Cryptography via Burnside Groups

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Based on work w/ G.Baumslag, N.Fazio, K.Iga, L.Perret, V.Shpilrain and W.E.Skeith III

Talk Preview

Goal

Identify **viable** intractability assumptions from combinatorial group theory

- Evidence of (average-case) hardness (random self-reducibility)
- Cryptographically useful

Approach

- Generalize well-established crypto assumptions (LPN/LWE) to a group-theoretic setting
- Study instantiation in suitable non-commutative groups

Outline

1. **Background**
   - Burnside Groups (B_n)
   - Learning Burnside Homomorphisms with Noise (B_n-LHN)

2. **Random Self-Reducibility of B_n-LHN**

3. **Cryptography (Minicrypt) via Burnside Groups**

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3. Cryptography (Minicrypt) via Burnside Groups

Burnside Problem (Informal)

- Are groups whose elements all have **finite** order necessarily **finite**?
- What is their combinatorial structure?

Free Burnside group of exponent m

- $B(n,m)$: "Most generic" group with n generators where the order of **all** elements divides m
  - Generators $x_1, \ldots, x_n$ (like indeterminates in a multivariate poly)
  - Elements are sequences of x_i and x_i^{-1}
  - Empty sequence is the identity element of the group
  - Exponent condition: For every $w \in B(n,m)$ it holds that $\boxed{w^m = 1}$
- Examples:
  - $x_1 x_4^{-1} x_1 \in B(4,3), \quad x_1^{-1} x_4^{-1} \in B(4,3)$
  - $x_1^2 = x_1^{-1}$, but $x_1 x_4^{-1} x_1 \neq x_1^{-1} x_4^{-1} = x_1 x_1 x_4^{-1}$ ($B(4,3)$ is not abelian)
  - On the other hand:

$$x_1 x_4^{-1} x_1 = x_4 x_1^{-1} x_4, \qquad \text{since } x_1 x_4^{-1} x_1 x_4^{-1} x_1 x_4^{-1} = (x_1 x_4^{-1})^3 = 1$$

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Burnside Groups (cont'd)

- Characterizing $B(n, m)$ not so easy ...

$B(n, 2)$	Finite and abelian, isomorphic to $(\mathbb{F}_2^n, +)$
$B(n, 3)$	Finite, non-commutative, much larger than $(\mathbb{F}_3^n, +)$
$B(n, 4)$	Finite
$B(n, 5)$	**Unknown**
$B(n, 6)$	Finite
$B(n, 7)$	**Unknown**
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$B(n, m)$, m "large"	Infinite

- Will focus on $B(n, 3)$ (simplest case beyond vector spaces)
  - Notation: $B_n \doteq B(n, 3)$

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- B_n: "Most generic" group with n generators where the order of **all** non-identity elements is 3
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- **Q**: "Most generic"!?

  A: The only non-trivial identities in B_n are those implied by $(\star)$

$\Rightarrow$ B_n non-commutative

- $x_i x_j \neq x_j x_i$ for any two distinct generators $(i \neq j)$

$\Rightarrow$ Group operation in B_n defined "formally"

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Basic Commutators

- In B_n, $x_i x_j \neq x_j x_i$ for any two distinct generators ($i \neq j$)
- However, always possible to get $x_i x_j = x_j x_i [x_i, x_j]$ by defining

$$[x_i, x_j] \doteq x_i^{-1} x_j^{-1} x_i x_j$$

  Call $[x_i, x_j]$ a **2-commutator**
- Similarly, define a **3-commutator** $[x_i, x_j, x_k]$ as

$$[x_i, x_j, x_k] \doteq [[x_i, x_j], x_k]$$

- In general, may define **ℓ-commutators** inductively, but in B_n all ℓ-commutators vanish for $\ell \geq 4$,

$$[x_i, x_j, x_k, x_h] = 1$$

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Commutators Identities in B_n

- $[x_i, x_j, x_k, x_h] = 1$ implies:
  - 3-commutators commute with all $w \in B_n$:

$$[x_i, x_j, x_k]w = w[x_i, x_j, x_k]$$

  - 2-commutators commute among themselves:

$$[x_k, x_h][x_i, x_j] = [x_i, x_j][x_k, x_h]$$

- Other commutator identities in B_n:

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Normal Form in B_n

- In general, elements in non-commutative groups may have multiple equivalent forms
  - *E.g.*, $x_i x_j^{-1} x_i = x_j x_i^{-1} x_j$
- In B_n, commutator identities imply that any $w \in B_n$ can always be written uniquely as:

$$w = \prod_{i=1}^{n} x_i^{\alpha_i} \prod_{i<j} [x_i, x_j]^{\beta_{i,j}} \prod_{i<j<k} [x_i, x_j, x_k]^{\gamma_{i,j,k}}$$

where $\alpha_i, \beta_{i,j}, \gamma_{i,j,k} \in \{-1, 0, 1\}$, for all $1 \leq i < j < k \leq n$

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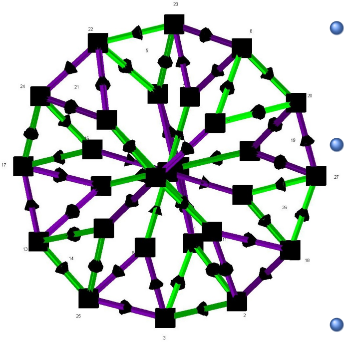
Example: The Structure of B_2



- Cayley graph of B_2 (left): nodes $\equiv$ elements; edges $\equiv$ multiplication by a generator (green: x_1; purple: x_2)
- B_2 has 27 elements, of the form

$$x_1^{\alpha_1} x_2^{\alpha_2} [x_1, x_2]^{\beta_{1,2}}, \alpha_1, \alpha_2, \beta_{1,2} \in \mathbb{F}_3$$

