A Simple (Leveled) Fully Homomorphic Encryption Scheme

And Thoughts on Bootstrapping

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Our Results

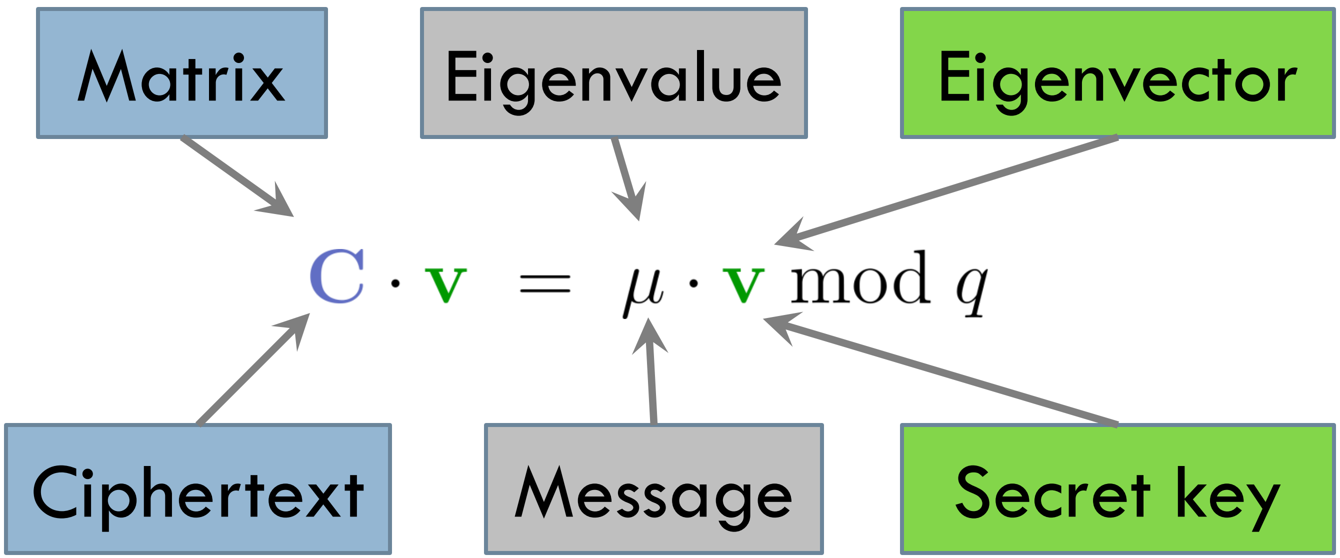
"Leveled" FHE from LWE, with nice properties:

- "Leveled" FHE: Can't go an unbounded # of levels. Can set params to enable any poly(λ) # of levels.
- Conceptual Simplicity: Ciphertexts are matrices. To add or multiply, just add or multiply matrices.
- Asymptotic Advantage: n^{ω} computation per mult
  - $\omega < 2.3727$ is the matrix multiplication constant
  - Previous schemes: "Relinearization" takes n^3 computation

Keep Good Parts of Previous Schemes

- Leveled FHE without bootstrapping [BGV12]
- Security: Based on LWE for quasi-polynomial factors (if you use bootstrapping) [BGV12]

Main Idea: Warm-Up (Toy Scheme)



- **Homomorphism:** Add or multiply ciphertexts.
  Suppose $\mathbf{C_1} \cdot \mathbf{v} = \mu_1 \cdot \mathbf{v} \bmod q$ and $\mathbf{C_2} \cdot \mathbf{v} = \mu_2 \cdot \mathbf{v} \bmod q$.
  Then $\mathbf{C_1} \cdot \mathbf{C_2} \cdot \mathbf{v} = \mu_1 \cdot \mu_2 \cdot \mathbf{v} \bmod q$

Insecurity of Toy Scheme

- Attack using encryptions of 0:

$$\mathbf{C} \cdot \mathbf{v} \;=\; \mathbf{0} \bmod q$$

  Search for $\mathbf{v}$ in the null space of $\mathbf{C}$.

