Lattice Cryptography: an introduction

Daniele Micciancio

Department of Computer Science and Engineering
University of California, San Diego

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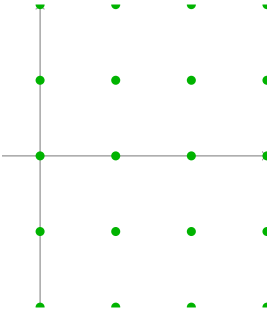
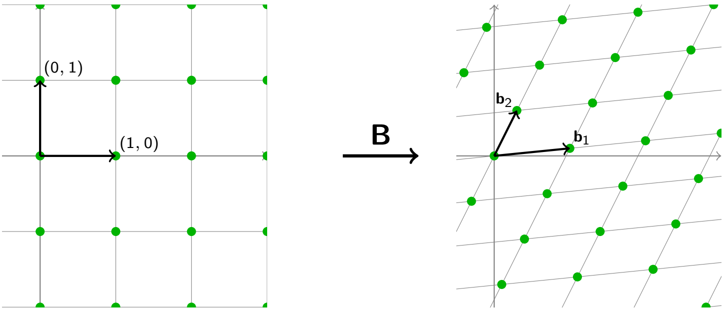
Point Lattices

- The simplest example of lattice is $\mathbb{Z}^n = \{(x_1, \ldots, x_n) \colon x_i \in \mathbb{Z}\}$



- The simplest example of lattice is $\mathbb{Z}^n = \{(x_1, \ldots, x_n) \colon x_i \in \mathbb{Z}\}$
- Other lattices are obtained by applying a linear transformation

$$\mathbf{B} \colon \mathbf{x} = (x_1, \ldots, x_n) \mapsto \mathbf{B}\mathbf{x} = x_1 \cdot \mathbf{b}_1 + \cdots + x_n \cdot \mathbf{b}_n$$

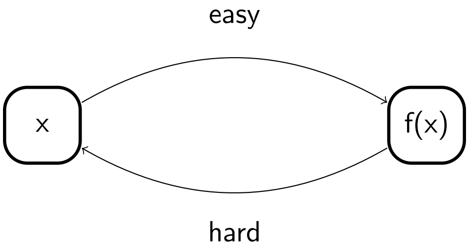


One-way Functions

Definition (One-Way Function (Informal))

An injective function $f : X \to Y$ is one-way if

- It is easy to compute, i.e., there is an efficient algorithm that on input x outputs $f(x)$
- It is hard to invert, i.e., there is no efficient algorithm that on input $f(x)$ outputs x



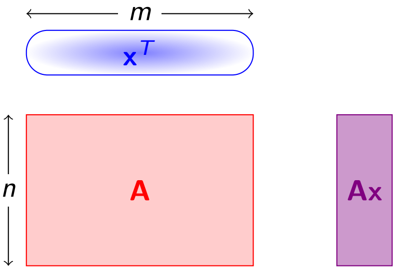
Outline

Modern Lattice Cryptography:

- The Short Integer Solusion (SIS) Function
  - Properties
  - Cryptographic Applications
- The Learning With Errors (LWE) Function
  - Properties
  - Cryptographic Applications
- Efficiency Considerations

Ajtai's one-way function (SIS)

- Parameters: $m, n, q \in \mathbb{Z}$
- Key: $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$
- Input: $\mathbf{x} \in \{0,1\}^m$



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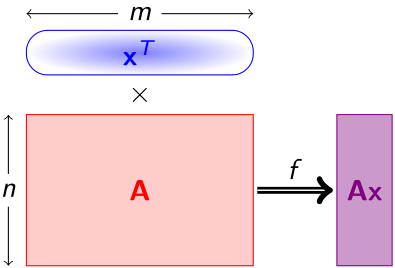

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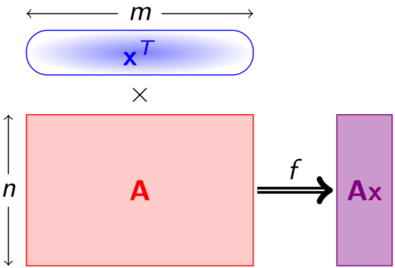

Theorem (A'96)

For $m > n \lg q$, if lattice problems (SIVP) are hard to approximate in the worst-case, then $f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x} \bmod q$ is a one-way function.

