Solving All Lattice Problems in Deterministic Single Exponential Time

Daniele Micciancio
(Joint work with P. Voulgaris, STOC 2010)

UCSD

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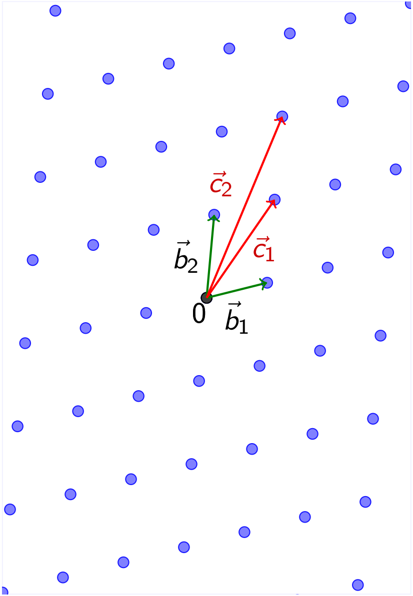
Lattices

- Traditional area of mathematics
  - Bridge between number theory and geometry
  - Studied by Lagrange, Gauss, ..., Minkowski, ...
- Key to many algorithmic applications
  - Cryptanalysis, Coding Theory, Integer Programming
- Foundation of Lattice based Cryptography
  - Exponentially hard to break, even by quantum adversary
  - Asymptotically fast and easily parallelizable cryptographic functions
  - Secure based on conjectured hardness of *worst-case* problems
  - Extremely versatile: CPA/CCA encryption, digital signature, ... group and ring signatures, threshold cryptography, IBE, ..., HIBE, ..., FHE, ...

Outline

Point Lattices



A lattice is the set of all integer linear combinations of (linearly independent) basis vectors $\mathbf{B} = \{\vec{b}_1, \ldots, \vec{b}_n\} \subset \mathbb{R}^n$:

$$\Lambda = \sum_{i=1}^{n} \vec{b}_i \cdot \mathbb{Z} = \{\mathbf{B}\vec{x} \colon \vec{x} \in \mathbb{Z}^n\}$$

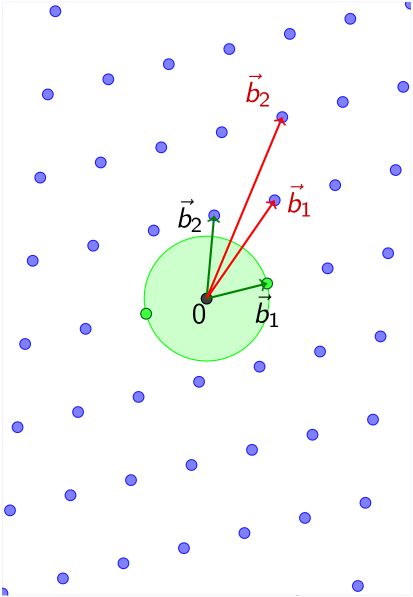
The same lattice has many bases

$$\Lambda = \sum_{i=1}^{n} \vec{c}_i \cdot \mathbb{Z}$$

Definition (Lattice)

Discrete additive subgroup of $\mathbb{R}^n$

Shortest Vector Problem (SVP)

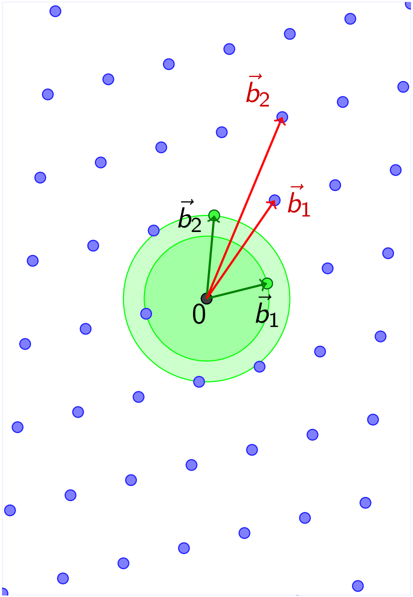


Definition (SVP)