- Attack using any encryptions: Find the eigenvalues and eigenvectors of $\mathbf{C}$ by solving $\mathbf{C}$'s characteristic polynomial: $\det(x \cdot \mathbf{I} - \mathbf{C}) = 0 \bmod q$.

Patching the Toy Scheme

- ~~Method 1: Use multilinear maps to encode (entries of) ciphertext matrix.~~
  - ~~Multilinear map encoding makes it hard to search the null space of $\mathbf{C}$ or compute high-degree determinants.~~
  - ~~Can get somewhat homomorphic encryption this way.~~
- Method 2: The Approximate Eigenvector Method

$$\mathbf{C} \cdot \mathbf{v} \;=\; \mu \cdot \mathbf{v} + \mathbf{e} \bmod q$$

- $\mathbf{e}$ is a noise vector with small coefficients ($\ll q$)
- $\mathbf{v}$ is an *approximate eigenvector*

Approximate Eigenvector Homomorphisms

$$\mathbf{C_1}\cdot\mathbf{v} \;=\; \mu_1\cdot\mathbf{v}+\mathbf{e_1} \bmod q, \qquad \mathbf{C_2}\cdot\mathbf{v} \;=\; \mu_2\cdot\mathbf{v}+\mathbf{e_2} \bmod q$$

- Addition: Set $\mathbf{C}^+ \leftarrow \mathbf{C_1} + \mathbf{C_2} \bmod q$.

$$\mathbf{C}^+ \cdot \mathbf{v} \;=\; (\mu_1 + \mu_2)\cdot \mathbf{v} + (\mathbf{e_1} + \mathbf{e_2}) \bmod q$$

- Multiplication: Set $\mathbf{C}^\times \leftarrow \mathbf{C_1} \times \mathbf{C_2} \bmod q$.

$$\begin{aligned}
\mathbf{C}^\times \cdot \mathbf{v} &= \mathbf{C_1}\cdot(\mu_2\cdot\mathbf{v}+\mathbf{e_2}) \\
&= \mu_2\cdot\mathbf{C_1}\cdot\mathbf{v}+\mathbf{C_1}\cdot\mathbf{e_2} \\
&= \mu_2\cdot(\mu_1\cdot\mathbf{v}+\mathbf{e_1})+\mathbf{C_1}\cdot\mathbf{e_2} \\
&= \mu_1\cdot\mu_2\cdot\mathbf{v}+(\mu_2\cdot\mathbf{e_1}+\mathbf{C_1}\cdot\mathbf{e_2})
\end{aligned}$$

New Noise

Controlling the Noise

New Noise

$$\mathbf{C}^{\times} \cdot \mathbf{v} = \mathbf{C_1} \cdot \mathbf{C_2} \cdot \mathbf{v} = \mu_1 \cdot \lambda_2 \cdot \mathbf{v} + (\mu_2 \cdot \mathbf{e_1} + \mathbf{C_1} \cdot \mathbf{e_2})$$

- **Keep messages small**:
  Easy! Restrict messages to $\{0,1\}$ and use NAND gates.
- **Keep ciphertext entries small**:
  Suppose $\mathbf{C}$ is a product of matrices.
  Can we "Flatten" $\mathbf{C}$ to make its entries small (in $\{0,1\}$)?
- **If we could flatten ciphertexts...**
  - Homomorphic Mults increase noise by factor of at most $n+1$.
  - Can evaluate depth $\Theta(\log_{n+1} q)$ before noise reaches q.
  - Set $q = 2^{n^{\Theta(1)}}$. Then we can evaluate polynomial depth, and obtain a (leveled) FHE scheme.

