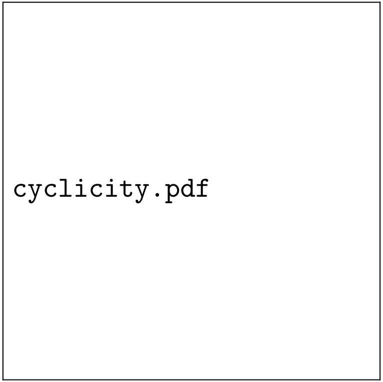

Figure : Naïve Indexing ($i = 1, 3, 5, \dots$)

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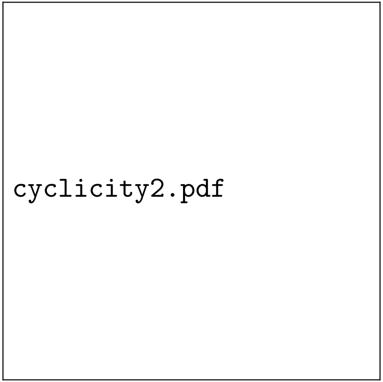

Figure : Multiplicative Indexing ($i = 3^0, 3^1, 3^2, \ldots$)

Dual of a Circulant Basis

Notice that $\mathbf{Z}_{ij} = \log|\sigma_j(1-\zeta^i)| = \log|1-\omega^{ij}|$:
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Fact

If $\mathbf{M}$ is a non-singular, G-circulant matrix, then

- *its eigenvalues are given by* $\lambda_\chi = \sum_{g \in G} \overline{\chi(g)} \cdot \mathbf{M}_{1,g}$
  where $\chi \in \widehat{G}$ *is a character* $G \to \mathbb{C}$
- *All the vectors of* $\mathbf{M}^\vee$ *have the same norm* $\|\mathbf{m}_i^\vee\|^2 = \sum_{\chi \in \widehat{G}} |\lambda_\chi|^{-2}$

Note: The characters of G can be extended to even Dirichlet characters mod m: $\chi : \mathbb{Z} \to \mathbb{C}$, by setting $\chi(a) = 0$ if $\gcd(a, m) > 1$.

Computing the Eigenvalues

We wish to give a lower bound on $|\lambda_\chi|$ where

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Something cute to be learned !

The equations looks not very algebraic (log ?), yet appears quite naturally... Surely mathematicians knows how to deal with this.

Indeed, computation of the volume of that basis appears in [Washington, 1997].

Computing the Eigenvalues

We wish to give a lower bound on $|\lambda_\chi|$ where

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We develop using the Taylor series

$$\log|1 - x| = -\sum_{k \geq 1} x^k / k$$

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$$\log |1 - x| = -\sum_{k \geq 1} x^k / k$$

and obtain

$$-\lambda_\chi = \sum_{a \in G} \sum_{k \geq 1} \overline{\chi(a)} \cdot \frac{\omega^{ka}}{k}.$$

Computing the Eigenvalues (continued)

We were trying to lower bound $|\lambda_\chi|$ where

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Fact (Separability of Gauss Sums)

If χ is a primitive Dirichlet character mod m *then*

$$\sum_{a\in\mathbb{Z}_m^\times} \overline{\chi(a)} \cdot \omega^{ka} = \chi(k) \cdot G(\chi) \quad \textit{where } |G(\chi)| = \sqrt{m}.$$

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For this talk, let's ignore non-primitive characters. We rewrite

$$|\lambda_\chi| = \sqrt{\frac{m}{2}} \cdot \left| \sum_{k \geq 1} \frac{\chi(k)}{k} \right|.$$

The Analytical Hammer

We were trying to lower bound $\left|\lambda_\chi\right| = \sqrt{\frac{m}{2}} \cdot \left|\sum_{k\geq 1} \frac{\chi(k)}{k}\right|$.
One recognizes a Dirichlet L-series

$$L(s, \chi) = \sum \frac{\chi(k)}{k^s}.$$

The Analytical Hammer

We were trying to lower bound $\left|\lambda_\chi\right| = \sqrt{\frac{m}{2}} \cdot \left|\sum_{k\geq 1} \frac{\chi(k)}{k}\right|$.
One recognizes a Dirichlet L-series

$$L(s,\chi) = \sum \frac{\chi(k)}{k^s}.$$

Theorem ([Littlewood, 1924, Youness et al., 2013])

Under the Generalized Riemann Hypothesis, for any primitive Dirichlet character χ mod m it holds that

$$1/\ell(m) \leq |L(1,\chi)| \leq \ell(m) \quad \textit{where } \ell(m) = C \ln\ln m$$

for some universal constant $C > 0$.