- Isomorphic to Heisenberg Group $H_1(\mathbb{F}_3)$:

$$\begin{pmatrix} 1 & \alpha_1 & \beta_{1,2} \\ 0 & 1 & \alpha_2 \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{F}_3)$$

- Beware of hasty generalization: for $n \geq 3$, $B_n \ncong H_m(\mathbb{F}_3)$
- No known $poly(n)$-order representation of B_n

Group operation in B_n

- Recall the normal form in B_n:

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- ... then reduce back to normal by reordering commutators via $O(n^3)$ three-stage **collecting process** (*next*)

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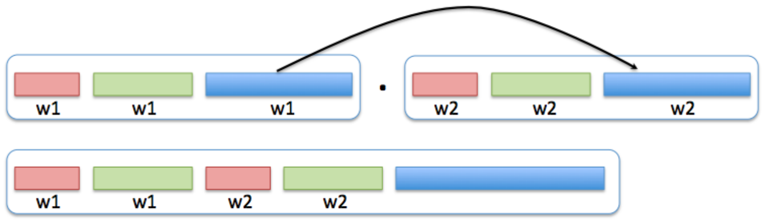
The Collecting Process (1/3)

Stage 1

Aggregate 3-commutators in w_1 and w_2, adding matching exponents mod 3

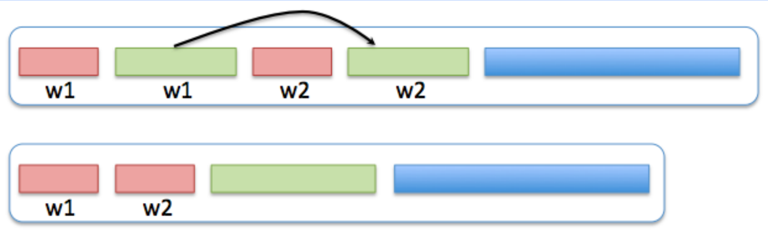


Time: $O(1)$ per 3-commutator, total $O(n^3)$

The Collecting Process (2/3)

Stage 2

Move 2-commutators in w_1 to the right of generators in w_2



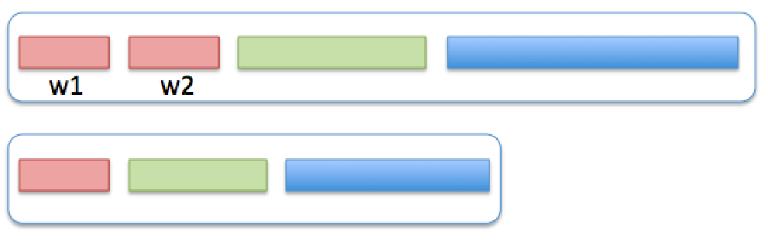
Each 2-commutator traveling right incurs $O(n)$ (constant-time) swaps with generators in w_2.

Time: $O(n)$ per 2-commutator, total $O(n^3)$

The Collecting Process (3/3)

Stage 3

Restore lexicographic order among generators



Fixing each out-of-order generator takes $O(n)$ swaps, and each swap creates a 2-commutator.
Before moving on to the next generator, these $O(n)$ 2-commutators must travel rightward (similarly to step 2 above), which takes $O(n^2)$ steps

Time: $O(n^2)$ per generator, total $O(n^3)$

Group operation in B_n: Example

$$x_1^{-1}x_3[x_2,x_3] \quad \cdot \quad x_1x_2[x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3][x_2,x_3,x_1]x_2[x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3][x_1,x_2,x_3]x_2[x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3][x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3]^{-1} =$$

$$x_1^{-1}x_3x_1x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_1^{-1}x_1x_3[x_3,x_1]x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3[x_1,x_3]^{-1}x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3x_2[x_1,x_3]^{-1}[x_1,x_3,x_2]^{-1}[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3x_2[x_1,x_3]^{-1}[x_1,x_2,x_3][x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3x_2[x_1,x_3]^{-1}[x_2,x_3][x_1,x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_2x_3[x_3,x_2][x_1,x_3]^{-1}[x_2,x_3] =$$

$$x_2x_3[x_2,x_3]^{-1}[x_1,x_3]^{-1}[x_2,x_3] =$$

$$x_2x_3[x_1,x_3]^{-1}[x_2,x_3]^{-1}[x_2,x_3] =$$

$$x_2x_3[x_1,x_3]^{-1}$$

Group operation in B_n: Example

$$x_1^{-1}x_3[x_2,x_3] \quad \cdot \quad x_1x_2[x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3][x_2,x_3,x_1]x_2[x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3][x_1,x_2,x_3]x_2[x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3][x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3]^{-1} =$$

$$x_1^{-1}x_3x_1x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_1^{-1}x_1x_3[x_3,x_1]x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3[x_1,x_3]^{-1}x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3x_2[x_1,x_3]^{-1}[x_1,x_3,x_2]^{-1}[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3x_2[x_1,x_3]^{-1}[x_1,x_2,x_3][x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3x_2[x_1,x_3]^{-1}[x_2,x_3][x_1,x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_2x_3[x_3,x_2][x_1,x_3]^{-1}[x_2,x_3] =$$

$$x_2x_3[x_2,x_3]^{-1}[x_1,x_3]^{-1}[x_2,x_3] =$$

$$x_2x_3[x_1,x_3]^{-1}[x_2,x_3]^{-1}[x_2,x_3] =$$

$$x_2x_3[x_1,x_3]^{-1}$$

Group operation in B_n: Example

$$x_1^{-1}x_3[x_2,x_3] \quad \cdot \quad x_1x_2[x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3][x_2,x_3,x_1]x_2[x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3]\color{blue}{[x_1,x_2,x_3]}\color{black}x_2[x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3][x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3]^{-1} =$$
$$x_1^{-1}x_3x_1x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_1^{-1}x_1x_3[x_3,x_1]x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3[x_1,x_3]^{-1}x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3x_2[x_1,x_3]^{-1}[x_1,x_3,x_2]^{-1}[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3x_2[x_1,x_3]^{-1}[x_1,x_2,x_3][x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3x_2[x_1,x_3]^{-1}[x_2,x_3][x_1,x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_2x_3[x_3,x_2][x_1,x_3]^{-1}[x_2,x_3] =$$
$$x_2x_3[x_2,x_3]^{-1}[x_1,x_3]^{-1}[x_2,x_3] =$$
$$x_2x_3[x_1,x_3]^{-1}[x_2,x_3]^{-1}[x_2,x_3] =$$
$$x_2x_3[x_1,x_3]^{-1}$$