How to Flatten Ciphertexts

- Notation: $\mathbf{a} \in \mathbb{Z}_q^k, \quad \ell = \lfloor \log q \rfloor + 1, \quad N = k \cdot \ell$
- **Some definitions:**
  - $\mathsf{BitDecomp}(\mathbf{a}) = (a_{1,0}, \ldots, a_{1,\ell-1}, \ldots, a_{k,0}, \ldots, a_{k,\ell-1}) \in \{0,1\}^N$.
    Each coefficient of $\mathbf{a}$ decomposed into bits, least to most significant.
  - $\mathsf{BitDecomp}^{-1}(\mathbf{b} \in \mathbb{Z}_q^N) = (\sum_j 2^j b_{1,j} \bmod q, \ldots, \sum_j 2^j b_{k,j} \bmod q)$.
    $\mathsf{BitDecomp}^{-1}$ is defined even on inputs not in image of $\mathsf{BitDecomp}$.
  - $\mathsf{Flatten}(\mathbf{b} \in \mathbb{Z}_q^N) = \mathsf{BitDecomp}(\mathsf{BitDecomp}^{-1}(\mathbf{b}))$.
    This is a vector with coefficients in $\{0,1\}$.
  - $\mathsf{Powersof2}(\mathbf{s}) = (s_1, 2s_1, \ldots, 2^{\ell-1}s_1, \ldots, s_k, 2s_k, \ldots, 2^{\ell-1}s_k) \bmod q$
- **Some obvious facts:**
  - For any $\mathbf{a}, \mathbf{s} \in \mathbb{Z}_q^k$, it holds that $\langle \mathbf{a}, \mathbf{s} \rangle = \langle \mathsf{BitDecomp}(\mathbf{a}), \mathsf{Powersof2}(\mathbf{s}) \rangle$.
  - For any $\mathbf{b} \in \mathbb{Z}_q^N$ and $\mathbf{s} \in \mathbb{Z}_q^k$,
    $\langle \mathbf{b}, \mathsf{Powersof2}(\mathbf{s}) \rangle = \langle \mathsf{BitDecomp}^{-1}(\mathbf{b}), \mathbf{s} \rangle = \langle \mathsf{Flatten}(\mathbf{b}), \mathsf{Powersof2}(\mathbf{s}) \rangle$

How to Flatten Ciphertexts II

For $\mathbf{b} \in \mathbb{Z}_q^N, \mathbf{s} \in \mathbb{Z}_q^k$, $\langle \mathbf{b}, \mathsf{Powersof2}(\mathbf{s}) \rangle = \langle \mathsf{Flatten}(\mathbf{b}), \mathsf{Powersof2}(\mathbf{s}) \rangle$

- **Give the approximate eigenvector a special form**:
  $\mathbf{v} = \mathsf{Powersof2}(\mathbf{s})$ for some $\mathbf{s}$.
- **Flattening a ciphertext**:
  - Suppose $\mathbf{C}^{\text{NAND}} = \mathrm{I}_N - \mathbf{C_1} \cdot \mathbf{C_2} \bmod q$, for a NAND gate.
  - Set $\mathbf{C_3} \leftarrow \mathsf{Flatten}(\mathbf{C}^{\text{NAND}})$, flattening each row of $\mathbf{C}^{\text{NAND}}$. Each coefficient of $\mathbf{C_3}$ is in $\{0, 1\}$.
  - Then, $\mathbf{C_3} \cdot \mathbf{v} = \mathbf{C}^{\text{NAND}} \cdot \mathbf{v} \bmod q$. We have not changed what is decrypted, or even increased the noise.
- **We have (leveled) FHE!**

KeyGen, Encrypt, and Decrypt

- **Setup**$(1^n, 1^L)$: Set basic parameters q, $\ell = \lfloor \log q \rfloor + 1$, $N = (n+1) \cdot \ell$, $m = O(n \log q)$.
- **KeyGen**(1^n): Generate secret key $\mathbf{s} = (1, \mathbf{t}) \in \mathbb{Z}_q^{n+1}$.
  Secret key: $\mathbf{v} \leftarrow \mathsf{Powersof2}(\mathbf{s}) \in \mathbb{Z}_q^N$.
  Public key: $\mathbf{A} \in \mathbb{Z}_q^{m \times (n+1)}$ is uniform except $\mathbf{e} \leftarrow \mathbf{A} \cdot \mathbf{s}$ "small".
- **Encrypt**$(\mathbf{A}, \mu \in \{0,1\})$: For random $\mathbf{R} \in \{0,1\}^{N \times m}$, output:

$$\mathbf{C} \leftarrow \mathsf{Flatten}(\mu \cdot \mathbf{I}_N + \mathsf{BitDecomp}(\mathbf{R} \cdot \mathbf{A}))$$

- **Decrypt**$(\mathbf{C}, \mathbf{v})$: Compute:

$$\mathbf{C} \cdot \mathbf{v} = \mu \cdot \mathbf{v} + \mathsf{BitDecomp}(\mathbf{R} \cdot \mathbf{A}) \cdot \mathbf{v} = \mu \cdot \mathbf{v} + \mathbf{R} \cdot \mathbf{A} \cdot \mathbf{s} = \mu \cdot \mathbf{v} + \mathit{small}$$

  Recover μ from $2^{\ell-1} \cdot \mu + \mathit{small}$.