Applications: OWF [A'96], Hashing [GGH'97], Commit [KTX'08], ID schemes [L'08], Signatures [LM'08,GPV'08,. . . ,DDLL'13] . . .

SIS: Properties and Applications

- Properties:
  1. Compression
  2. Regularity
  3. Homomorphism
- Applications:
  1. Collision Resistant Hashing
  2. Commitment Schemes
  3. Digital Signatures

SIS Property 1: Compression

SIS Function

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Main security parameter: n. (Security largely independent of m.)

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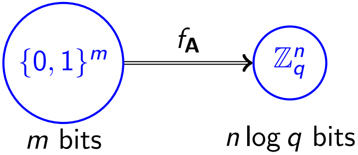

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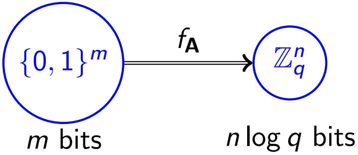

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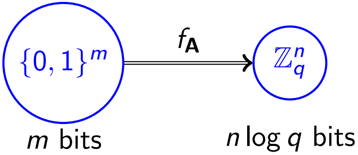

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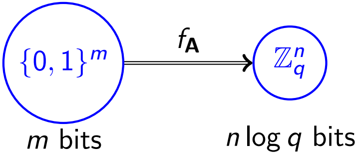

Ajtai's theorem requires $(m > n \lg q)$

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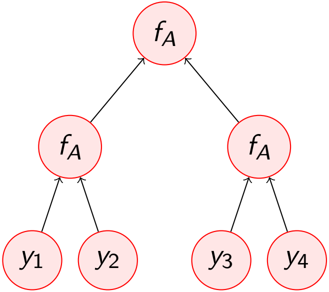
Classic application: Merkle Trees

- Leaves are user data
- Each internal node is the hash of its children
- Root r commits to all $y_1, \ldots, y_n$
- Each y_i can be shown to be consistent with r by revealing $\log(n)$ values



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If $f_{\mathbf{A}} \colon \{0, \pm 1\}^m \to \mathbb{Z}_q^n$ *is one-way, then* $f_{\mathbf{A}} \colon \{0, 1\}^m \to \mathbb{Z}_q^n$ *is collision resistant.*

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- If $x'_i = 1$ and $x_i = 0$, then $\mathbf{A}(\mathbf{x} - \mathbf{x}') = \mathbf{y}$

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Pairwise independence:

- Fix $\mathbf{x_1} \neq \mathbf{x_2} \in \{0,1\}^m$,
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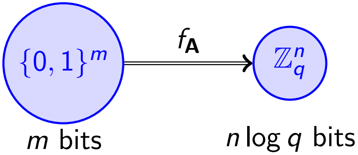

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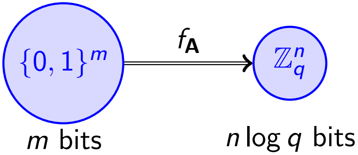

Lemma (Leftover Hash Lemma)

Pairwise Indepencence + Compression $\implies$ *Regular*

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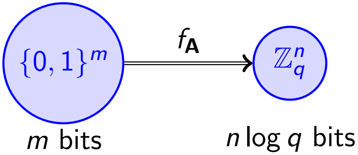

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$f_{\mathbf{A}} : (U(\{0,1\}^n)) \approx U(\mathbb{Z}_q^n)$ maps uniform to uniform.