Given a lattice $\mathcal{L}(\mathbf{B})$, find a (nonzero) lattice vector $\mathbf{B}\vec{x}$ (with $\vec{x} \in \mathbb{Z}^k$) of minimal length $\|\mathbf{B}\vec{x}\|$

- Input: A lattice basis $\mathbf{B}$
- Output: A shortest nonzero vector $\vec{s} \in \Lambda$
- The problem is hard when dimension n is high and basis is skewed
- Shortest vector can be much shorter than basis vectors

Shortest Independent Vectors Problem (SIVP)

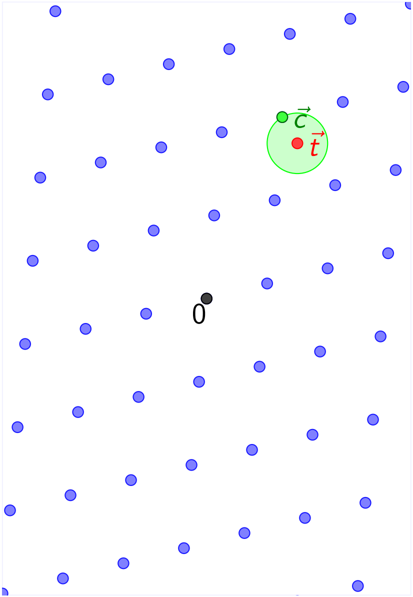


Definition (SIVP)

Given a lattice $\mathcal{L}(\mathbf{B})$, find n linearly independent lattice vectors $\vec{s}_1, \ldots, \vec{s}_n$ of minimal length $\max_i \|\vec{s}_i\|$

- Input: A lattice basis $\mathbf{B}$
- Output: n shortest linearly independent lattice vectors $\vec{s}_1, \ldots, \vec{s}_n \in \Lambda$
- The problem is hard when dimension n is high and basis is skewed

Closest Vector Point (CVP)



Inhomogeneous version of SVP

Definition (CVP)

Given a lattice $\mathcal{L}(\mathbf{B})$ and a target point $\vec{t}$, find a lattice vector $\mathbf{B}\vec{x}$ which minimizes the distance $\|\mathbf{B}\vec{x} - \vec{t}\|$

- Input: A lattice $\Lambda(\mathbf{B})$, and a target vector $\vec{t}$
- Output: A closest lattice point $\vec{c} \in \Lambda$
- NP-hard [vEB'81], even for fixed lattice [M'01]

Lattice problems, Cryptography, Algorithms

Approximating SVP, SIVP, CVP

- Best known polynomial time algorithm only find poor ($2^{\omega(n/\log n)}$) approximations
- Lattice based cryptography is based on the conjectured hardness of finding good ($n^{O(1)}$) approximate solutions

Solving SVP, SIVP, CVP exactly

- NP-hard: no subexponential time solution is expected
- Best known exact algorithms run in exponential time $2^{\Omega(n)}$

Applications of exact SVP, SIVP, CVP

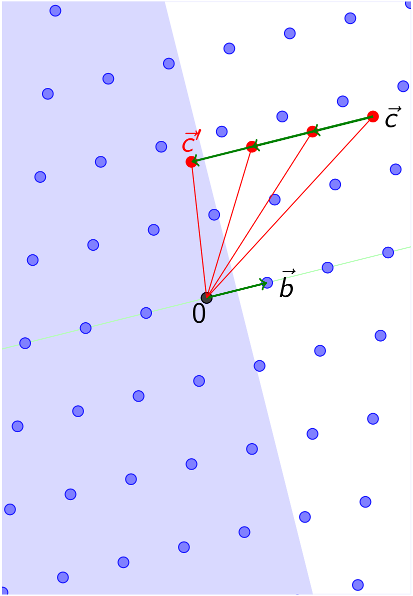
- Some applications involve low dimensional lattices
- Efficient approximation algorithms are based on exact solution of small dimensional subproblems