Reduction to LWE

- **Search:** Find $\mathbf{t} \in \mathbb{Z}_q^n$ given noisy inner products

$$\mathbf{a_1} \in \mathbb{Z}_q^n, \qquad b_1 = \langle \mathbf{a_1}, \mathbf{t} \rangle + e_1 \bmod q$$
$$\mathbf{a_2} \in \mathbb{Z}_q^n, \qquad b_2 = \langle \mathbf{a_2}, \mathbf{t} \rangle + e_2 \bmod q$$
$$\vdots$$

$\mathbf{a_i}$'s uniform, e_i's are "small" errors (much smaller than q)

- **Decision:** Distinguish $(\mathbf{a_i}, b_i)$ from uniform $(\mathbf{a_i}, b_i)$

Reduction to LWE

- **LWE**: Distinguish whether:
  1. $\mathbf{A} \in \mathbb{Z}_q^{m\times(n+1)}$ is uniform, or
  2. $\mathbf{A} \in \mathbb{Z}_q^{m\times(n+1)}$ is uniform conditioned on $\mathbf{A}\cdot\mathbf{s} \bmod q$ being "small" for some vector $\mathbf{s} = (1, -\mathbf{t}) \in \mathbb{Z}_q^{n+1}$.
- **Public key**: $\mathbf{A} \in \mathbb{Z}_q^{m\times(n+1)}$ is uniform except $\mathbf{e} \leftarrow \mathbf{A}\cdot\mathbf{s}$ "small". Indistinguishable from uniform under LWE.
- **Ciphertexts**: Recall $\mathbf{C} \leftarrow \mathsf{Flatten}(\mu \cdot \mathbf{I}_N + \mathsf{BitDecomp}(\mathbf{R}\cdot\mathbf{A}))$
  - $\mathbf{R}\cdot\mathbf{A}$ looks uniform by LWE and the leftover hash lemma.
  - Rows of $\mathbf{R}\cdot\mathbf{A}$ are "encryptions of 0" in Regev's scheme.
  - $\mathsf{BitDecomp}^{-1}(\mathbf{C})$ hides μ. Therefore so does $\mathbf{C}$.

Review of the Scheme

- **Approximate Eigenvector Method**:
  - $\mathbf{C} \cdot \mathbf{v} = \mu \cdot \mathbf{v} + \mathbf{e} \bmod q$
  - Secret key is $\mathbf{v} = \mathsf{Powersof2}(\mathbf{s})$, an approximate eigenvector
  - μ is the message that is encrypted
  - Homomorphic ops: Add or multiply ciphertexts, then $\mathsf{Flatten}$.
  - Bottom line: We get leveled FHE based on LWE with asymptotically better performance.

Noisiness of Ciphertexts

- Ciphertext noise grows exponentially with depth.
- Hence log q and dimension of ciphertext matrices grow linearly with depth.

Ciphertext Size Reduction

- Modulus reduction [BV11b, BGV12]:
  - Suppose c encrypts m – that is, $m = [[\langle c,v\rangle]_q]_2$.
  - Let's pick $p<q$ and set $c^* = (p/q)\cdot c$, rounded.
  - Maybe it is true that:
    - c^* encrypts m: $m = [[\langle c^*,v\rangle]_p]_2$ (new inner modulus).
    - $|[\langle c,v\rangle]_p| \approx (p/q) \cdot |[\langle c,v\rangle]_q|$ (noise is smaller).
  - This really shouldn't work… but it does…
- Also, dimension reduction: won't go over this.