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- Security properties:
  - Hiding: $c = C(m; \$)$ is independent of m
  - Binding: hard to find $m \neq m'$ and r, r' such that $C(m; r) = C(m'; r')$.

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- Binding Property: Finding $(m, r) \neq (m', r')$ such that $C(\mathbf{m}, \mathbf{r}) = C(\mathbf{m}', \mathbf{r}')$ breaks the collision resistance of $f_{[\mathbf{A_1}, \mathbf{A_2}]}$

SIS Property 3: (Approximate) Linear Homomorphism

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  - Domain of $f_{\mathbf{A}}$ is not closed under $+$
- $f_{\mathbf{A}}$ is also key-homomorphic:

$$f_{\mathbf{A}_1}(\mathbf{x}) + f_{\mathbf{A}_2}(\mathbf{x}) = f_{\mathbf{A}_1+\mathbf{A}_2}(\mathbf{x})$$

(One-Time) Digital Signatures

- Digital Signature Scheme:
  - Key Generation Algorithm: $(pk, sk) \leftarrow KeyGen$
  - Signing Algorithm: $Sign(sk, m) = \sigma$
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  4. ...and outputs forgery $(m', \sigma') \leftarrow Adv(\sigma)$.

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  5. Adversary wins if $Verify(m', \sigma')$ and $m \neq m'$.

(One-Time) Digital Signatures

- Digital Signature Scheme:
  - Key Generation Algorithm: $(pk, sk) \leftarrow KeyGen$
  - Signing Algorithm: $Sign(sk, m) = \sigma$
  - Verification Algorithm: $Verify(pk, m, \sigma)$
- (One-Time) Security:
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- General Signatures: Adversary is allowed an arbitrary number of signature queries

SIS Application 3: One-Time Signatures

- Extend $f_{\mathbf{A}}$ to matrices $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_l]$:

$$f_{\mathbf{A}}(\mathbf{X}) = [f_{\mathbf{A}}(\mathbf{x}_1), \ldots, f_{\mathbf{A}}(\mathbf{x}_l)] = \mathbf{A}\mathbf{X} \pmod{q}$$

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$$f_{\mathbf{A}}(\sigma) = f_{\mathbf{A}}(\mathbf{Xm} + \mathbf{x}) = f_{\mathbf{A}}(\mathbf{X})\mathbf{m} + f_{\mathbf{A}}(\mathbf{x}) = \mathbf{Ym} + \mathbf{y}$$

Learning with errors (LWE)

- $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$, $\mathbf{s} \in \mathbb{Z}_q^n$, $\mathbf{e} \in \mathcal{E}^m$.
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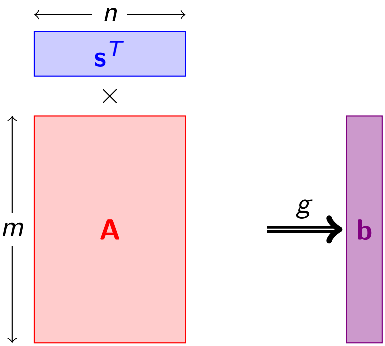

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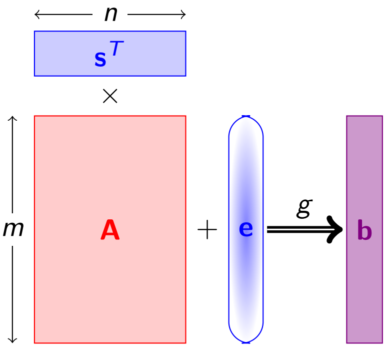

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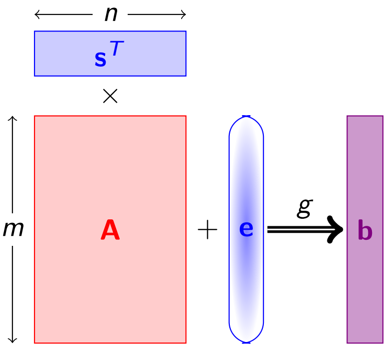
Theorem (Regev'05)

The function $g_{\mathbf{A}}(\mathbf{s},\mathbf{e})$ is hard to invert on the average, assuming SIVP is hard to approximate in the worst-case even for quantum computers.