Group operation in B_n: Example

$$
\begin{aligned}
& x_1^{-1}x_3[x_2,x_3] \quad \cdot \quad x_1x_2[x_1,x_2,x_3] = \\
& x_1^{-1}x_3x_1[x_2,x_3][x_2,x_3,x_1]x_2[x_1,x_2,x_3] = \\
& x_1^{-1}x_3x_1[x_2,x_3][x_1,x_2,x_3]x_2[x_1,x_2,x_3] = \\
& x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3][x_1,x_2,x_3] = \\
& x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3]^{-1} = \\
& x_1^{-1}x_3x_1x_2[x_2,x_3][x_1,x_2,x_3]^{-1} = \\
& x_1^{-1}x_1x_3[x_3,x_1]x_2[x_2,x_3][x_1,x_2,x_3]^{-1} = \\
& x_3[x_1,x_3]^{-1}x_2[x_2,x_3][x_1,x_2,x_3]^{-1} = \\
& x_3x_2[x_1,x_3]^{-1}[x_1,x_3,x_2]^{-1}[x_2,x_3][x_1,x_2,x_3]^{-1} = \\
& x_3x_2[x_1,x_3]^{-1}[x_1,x_2,x_3][x_2,x_3][x_1,x_2,x_3]^{-1} = \\
& x_3x_2[x_1,x_3]^{-1}[x_2,x_3][x_1,x_2,x_3][x_1,x_2,x_3]^{-1} = \\
& x_2x_3[x_3,x_2][x_1,x_3]^{-1}[x_2,x_3] = \\
& x_2x_3[x_2,x_3]^{-1}[x_1,x_3]^{-1}[x_2,x_3] = \\
& x_2x_3[x_1,x_3]^{-1}[x_2,x_3]^{-1}[x_2,x_3] = \\
& x_2x_3[x_1,x_3]^{-1}
\end{aligned}
$$

Group operation in B_n: Example

$$x_1^{-1}x_3[x_2,x_3] \quad \cdot \quad x_1x_2[x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3][x_2,x_3,x_1]x_2[x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3][x_1,x_2,x_3]x_2[x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3][x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3]^{-1} =$$
$$x_1^{-1}x_3x_1x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_1^{-1}x_1x_3[x_3,x_1]x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3[x_1,x_3]^{-1}x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3x_2[x_1,x_3]^{-1}[x_1,x_3,x_2]^{-1}[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3x_2[x_1,x_3]^{-1}[x_1,x_2,x_3][x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3x_2[x_1,x_3]^{-1}[x_2,x_3][x_1,x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_2x_3[x_3,x_2][x_1,x_3]^{-1}[x_2,x_3] =$$
$$x_2x_3[x_2,x_3]^{-1}[x_1,x_3]^{-1}[x_2,x_3] =$$
$$x_2x_3[x_1,x_3]^{-1}[x_2,x_3]^{-1}[x_2,x_3] =$$
$$x_2x_3[x_1,x_3]^{-1}$$

Group operation in B_n: Example

$$x_1^{-1}x_3[x_2,x_3] \quad \cdot \quad x_1x_2[x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3][x_2,x_3,x_1]x_2[x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3][x_1,x_2,x_3]x_2[x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3][x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3]^{-1} =$$
$$x_1^{-1}x_3x_1x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_1^{-1}x_1x_3[x_3,x_1]x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3[x_1,x_3]^{-1}x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3x_2[x_1,x_3]^{-1}[x_1,x_3,x_2]^{-1}[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3x_2[x_1,x_3]^{-1}[x_1,x_2,x_3][x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3x_2[x_1,x_3]^{-1}[x_2,x_3][x_1,x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_2x_3[x_3,x_2][x_1,x_3]^{-1}[x_2,x_3] =$$
$$x_2x_3[x_2,x_3]^{-1}[x_1,x_3]^{-1}[x_2,x_3] =$$
$$x_2x_3[x_1,x_3]^{-1}[x_2,x_3]^{-1}[x_2,x_3] =$$
$$x_2x_3[x_1,x_3]^{-1}$$