Modulus Reduction Magic Trick

- Scaling lemma: Let $p < q$ be odd moduli.
  - Given c with $m = [[\langle c,s\rangle]_q]_2$. Set $c' = (p/q)c$. Set c'' to be
    - the integer vector closest to c', such that $c'' = c \bmod 2$.
  - If $|[\langle c,s\rangle]_q| < q/2 - (q/p)\cdot \ell_1(s)$, then:
    - c'' is a valid encryption of m with possibly much less noise!
    - $m = [[\langle c'',s\rangle]_p]_2$, and $|[\langle c'',s\rangle]_p| < (p/q) \cdot |[\langle c,s\rangle]_q| + \ell_1(s)$

Annotated Proof

1. For some k, $[\langle c,s\rangle]_q = \langle c,s\rangle - kq$.
2. $(p/q)|[\langle c,s\rangle]_q| = \langle c',s\rangle - kp$.
3. $|\langle c''-c',s\rangle| < \ell_1(s)$.

1. Imagine $\langle c,s\rangle$ is close to kq.
2. Then $\langle c',s\rangle$ is close to kp.
3. $\langle c'',s\rangle$ also close to kp if s is small.

4. Thus, $|\langle c'',s\rangle - kp| < (p/q)\ |[\langle c,s\rangle]_q| + \ell_1(s) < p/2$.
5. So, $[\langle c'',s\rangle]_p = \langle c'',s\rangle - kp$.
6. Since $c' = c$ and $p = q \bmod 2$, we have $[[\langle c'',s\rangle]_p]_2, = [[\langle c,s\rangle]_q]_2$.

Modulus Reduction: Shortcomings

- Reduces size of modulus (q to p) and size of ciphertext
- *Does not* reduce *ratio* of modulus to noise.

Thoughts on Bootstrapping

Bootstrapping: What Is It?

- So far, we can evaluate bounded depth funcs F:

x_1
x_2
…
x_t

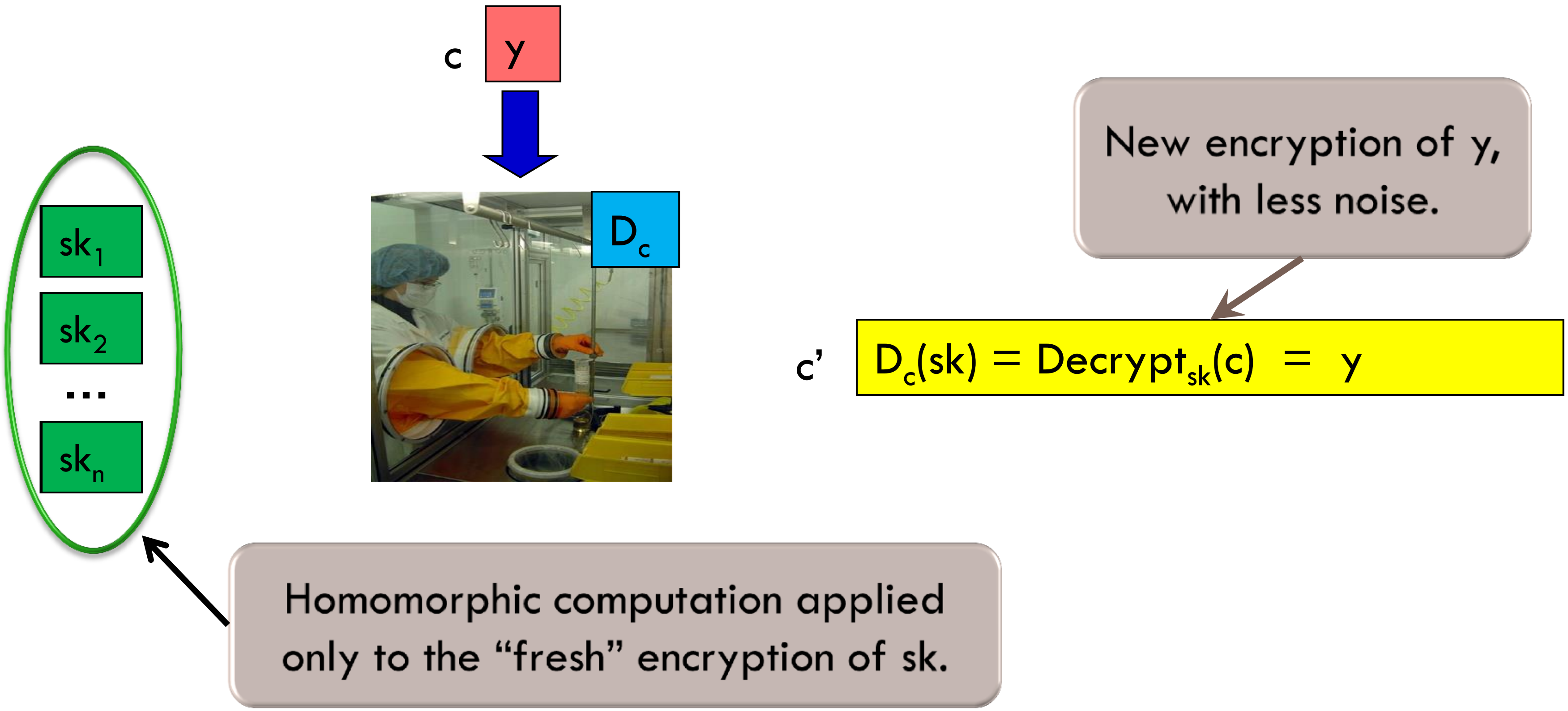
F

c $F(x_1, x_2, \ldots, x_t)$

- We have a noisy evaluated ciphertext c.
- We want to get a less noisy c' that encrypts the same value, but with less noise.
  - Modulus reduction is not enough…
- Bootstrapping *refreshes* ciphertexts, using the *encrypted secret key*.

Bootstrapping: What Is It?

- For ciphertext c, consider $D_c(sk) = \mathrm{Decrypt}_{sk}(c)$
  - Suppose $D_c(\cdot)$ is a low-depth polynomial in sk.
- Include in the public key also $\mathrm{Enc}_{pk}(sk)$.



Bootstrapping: A Mixed Blessing

- Good news: Gives us unbounded depth

- Bad news: Computationally very expensive!
  - Involves running Decrypt circuit *homomorphically*.
  - Decrypt is rather expensive already. Why?
    - Decryption formula must have high (polynomial) degree (log depth).
  - Decrypting with the overhead of homomorphic encryption is too much.

Gentry-Halevi Implementation (Eurocrypt '11): The Somewhat Homomorphic Scheme

Dimension	KeyGen	Enc (amortized)	Dec
512 200,000-bit integers	0.16 sec	4 millisec	4 millisec
2048 800,000-bit integers	1.25 sec	60 millisec	23 millisec
8192 3,200,000-bit integers	10 sec	0.7 sec	0.12 sec
32728 13,000,000-bit integers	95 sec	5.3 sec	0.6 sec

Gentry-Halevi Implementation (Eurocrypt '11): The FHE Scheme

Dimension	KeyGen	PK size	Re-Crypt
512 200,000-bit integers	2.4 sec	17 MByte	6 sec
2048 800,000-bit integers	40 sec	70 MByte	31 sec
8192 3,200,000-bit integers	8 min	285 MByte	3 min
32728 13,000,000-bit integers	2 hours	2.3 GByte	30 min

We Want a New Approach for FHE

- Do we really need "noisy" ciphertexts?
- Can we "refresh" ciphertexts (reduce their noise) without "bootstrapping", or a radically streamlined version of it?
- Can we at least allow q to be only polynomial in the security parameter (rather than quasi-polynomial)?

"Polly Cracker": An Attempt at No-Noise FHE [Fellows-Koblitz '93]

Main Idea

Encryptions of 0 evaluate to 0 at the secret key.

- KeyGen: Secret = some point $\mathbf{s} = (s_1, \ldots, s_n) \in Z_q^n$.
  Public key: Polynomials $\{a_i(x_1,\ldots,x_n)\}$ s.t. $a_i(\mathbf{s})=0 \bmod q$.
- Encrypt: From $\{a_i\}$, generate a *random* polynomial $b(\mathbf{x})$ such that $b(\mathbf{s}) = 0 \bmod q$. For m in $\{0,1\}$, ciphertext is:

$$c(\mathbf{x}) = m + b(\mathbf{x}) \bmod q.$$

- Decrypt: Evaluate ciphertext at secret: $c(\mathbf{s})=m \bmod q$.
- ADD and MULT: Output sum or product of ciphertexts.