LWE: Properties and Applications

- Properties
  1. Injectivity
  2. Pseudorandomness
  3. Homomorphism
- Applications
  1. Symmetric Key Encryption
  2. Public Key Encryption

LWE Property 1: Injectivity

LWE Function

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Main security parameter: n. (Security largely independent of m.)

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$n \lg q + m \log |\mathcal{E}|$ bits $\qquad$ $m \log q$ bits

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- $g_{\mathbf{A}}$: $n \lg q + m \lg |\mathcal{E}|$ bits $\rightarrow$ $m \lg q$ bits.
- $g_{\mathbf{A}}$ expands the input roughly by a factor $\log q / \log |\mathcal{E}|$, and is injective with high probability

LWE: Learning Formulation

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Definition (Learning With Errors (search version))

Given samples $(\mathbf{a}_i, \mathbf{a}_i\mathbf{s} + x_i)$ for fixed $\mathbf{s}$, and random $\mathbf{a}_i \in \mathbf{Z}_q^n$, $\mathbf{x}_i \leftarrow \mathcal{E}$, learn $\mathbf{s}$.

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Definition (Pseudorandom Generator (PRG))

A function $f : X \to Y$ is a pseudorandom generator if for every efficient algorithm $\mathcal{D}$, $\Pr_{x \in X}\{\mathcal{D}(f(x)) = 1\} \approx \Pr_{y \in Y}\{\mathcal{D}(y) = 1\}$.

LWE Property 2: Pseudorandomness

Theorem (Pseudorandomness of LWE)

If (search) LWE is hard, then $g_{\mathbf{A}}(\mathbf{s}, \mathbf{x})$ is pseudorandomn.

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Symmetric Encryption

- Definition
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- Security
  1. Pick secret key $sk \leftarrow KeyGen$
  2. Adversary makes encryption queries $m_1, m_2, \ldots \leftarrow \mathcal{A}$
  3. Adversary cannot distinguish $Enc(sk, m_i)$ from $Enc(sk, 0)$

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LWE Property 3: Homomorphism

- The LWE function is linearly homomorphic

$$g_{\mathbf{A}_1}(\mathbf{s}, \mathbf{x}_1) + g_{\mathbf{A}_2}(\mathbf{s}, \mathbf{x}_2) = g_{\mathbf{A}_1+\mathbf{A}_2}(\mathbf{s}, \mathbf{x}_1 + \mathbf{x}_2)$$

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- The errors x_i add up. Still, if initial x_i are small, and few ciphertexts are added, result is decryptable.

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$$\begin{aligned}\sum_i r_i \cdot p_i + m &= \sum_i r_i \cdot Enc(sk, 0) + m \\ &= Enc(sk, \sum_i r_i \cdot 0 + m) = Enc(sk, m)\end{aligned}$$

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\end{aligned}
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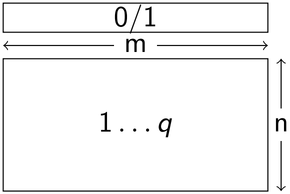
- Decryption: same as before
- if p_i has error x_i, then $E(pk, m)$ has error $\sum_i r_i x_i$

Efficiency of Ajtai's function

- $q = n^{O(1)}$, $m = O(n \log n) > n \log_2 q$
- E.g., $n = 64$, $q = 2^8$, $m = 1024$
- $f_{\mathbf{A}}$ maps 1024 bits to 512.