Group operation in B_n: Example

$$
\begin{aligned}
&x_1^{-1} x_3 [x_2, x_3] \quad \cdot \quad x_1 x_2 [x_1, x_2, x_3] = \\
&x_1^{-1} x_3 x_1 [x_2, x_3][x_2, x_3, x_1] x_2 [x_1, x_2, x_3] = \\
&x_1^{-1} x_3 x_1 [x_2, x_3][x_1, x_2, x_3] x_2 [x_1, x_2, x_3] = \\
&x_1^{-1} x_3 x_1 [x_2, x_3] x_2 [x_1, x_2, x_3][x_1, x_2, x_3] = \\
&x_1^{-1} x_3 x_1 [x_2, x_3] x_2 [x_1, x_2, x_3]^{-1} = \\
&x_1^{-1} x_3 x_1 x_2 [x_2, x_3][x_1, x_2, x_3]^{-1} = \\
&x_1^{-1} x_1 x_3 [x_3, x_1] x_2 [x_2, x_3][x_1, x_2, x_3]^{-1} = \\
&x_3 [x_1, x_3]^{-1} x_2 [x_2, x_3][x_1, x_2, x_3]^{-1} = \\
&x_3 x_2 [x_1, x_3]^{-1} [x_1, x_3, x_2]^{-1} [x_2, x_3][x_1, x_2, x_3]^{-1} = \\
&x_3 x_2 [x_1, x_3]^{-1} [x_1, x_2, x_3][x_2, x_3][x_1, x_2, x_3]^{-1} = \\
&x_3 x_2 [x_1, x_3]^{-1} [x_2, x_3][x_1, x_2, x_3][x_1, x_2, x_3]^{-1} = \\
&x_2 x_3 [x_3, x_2][x_1, x_3]^{-1} [x_2, x_3] = \\
&x_2 x_3 [x_2, x_3]^{-1} [x_1, x_3]^{-1} [x_2, x_3] = \\
&x_2 x_3 [x_1, x_3]^{-1} [x_2, x_3]^{-1} [x_2, x_3] = \\
&x_2 x_3 [x_1, x_3]^{-1}
\end{aligned}
$$

Group operation in B_n: Example

$$x_1^{-1}x_3[x_2,x_3] \quad \cdot \quad x_1x_2[x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3][x_2,x_3,x_1]x_2[x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3][x_1,x_2,x_3]x_2[x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3][x_1,x_2,x_3] =$$
$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3]^{-1} =$$
$$x_1^{-1}x_3x_1x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_1^{-1}x_1x_3[x_3,x_1]x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3[x_1,x_3]^{-1}x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3x_2[x_1,x_3]^{-1}[x_1,x_3,x_2]^{-1}[x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3x_2[x_1,x_3]^{-1}[x_1,x_2,x_3][x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_3x_2[x_1,x_3]^{-1}[x_2,x_3][x_1,x_2,x_3][x_1,x_2,x_3]^{-1} =$$
$$x_2x_3[x_3,x_2][x_1,x_3]^{-1}[x_2,x_3] =$$
$$x_2x_3[x_2,x_3]^{-1}[x_1,x_3]^{-1}[x_2,x_3] =$$
$$x_2x_3[x_1,x_3]^{-1}[x_2,x_3]^{-1}[x_2,x_3] =$$
$$x_2x_3[x_1,x_3]^{-1}$$

Group operation in B_n: Example

$$x_1^{-1}x_3[x_2,x_3] \quad \cdot \quad x_1x_2[x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3][x_2,x_3,x_1]x_2[x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3][x_1,x_2,x_3]x_2[x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3][x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3]^{-1} =$$

$$x_1^{-1}x_3x_1x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_1^{-1}x_1x_3[x_3,x_1]x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3[x_1,x_3]^{-1}x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3x_2[x_1,x_3]^{-1}[x_1,x_3,x_2]^{-1}[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3x_2[x_1,x_3]^{-1}[x_1,x_2,x_3][x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3x_2[x_1,x_3]^{-1}[x_2,x_3][x_1,x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_2x_3[x_3,x_2][x_1,x_3]^{-1}[x_2,x_3] =$$

$$x_2x_3[x_2,x_3]^{-1}[x_1,x_3]^{-1}[x_2,x_3] =$$

$$x_2x_3[x_1,x_3]^{-1}[x_2,x_3]^{-1}[x_2,x_3] =$$

$$x_2x_3[x_1,x_3]^{-1}$$

Group operation in B_n: Example

$$
\begin{aligned}
&x_1^{-1} x_3 [x_2, x_3] \quad \cdot \quad x_1 x_2 [x_1, x_2, x_3] = \\
&x_1^{-1} x_3 x_1 [x_2, x_3][x_2, x_3, x_1] x_2 [x_1, x_2, x_3] = \\
&x_1^{-1} x_3 x_1 [x_2, x_3][x_1, x_2, x_3] x_2 [x_1, x_2, x_3] = \\
&x_1^{-1} x_3 x_1 [x_2, x_3] x_2 [x_1, x_2, x_3][x_1, x_2, x_3] = \\
&x_1^{-1} x_3 x_1 [x_2, x_3] x_2 [x_1, x_2, x_3]^{-1} = \\
&x_1^{-1} x_3 x_1 x_2 [x_2, x_3][x_1, x_2, x_3]^{-1} = \\
&x_1^{-1} x_1 x_3 [x_3, x_1] x_2 [x_2, x_3][x_1, x_2, x_3]^{-1} = \\
&x_3 [x_1, x_3]^{-1} x_2 [x_2, x_3][x_1, x_2, x_3]^{-1} = \\
&x_3 x_2 [x_1, x_3]^{-1} [x_1, x_3, x_2]^{-1} [x_2, x_3][x_1, x_2, x_3]^{-1} = \\
&x_3 x_2 [x_1, x_3]^{-1} {\color{blue}[x_1, x_2, x_3]} [x_2, x_3][x_1, x_2, x_3]^{-1} = \\
&x_3 x_2 [x_1, x_3]^{-1} [x_2, x_3][x_1, x_2, x_3][x_1, x_2, x_3]^{-1} = \\
&x_2 x_3 [x_3, x_2][x_1, x_3]^{-1} [x_2, x_3] = \\
&x_2 x_3 [x_2, x_3]^{-1} [x_1, x_3]^{-1} [x_2, x_3] = \\
&x_2 x_3 [x_1, x_3]^{-1} [x_2, x_3]^{-1} [x_2, x_3] = \\
&x_2 x_3 [x_1, x_3]^{-1}
\end{aligned}
$$