Geometric Conclusion

Theorem (Cramer, D. , Peikert, Regev)

Let $m = p^k$*, and* $\mathbf{B} = \left(\mathrm{Log}(b_j)\right)_{j \in G \setminus \{1\}}$ *be the canonical basis of* $\mathrm{Log}\, C$*. Then, all the vectors of* $\mathbf{B}^\vee$ *have the same norm and, under GRH, this norm is upper bounded as follows*

$$\left\|\mathbf{b}_j^\vee\right\|^2 \leq O\left(m^{-1} \cdot \log^3 m\right).$$

Overview

Proof Plan (Reminder)

1. Construct a basis $\mathbf{B}$ of the unit-log lattice $\operatorname{Log} R^\times$
   - Choose the Canonical Cyclotomics Units
   $$\mathbf{b}_j = \operatorname{Log} \frac{1-\zeta^j}{1-\zeta}$$
2. Prove that the basis is "good", that is $\|\mathbf{b}_j^\vee\|$ are all small
   - Proved
   $$\left\|\mathbf{b}_j^\vee\right\|^2 \leq O\left(m^{-1} \cdot \log^3 m\right)$$
3. Prove that $\mathbf{e} = \operatorname{Log} g$ is small enough

Scaling Invariance

Lets assume the embeddings $(\sigma_i(g))$ are i.i.d. of distribution $\mathcal{D}$.

$$\mathrm{Log}\,(s \cdot \mathcal{D}^n) \simeq (1, 1, \ldots 1) \cdot \log s + \mathrm{Log}\,\mathcal{D}^n$$

Heuristic argument

Using scaling, assume that $\mathbb{E}[\operatorname{Log}\mathcal{D}^m] = \mathbf{0}$.

- Let $\mathbf{e} \leftarrow \operatorname{Log}\mathcal{D}^m$ ($\mathbf{e} = \operatorname{Log} g$)
- Each coordinate $\operatorname{Log}\mathcal{D}$ of $\mathbf{e}$ are independents, centered, of variance V
- For any $\mathbf{b}$, the variance of $\langle \mathbf{b}, \mathbf{e} \rangle$ is $V \cdot \|\mathbf{b}\|$
- By Markov Inequality, for a fixed i it should hold that

$$|\langle \mathbf{b}_i^\vee, \mathbf{e} \rangle| \leq 1/2$$

except with $o(1)$ probability (recall we've proved that $\|\mathbf{b}_i^\vee\| = o(1)$)

Conclusion from better tail bounds

The previous argument does not allows to conclude simultanously on all i's. We fill this gap using stronger tail bounds, form the theory of sub-exponential random variables [Vershynin, 2012]

Theorem (Cramer, D. , Peikert, Regev)

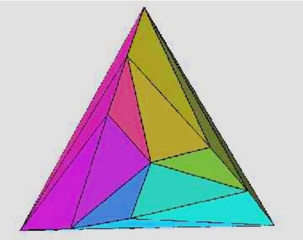
If g follows a Continuous Normal Distribution, then for $\mathbf{e} = \operatorname{Log} g$, we have $|\langle \mathbf{b}_i^\vee, \mathbf{e}\rangle| \leq 1/2$ for all i's except with negligible probability.

Corollary

If g follows a Discrete Normal Distribution of parameter $\sigma \geq poly(m)$, then for $\mathbf{e} = \operatorname{Log} g$, we have $|\langle \mathbf{b}_i^\vee, \mathbf{e}\rangle| \leq 1/2$ for all i's except with probability $1/n^{\Theta(1)}$.

Thanks

Figure : The Shintani Domain of $\mathbb{Z}[\zeta_7 + \bar{\zeta_7}]$. Credit: Paul Gunells
`http://people.math.umass.edu/~gunnells/pictures/pictures.html`



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