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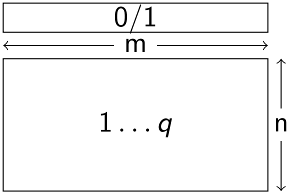

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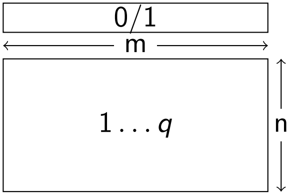
- Usable, but inefficient
  - Source of inefficiency: quadratic dependency in n



Problem

Can we do better than $O(n^2)$ complexity?

Efficient lattice based hashing

Idea

Use structured matrix

$$\mathbf{A} = [\mathbf{A}^{(1)} \mid \ldots \mid \mathbf{A}^{(m/n)}]$$

where $\mathbf{A}^{(i)} \in \mathbb{Z}_q^{n \times n}$ is circulant

$$\mathbf{A}^{(i)} = \begin{bmatrix} a_1^{(i)} & a_n^{(i)} & \cdots & a_2^{(i)} \\ a_2^{(i)} & a_1^{(i)} & \cdots & a_3^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ a_n^{(i)} & a_{n-1}^{(i)} & \cdots & a_1^{(i)} \end{bmatrix}$$

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- Proposed by [M02], where it is proved that $f_{\mathbf{A}}$ is one-way under plausible complexity assumptions
- Similar idea first used by NTRU public key cryptosystem (1998), but with no proof of security
- Wishful thinking: finding short vectors in $\Lambda_q^{\perp}(\mathbf{A})$ is hard, and therefore $f_{\mathbf{A}}$ is collision resistant

Can you find a collision?

1	4	3	8	6	4	9	0	2	6	4	5	3	2	7	1
8	1	4	3	0	6	4	9	5	2	6	4	1	3	2	7
3	8	1	4	9	0	6	4	4	5	2	6	7	1	3	2
4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3

Can you find a collision?

1	0	0	-1	-1	1	1	0	0	0	1	1	1	0	-1	0	
1	4	3	8	6	4	9	0	2	6	4	5	3	2	7	1	5
8	1	4	3	0	6	4	9	5	2	6	4	1	3	2	7	4
3	8	1	4	9	0	6	4	4	5	2	6	7	1	3	2	8
4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3	6

Can you find a collision?

$$\begin{array}{|cccc|cccc|cccc|cccc|}
? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\
\hline
1 & 4 & 3 & 8 & 6 & 4 & 9 & 0 & 2 & 6 & 4 & 5 & 3 & 2 & 7 & 1 \\
8 & 1 & 4 & 3 & 0 & 6 & 4 & 9 & 5 & 2 & 6 & 4 & 1 & 3 & 2 & 7 \\
3 & 8 & 1 & 4 & 9 & 0 & 6 & 4 & 4 & 5 & 2 & 6 & 7 & 1 & 3 & 2 \\
4 & 3 & 8 & 1 & 4 & 9 & 0 & 6 & 6 & 4 & 5 & 2 & 2 & 7 & 1 & 3 \\
\hline
\end{array}
\quad
\begin{array}{|c|}
\\
\hline
0 \\
0 \\
0 \\
0 \\
\hline
\end{array}$$

Can you find a collision?

$$
\left|\begin{array}{cccc|cccc|cccc|cccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 4 & 3 & 8 & 6 & 4 & 9 & 0 & 2 & 6 & 4 & 5 & 3 & 2 & 7 & 1 \\
8 & 1 & 4 & 3 & 0 & 6 & 4 & 9 & 5 & 2 & 6 & 4 & 1 & 3 & 2 & 7 \\
3 & 8 & 1 & 4 & 9 & 0 & 6 & 4 & 4 & 5 & 2 & 6 & 7 & 1 & 3 & 2 \\
4 & 3 & 8 & 1 & 4 & 9 & 0 & 6 & 6 & 4 & 5 & 2 & 2 & 7 & 1 & 3
\end{array}\right|
\qquad
\left|\begin{array}{c} \\ \hline \\ \\ \\ \end{array}\right|
$$

$$
\boxed{\begin{array}{c} 6 \\ 6 \\ 6 \\ 6 \end{array}}
\qquad
\boxed{\begin{array}{c} 9 \\ 9 \\ 9 \\ 9 \end{array}}
\qquad
\boxed{\begin{array}{c} 7 \\ 7 \\ 7 \\ 7 \end{array}}
\qquad
\boxed{\begin{array}{c} 3 \\ 3 \\ 3 \\ 3 \end{array}}
$$

Can you find a collision?