Group operation in B_n: Example

$$x_1^{-1}x_3[x_2,x_3] \quad \cdot \quad x_1x_2[x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3][x_2,x_3,x_1]x_2[x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3][x_1,x_2,x_3]x_2[x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3][x_1,x_2,x_3] =$$

$$x_1^{-1}x_3x_1[x_2,x_3]x_2[x_1,x_2,x_3]^{-1} =$$

$$x_1^{-1}x_3x_1x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_1^{-1}x_1x_3[x_3,x_1]x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3[x_1,x_3]^{-1}x_2[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3x_2[x_1,x_3]^{-1}[x_1,x_3,x_2]^{-1}[x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3x_2[x_1,x_3]^{-1}[x_1,x_2,x_3][x_2,x_3][x_1,x_2,x_3]^{-1} =$$

$$x_3x_2[x_1,x_3]^{-1}[x_2,x_3][x_1,x_2,x_3][x_1,x_2,x_3]^{-1} =$$

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Group operation in B_n: Example

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Burnside Groups: Recap

- Compact normal form:

$$\prod_{i=1}^{n} x_i^{\alpha_i} \prod_{i<j} [x_i, x_j]^{\beta_{i,j}} \prod_{i<j<k} [x_i, x_j, x_k]^{\gamma_{i,j,k}}$$

$\Rightarrow$ $|B_n| = 3^{n+\binom{n}{2}+\binom{n}{3}}$

- Efficient ($O(n^3)$) group operation
  - Cubic in security parameter, but linear in input size
  - Similar (somewhat simpler) process to compute inverses (omitted)
- Non-commutative, but enjoys several useful identities
  - $www = 1$ for any $w \in B_n$
  - $[x_i, x_j, x_k, x_h] = 1$ for any choice of generators

Q: What computational tasks are hard over Burnside groups?!

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Q: What computational tasks are hard over Burnside groups?!

Learning With Errors (LWE)

The LWE Setting

- $\mathbf{s} \in \mathbb{F}_q^n$
- Ψ_n: a discrete gaussian distribution over $\mathbb{F}_q$ centered at 0
- $\mathbf{A}_{\mathbf{s}}^{\Psi_n}$: distribution on $\mathbb{F}_q^n \times \mathbb{F}_q$ whose samples are pairs $(\mathbf{a}, b)$ where $\mathbf{a} \xleftarrow{\$} \mathbb{F}_q^n$, $b = \mathbf{s} \cdot \mathbf{a} + e$, $e \xleftarrow{\$} \Psi_n$

$$
\begin{array}{ccc}
\mathbb{F}_q^n & \ni & \mathbf{a} \\
\mathbf{s} \cdot \_ \downarrow & & \downarrow \approx \mathbf{s} \cdot \mathbf{a} \\
\mathbb{F}_q & \ni & b = \mathbf{s} \cdot \mathbf{a} + e, \quad e \xleftarrow{\$} \Psi_n
\end{array}
$$

LWE Assumption

$$\mathbf{A}_{\mathbf{s}}^{\Psi_n} \underset{\text{PPT}}{\approx} \mathbf{U}(\mathbb{F}_q^n \times \mathbb{F}_q)$$

LWE over Groups: Learning Homomorphisms w/ Noise

Vector Spaces | **Groups**

$$
\begin{array}{ccc}
\mathbb{F}_q^n & \ni & \mathbf{a} \\
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\end{array}
\qquad\Bigg|\qquad
\begin{array}{ccc}
G_n & \ni & a \\
\varphi \downarrow & & \downarrow \approx \varphi(a) \\
P_n & \ni & b = \varphi(a)e
\end{array}
$$

Learning With Errors

secret linear functional $\mathbf{s}\cdot\_$

Discrete gaussian noise e

Learning Homomorphisms w/ Noise

secret (G_n, P_n)-homomorphism φ

"small" P_n-noise $e \xleftarrow{\$} \Psi_n$

Learning Homomorphisms with Noise (LHN)

The LHN Setting

- Groups G_n, P_n
- Distributions Γ_n, Ψ_n, Φ_n over G_n, P_n, $\mathrm{hom}(G_n, P_n)$, resp.
- $\mathbf{A}_{\varphi}^{\Psi_n}$ (for $\varphi \in \mathrm{hom}(G_n, P_n)$): Distribution over $G_n \times P_n$ whose samples are pairs (a, b) where $a \xleftarrow{\$} \Gamma_n$, $e \xleftarrow{\$} \Psi_n$, $b = \varphi(a)e$

$$\begin{array}{ccccc} G_n & \ni & a & & \\ \varphi \big\downarrow & & \big\downarrow \approx \varphi(a) & & \\ P_n & \ni & b & = & \varphi(a)e \end{array}$$

LHN Assumption

$$\mathbf{A}_{\varphi}^{\Psi_n} \underset{\text{PPT}}{\approx} \mathbf{U}(G_n \times P_n), \qquad \varphi \xleftarrow{\$} \Phi_n$$

LWE As an Instance of LHN

- $G_n := (\mathbb{F}_p^n, +)$ and $\Gamma_n := \mathbf{U}(\mathbb{F}_p^n)$
- $P_n := (\mathbb{F}_p, +)$ and $\Psi_n :=$ discrete gaussian
- $\varphi := \mathbf{s} \cdot \_$ and $\Phi_n := \mathbf{U}(\mathrm{hom}(\mathbb{F}_p^n, \mathbb{F}_p))$

$$\begin{array}{ccc} \mathbb{F}_p^n & \ni & \mathbf{a} \\ \mathbf{s}\cdot\_ \downarrow & & \downarrow \approx \mathbf{s}\cdot\mathbf{a} \\ \mathbb{F}_p & \ni & b \\ & & \| \\ & & \mathbf{s}\cdot\mathbf{a} + e \end{array} \qquad \Bigg| \qquad \begin{array}{ccc} G_n & \ni & a \\ \varphi \downarrow & & \downarrow \approx \varphi(a) \\ P_n & \ni & b \\ & & \| \\ & & \varphi(a)e \end{array}$$