Polly Cracker Cryptanalysis

- An Attack if # of monomials in ciphertexts is small:
  - Collect lots of encryptions $\{c_i\}$ of 0.
  - If the challenge ciphertext also encrypts 0, it will likely be in linear span of the given encryptions of 0.
    - Use Gaussian elimination (linear algebra).
- Avoiding the attack:
  - Can # of monomials in ciphertext be exponential?
  - But ciphertext can be efficiently represented?
  - Without introducing other attacks?

Noisy Polly Cracker: A Framework for Most Somewhat Homomorphic Schemes

Main Idea

Encryptions of 0 evaluate to something small and even (smeven) at the secret key.

- KeyGen: Secret = some point $\mathbf{s} = (s_1, \ldots, s_n) \in Z_q^{\,n}$. gcd(q,2)=1. Public key: Polynomials $\{a_i(x_1,\ldots,x_n)\}$ s.t. $a_i(\mathbf{s})=2e_i \bmod q$, $|e_i| \ll q$.
- Encrypt: From $\{a_i\}$, generate a *random* polynomial $b(\mathbf{x})$ such that $b(\mathbf{s})$ = smeven mod q. For m in {0,1}, ciphertext is:

$$c(\mathbf{x}) = m + b(\mathbf{x}) \bmod q.$$

- Decrypt: Evaluate ciphertext at secret: $c(\mathbf{s})$=m+smeven mod q. Then, reduce mod 2 to get m.
- ADD and MULT: Output sum or product of ciphertexts.

Noisy Polly Cracker: A Framework for Most Somewhat Homomorphic Schemes

<u>Main Idea</u>

Encryptions of 0 evaluate to something small and even (smeven) at the secret key.

- KeyGen: Secret = some point $\mathbf{s} = (s_1, \ldots, s_n) \in Z_q^n$. gcd(q,2)=1. Public key: Polynomials $\{a_i(x_1,\ldots,x_n)\}$ s.t. $a_i(\mathbf{s}$

  $\{a_i\}$, generate a *random* polyn

  mod q. For m in {0,1}, ciphertext is:

$$c(\mathbf{x}) = m + b(\mathbf{x}) \bmod q.$$

- Decrypt: <u>Evaluate ciphertext at secret</u>: $c(\mathbf{s})$=m+smeven mod q. Then, reduce mod 2 to get m.
- ADD and MULT: Output sum or product of ciphertexts.

ADDs and MULTs make the "noise" grow.

We call [c($\mathbf{s}$) mod q] the "noise" of the ciphertext.

Confining Noise to Tight Orbits

- Ciphertexts have “noise”
- But want that noise doesn’t grow with # of operations
- Noise remains always in one of two distinct orbits O_0 and O_1, depending on which bit is encrypted.
- Noise maintains high entropy, without growing larger.
  - Can we find make the following maps efficiently computable, even when the orbits have high entropy, and when distinguishing elements of the two orbits is hard?

$$f_{ADD} : O_{m1} \times O_{m2} \rightarrow O_{m1+m2}$$

$$f_{MULT} : O_{m1} \times O_{m2} \rightarrow O_{m1 \times m2}$$

Confining Noise to Tight Orbits

- An Obstacle?
  - (Cohen, Shpilka, Tal): Other than linear polynomials, the min degree of a polynomial $f : [1,n] \to [1,n]$ is $n-o(n)$.
  - Suggests perhaps f_{ADD} and f_{MULT} must have very high degree – not a "simple" transformation.
- But is this really an obstacle?
  - Bootstrapping uses a polynomial of very high degree for free:
    - It decomposes a ciphertext into bits (mod 2) – this is a high-degree transformation viewed modulo $p \neq 2$.
    - Modulus reduction is also a "free" high-degree transformation.

Thank You! Questions?