1	1	1	1	-1	-1	-1	-1	0	0	0	0	1	1	1	1		
1	4	3	8	6	4	9	0	2	6	4	5	3	2	7	1		0
8	1	4	3	0	6	4	9	5	2	6	4	1	3	2	7		0
3	8	1	4	9	0	6	4	4	5	2	6	7	1	3	2		0
4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3		0

$$+1 \times \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \end{bmatrix} \quad -1 \times \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix} \quad +0 \times \begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \end{bmatrix} \quad +1 \times \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

Remarks about proofs of security

- This function is essentially the compression function of hash function LASH, modeled after NTRU
- You can still "prove" security based on average case assumption: Breaking the above hash function is as hard as finding short vectors in a random lattice $\Lambda([\mathbf{A}^{(1)}|\ldots|\mathbf{A}^{(m/n)}])$
- ...but we know the function is broken: The underlying random lattice distribution is weak!
- Conclusion: Assuming that a problem is hard on average-case is a really tricky business!

Can you find a collision now?

?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?
1	-4	-3	-8	6	-4	-9	-0	2	-6	-4	-5	3	-2	-7	-1
8	1	-4	-3	0	6	-4	-9	5	2	-6	-4	1	3	-2	-7
3	8	1	-4	9	0	6	-4	4	5	2	-6	7	1	3	-2
4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3

Can you find a collision now?

?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?
1	-4	-3	-8	6	-4	-9	-0	2	-6	-4	-5	3	-2	-7	-1
8	1	-4	-3	0	6	-4	-9	5	2	-6	-4	1	3	-2	-7
3	8	1	-4	9	0	6	-4	4	5	2	-6	7	1	3	-2
4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3

Theorem (trivial)

Finding collisions on the average is at least as hard as finding short vectors in the corresponding random lattices

Can you find a collision now?

?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?
1	-4	-3	-8	6	-4	-9	-0	2	-6	-4	-5	3	-2	-7	-1
8	1	-4	-3	0	6	-4	-9	5	2	-6	-4	1	3	-2	-7
3	8	1	-4	9	0	6	-4	4	5	2	-6	7	1	3	-2
4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3

Theorem (trivial)

Finding collisions on the average is at least as hard as finding short vectors in the corresponding random lattices

Theorem (Lyubashevsky&Micciancio)

Provably collision resistant, assuming the worst case hardness of approximating SVP and SIVP over anti-cyclic lattices.

Efficiency of anti-cyclic hashing

- Key size: $(m/n) \cdot n \log q = m \cdot \log q = \tilde{O}(n)$ bits
- Anti-cyclic matrix-vector multiplication can be computed in quasi-linear time $\tilde{O}(n)$ using FFT
- The resulting hash function can also be computed in $\tilde{O}(n)$ time
- For approximate choice of parameters, this can be very practical (SWIFFT [LMPR])
- The hash function is linear: $\mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y}$
- This can be a feature rather than a weakness

Conclusion

- Simple SIS/LWE functions
- Useful homomorphic properties $\Rightarrow$ Cryptographic applications
- Cyclic/Anticycic matrices (RingSIS/RingLWE):
  - key to efficiency in practice
  - technique pervasively used by all practical instantiations of lattice cryptography
- Question: Are these functions secure?
  - We think so, and that's where lattices come into the picture
  - ... but that's another story