B_n-LHN: Instantiating LHN over Burnside Groups

- $G_n := B_n$, $P_n := B_r$ (r small constant, *e.g.*, $r = 4$)
- $\Gamma_n := \mathbf{U}(B_n)$
- $\Phi_n := \mathbf{U}(\hom(B_n, B_r))$
- $\Psi_n := \left[\mathbf{v} \xleftarrow{\$} \mathbf{U}(\mathbb{F}_3^r),\ \sigma \xleftarrow{\$} S_r : \prod_{i=1}^{r} x_{\sigma(i)}^{v_i}\right]$ (S_r: **r-permutations**)

  (dist. over B_r-elements of Cayley-norm $\leq r =: \mathcal{B}_r$)

$$B_n \xrightarrow{\approx \varphi \xleftarrow{\$} \hom(B_n, B_r)} B_r$$

$$a \xleftarrow{\$} \mathbf{U}(B_n) \longmapsto \varphi(a)e, \quad (e \xleftarrow{\$} \Psi_n)$$

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B_n-LHN Assumption

$$\mathbf{A}_\varphi^{\mathcal{B}_r} \underset{\text{PPT}}{\approx} \mathbf{U}(B_n \times B_r), \qquad \textbf{any} \qquad \varphi \in \mathrm{Epi}(B_n, B_r)$$

Outline

Random Self-Reducibility (RSR) of B_n-LHN

- Worst-case-to-average-case reduction for B_n-LHN: Solving **random** instances not easier than solving an **arbitrary** instance

- Why does random self-reducibility matter?
  - Hallmark of robust crypto assumptions (SIS, LWE, DLog, RSA)
  - Desirable “all-or-nothing” hardness property: Either the problem is easy for (almost) all keys, or it is intractable for (almost) all keys
  - Critical for actual cryptosystems: Generation of cryptographic keys amounts to sampling **hard instances** of underlying computational problem: by RSR ensures random instance suffices

Understanding Burnside Homomorphisms

- In B_n-LHN, secret key is a (B_n, B_r)-homomorphism φ

$\Rightarrow$ Need to study $\hom(B_n, B_r)$

- Key fact: All Burnside groups are **relatively free**
  - For any group P of exponent 3, any mapping of generators $x_1, \ldots, x_n$ into P extends uniquely to a (B_n, P)-homomorphism
  - So $|hom(B_n, P)| = |P|^n$
  - For $P = B_r$ $(r \ll n)$, $|\hom(B_n, B_r)| = 3^{\left(r+\binom{r}{2}+\binom{r}{3}\right)n}$

$\Rightarrow$ The key space in B_n-LHN is exponential in n (security parameter)

Abelianization in B_n

- Abelianization of B_n $\equiv$ Quotient by its **commutator subgroup**:

$$[B_n, B_n] \doteq \{\prod_i v_i^{-1} w_i^{-1} v_i w_i : v_i, w_i \in B_n\}$$

$$B_n/[B_n, B_n] \cong (\mathbb{F}_3^n, +)$$

- Abelianization **map** $\rho_n : B_n \to B_n/[B_n, B_n] \cong (\mathbb{F}_3^n, +)$

$$\rho_n : \prod_{i=1}^{n} x_i^{\alpha_i} \prod_{i<j} [x_i, x_j]^{\beta_{i,j}} \prod_{i<j<k} [x_i, x_j, x_k]^{\gamma_{i,j,k}} \mapsto (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

- Abelianization of a (B_n, B_r)-**homomorphism** φ

$$\begin{array}{ccc} B_n & \xrightarrow{\varphi} & B_r \\ \downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle \rho_r} \\ (\mathbb{F}_3^n, +) & \xrightarrow[\overline{\varphi}]{} & (\mathbb{F}_3^r, +) \end{array}$$

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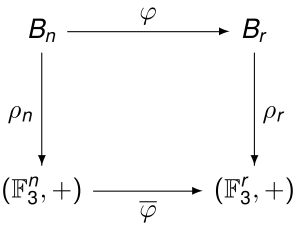
- Abelianization **map** $\rho_n : B_n \to B_n/[B_n, B_n] \cong (\mathbb{F}_3^n, +)$

$$\rho_n : \prod_{i=1}^{n} x_i^{\alpha_i} \prod_{i<j} [x_i, x_j]^{\beta_{i,j}} \prod_{i<j<k} [x_i, x_j, x_k]^{\gamma_{i,j,k}} \mapsto (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

- Abelianization of a (B_n, B_r)-**homomorphism** φ



Abelianizing B_n-LHN *vs.* LWE with $p = 3$

- **Q**: Does abelianization reduce B_n-LHN to LWE over $\mathbb{F}_3$?

- Recall: $a \xleftarrow{\$} \mathbf{U}(B_n)$, $e = \prod_{i=1}^{r} x_{\sigma(i)}^{v_i}$ $\quad (v_1, \ldots, v_r) \xleftarrow{\$} \mathbf{U}(\mathbb{F}_3^r)$, $\sigma \xleftarrow{\$} S_r$

Abelianizing B_n-LHN *vs.* LWE with $p = 3$

- **Q**: Does abelianization reduce B_n-LHN to LWE over $\mathbb{F}_3$?

$$\mathbf{A}_{\varphi}^{\mathcal{B}_r} \quad [\, i.e., (a, \varphi(a)e) \,] \quad \underset{\text{PPT}}{\approx} \qquad \mathbf{U}(B_n \times B_r)$$

- Recall: $a \xleftarrow{\$} \mathbf{U}(B_n), e = \prod_{i=1}^{r} x_{\sigma(i)}^{v_i} \qquad (v_1, \ldots, v_r) \xleftarrow{\$} \mathbf{U}(\mathbb{F}_3^r),\ \sigma \xleftarrow{\$} S_r$
- Top row represents the B_n-LHN assumption

Abelianizing B_n-LHN *vs.* LWE with $p = 3$

- **Q**: Does abelianization reduce B_n-LHN to LWE over $\mathbb{F}_3$?

$$\begin{array}{ccc} \mathbf{A}_{\varphi}^{\mathcal{B}_r} \quad [\, i.e.,(a, \varphi(a)e) \,] & \underset{\text{PPT}}{\approx} & \mathbf{U}(B_n \times B_r) \\ \Big\downarrow \rho & & \Big\downarrow \rho \\ [\, \rho(a), \overline{\varphi}(\rho(a)) + \rho(e) \,] & & \mathbf{U}(\mathbb{F}_3^n \times \mathbb{F}_3^r) \end{array}$$

- Recall: $a \xleftarrow{\$} \mathbf{U}(B_n), e = \prod_{i=1}^r x_{\sigma(i)}^{v_i} \qquad (v_1, \ldots, v_r) \xleftarrow{\$} \mathbf{U}(\mathbb{F}_3^r),\ \sigma \xleftarrow{\$} S_r$
- Top row represents the B_n-LHN assumption
- Bottom row shows the result of abelianization

Abelianizing B_n-LHN *vs.* LWE with $p = 3$

- **Q**: Does abelianization reduce B_n-LHN to LWE over $\mathbb{F}_3$?

$$
\begin{array}{ccc}
\mathbf{A}_{\varphi}^{\mathcal{B}_r} \quad [\,i.e.,(a, \varphi(a)e)\,] & \underset{\text{PPT}}{\approx} & \mathbf{U}(B_n \times B_r) \\
\Big\downarrow \rho & & \Big\downarrow \rho \\
\mathbf{A}_{\overline{\varphi}}^{\mathbf{U}(\mathbb{F}_3^r)} = \mathbf{U}(\mathbb{F}_3^n) \times \mathbf{U}(\mathbb{F}_3^r) & \equiv & \mathbf{U}(\mathbb{F}_3^n \times \mathbb{F}_3^r)
\end{array}
$$

- Recall: $a \xleftarrow{\$} \mathbf{U}(B_n), e = \prod_{i=1}^{r} x_{\sigma(i)}^{v_i}$ $\quad (v_1, \ldots, v_r) \xleftarrow{\$} \mathbf{U}(\mathbb{F}_3^r),\ \sigma \xleftarrow{\$} S_r$
- Top row represents the B_n-LHN assumption
- Bottom row shows the result of abelianization
- Bottom distributions **identical**—cannot be distinguished!

$\Rightarrow$ Abelianization does not help recognize B_n-LHN instances

RSR for B_n-LHN: Intuition

Two main steps:

1. Start with a generic partial key-randomization trick
2. Show that this randomization is complete in the case of B_n-LHN with **surjective** secret key ($\varphi \in \mathrm{Epi}(B_n, B_r)$)

Step 1: Domain Reshuffling

Lemma

Let α be a G_n-permutation, and $(a, b) \in G_n \times P_n$ be an LHN-instance sampled according to $\mathbf{A}_{\varphi}^{\Psi_n}$ ($b = \varphi(a)e$ for $e \stackrel{\$}{\leftarrow} \Psi_n$). Let $a' \doteq \alpha^{-1}(a)$. Then $(a', b) \in G_n \times P_n$ is sampled according to $\mathbf{A}_{\varphi\circ\alpha}^{\Psi_n}$

Proof.

Observe that

$$\begin{aligned}(a', b) &= (a', \varphi(a) \cdot e) \\ &= (a', \varphi \circ \alpha(\alpha^{-1}(a)) \cdot e) \\ &= (a', \varphi \circ \alpha(a') \cdot e)\end{aligned}$$

Step 1: Domain Reshuffling

Lemma

Let α be a G_n-permutation, and $(a, b) \in G_n \times P_n$ be an LHN-instance sampled according to $\mathbf{A}_{\varphi}^{\Psi_n}$ ($b = \varphi(a)e$ for $e \stackrel{\$}{\leftarrow} \Psi_n$). Let $a' \doteq \alpha^{-1}(a)$. Then $(a', b) \in G_n \times P_n$ is sampled according to $\mathbf{A}_{\varphi\circ\alpha}^{\Psi_n}$

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∎

Step 2: Completeness for Surjections

- Domain Reshuffling provides some partial randomization for an instantiation of the abstract LHN problem
  - For any $\mathbf{A}_{\varphi}^{\Psi_n}$, can transform an $\mathbf{A}_{\varphi}^{\Psi_n}$-instance into an $\mathbf{A}_{\varphi\circ\alpha}^{\Psi_n}$-instance, for any permutation α

- In the case of B_n-LHN, this simple randomization is complete for the set of **surjective** homomorphisms:

Lemma

$$(\forall\varphi, \varphi^* \in \mathrm{Epi}(B_n, B_r))(\exists\alpha \in \mathrm{Aut}(B_n))[\varphi^* = \varphi \circ \alpha]$$

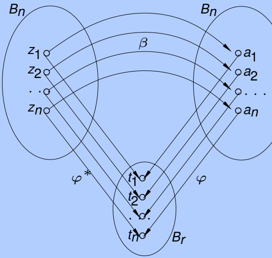
Proving Completeness

Claim

Given an arbitrary epimorphism φ and a target epimorphism φ^*, there exist an automorphism α such that $\varphi^* = \varphi \circ \alpha$

Proof Idea

- Freeness of $B_n \Rightarrow \exists\, \beta \in \hom(B_n, B_n)$ such that $\varphi^* = \varphi \circ \beta$



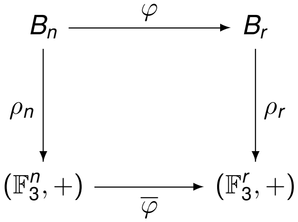
- **Technical hurdle**: β need not be an automorphism!
- **Solution**: "Patch" β into $\alpha \in \mathrm{Aut}(B_n)$

Proving Transitivity

"Patching argument" (omitted) hinges upon following technical lemma:

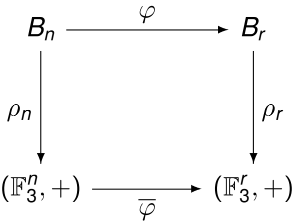
Lemma

Surjections $\varphi : B_n \to B_r$ are precisely the maps whose abelianization $\overline{\varphi}$ is also surjective



Proof ($\varphi \in \text{Epi}(B_n, B_r) \implies \overline{\varphi} \in \text{Epi}(\mathbb{F}_3^n, \mathbb{F}_3^r)$): Diagram chase

Proving Transitivity (cont'd)



Proof ($\overline{\varphi} \in \mathrm{Epi}(\mathbb{F}_3^n, \mathbb{F}_3^r) \implies \varphi \in \mathrm{Epi}(B_n, B_r)$)

- Let $\{x_1, \ldots, x_n\}$ be B_n gener's; define $y_i = \varphi(x_i)$ and $t_i = \rho_r(y_i)$
- Thesis amounts to proving $\{y_1, \ldots, y_n\}$ generates B_r
- By nilpotency of B_r (*cf.* next Lemma), suffices to show $\{t_1, \ldots, t_n\}$ generates $\mathbb{F}_3^r$
- Diagram chase shows $\rho_r \circ \varphi$ surj. $\Rightarrow \{t_1, \ldots, t_n\}$ generates $\mathbb{F}_3^r$ ∎

Proving Transitivity: Generating Sets of B_r

Lemma

Let G be a nilpotent group. If $\{y_1, \ldots, y_m\}$ generates G modulo the commutator subgroup $[G, G]$, then $\{y_1, \ldots, y_m\}$ generates G.

Since B_r has nilpotency class 3, and $B_r/[B_r, B_r] \cong \mathbb{F}_3^r$, we get:

Corollary

Let $\rho_r : B_r \to \mathbb{F}_3^r$ denote abelianization, and $y_1, \ldots, y_m \in B_r$. Then $\{y_1, \ldots, y_m\}$ generates B_r iff $\{\rho_r(y_1), \ldots, \rho_r(y_m)\}$ generates $\mathbb{F}_3^r$.

Outline

1. **Background**
   - Burnside Groups (B_n)
   - Learning Burnside Homomorphisms with Noise (B_n-LHN)
2. **Random Self-Reducibility of B_n-LHN**
3. **Cryptography (Minicrypt) via Burnside Groups**

Encryption

Fix an element $\tau \in B_r$ such that the shortest sequence of x_i and x_i^{-1} to express it is *"large"* (**Cayley norm** $\|\cdot\|_C$)

$$t \in \{0,1\} : \quad \mathrm{Enc}_\varphi(t) = (a,\ b\tau^t) \qquad a \xleftarrow{\$} B_n, e \xleftarrow{\$} \mathcal{B}_r, b = \varphi(a)e$$

Decryption

$$\mathrm{Dec}_\varphi(a, b') = \begin{cases} 0 & \text{if } \|\varphi(a)^{-1}b'\|_C \text{ "small"} \\ 1 & \text{o/w} \end{cases}$$

B_n-Based Public-Key Cryptosystem?

Challenge: Control noise in products of $\varphi(a_i)e_i$'s

B_n-Based Symmetric-Key Cryptosystem

Encryption

Fix an element $\tau \in B_r$ such that the shortest sequence of x_i and x_i^{-1} to express it is *"large"* (**Cayley norm** $\|\cdot\|_C$)

$$t \in \{0,1\}: \quad \mathrm{Enc}_\varphi(t) = (a,\ b\tau^t) \qquad a \xleftarrow{\$} B_n, e \xleftarrow{\$} \mathcal{B}_r, b = \varphi(a)e$$

Decryption

$$\mathrm{Dec}_\varphi(a, b') = \begin{cases} 0 & \text{if } \|\varphi(a)^{-1}b'\|_C \text{ "small"} \\ 1 & \text{o/w} \end{cases}$$

B_n-Based Public-Key Cryptosystem?

Challenge: Control noise in products of $\varphi(a_i)e_i$'s

Encryption

Fix an element $\tau \in B_r$ such that the shortest sequence of x_i and x_i^{-1} to express it is *"large"* (**Cayley norm** $\|\cdot\|_C$)

$$t \in \{0,1\}: \quad \mathrm{Enc}_\varphi(t) = (a,\ b\tau^t) \qquad a \xleftarrow{\$} B_n, e \xleftarrow{\$} \mathcal{B}_r, b = \varphi(a)e$$

Decryption

$$\mathrm{Dec}_\varphi(a, b') = \begin{cases} 0 & \text{if } \|\varphi(a)^{-1}b'\|_C \text{ "small"} \\ 1 & \text{o/w} \end{cases}$$

B_n-Based Public-Key Cryptosystem?

Challenge: Control noise in products of $\varphi(a_i)e_i$'s

Summary

- Algebraic generalization of the LWE problem to an abstract group-theoretic setting
- Exploration of the cryptographic viability of Burnside groups
  - Technical lemmas about homomorphisms between Burnside groups of exponent three
- Evidence to the hardness of the B_n-LHN problem of
  - Random Self-Reducibility:
    Solving random instances is as hard as solving arbitrary ones

Thank You!