How fast we we solve SVP, SIVP, CVP? (E.g., $2^{n/2} < 2^{100 \cdot n} < n^n$)

Complexity of SVP, SIVP, CVP

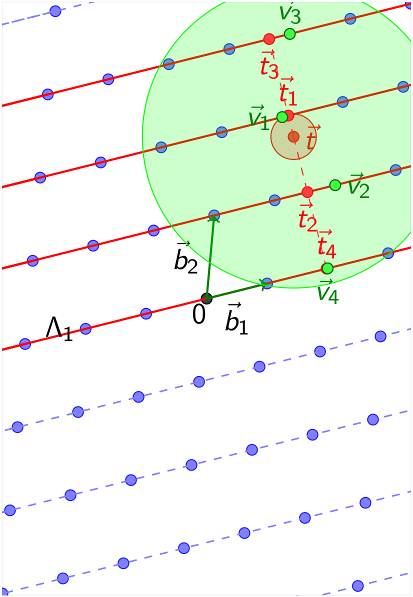
- Efficient (dimension preserving) reductions
  - SVP, SIVP $\leq$ CVP [GMSS'99, M'08]
- Fastest previous algorithm
  - SVP,SIVP,CVP ,IP: [Kannan'87] runs in $n^{O(n)}$ time
  - SVP: [AKS'01] runs in randomized $2^{O(n)}$ time and space
  - Algorithms work in any ℓ_p norm [BN'07]
- Questions
  - Can CVP, SIVP also be solved in $2^{c \cdot n}$ time? Yes! (for ℓ_2)
  - What is the smallest constant c? [NV'09,MP'10,PS'10]: $c < 2.5$ for SVP in ℓ_2. $c \leq 2$ for SVP,SIVP,CVP!
  - Is randomization and exponential space useful/necessary? Randomization is not!
  - What about other norms and Integer Programming (IP)?

Size Reduction



- $\vec{b}$: (short) lattice vector
- $\vec{c}$: arbitrary point
- Can make $\vec{c}$ shorter by subtracting $\vec{b}$ from it
- Repeat until $\vec{c}$ closer to $\vec{0}$ than to $\vec{b}$ or $-\vec{b}$
- Remarks
  - $\vec{c} - \vec{c}' \in \Lambda$
  - Key step in [LLL'82] basis reduction algorithm
  - Technique is used in most other lattice algorithms

Rank reduction: $CVP(\Lambda_n) \leq 2^n \cdot CVP(\Lambda_{n-1})$



- Goal: Solve $CVP(\Lambda_n, \vec{t})$
- Partition Λ_n into layers of the form: $\Lambda_{n-1} + c\vec{b}_n$, $c = 2, 1, 3, 0, \ldots$
- Find lattice point $\vec{v}_i$ in each layer closest to (the projection of) $\vec{t}$
- Only need to consider nearby layers
  - Dual LLL: 2^n layers
  - Dual SVP: n layers
- Select the best solution $\vec{v}_1$
- Notice: All layers contain same lattice Λ_{n-1}

Solving CVP by rank reduction

- Rank reduction $CVP(\Lambda_n) \leq k \cdot CVP(\Lambda_{n-1})$
  - LLL: $k = 2^n$, $T = 2^{n^2}$
  - SVP: $k = n$, $T = n^n$
- Iterate: $CVP(\Lambda_n) \leq k \cdot CVP(\Lambda_{n-1}) \leq \cdots \leq k^n CVP(\Lambda_1) = k^n$
- Our approach
  - Exploit the fact that recursive calls use the same lower dimensional sublattices
  - Preprocess the lattice to speed up the solution of many CVP instances

CVP with Preprocessing (CVPP)

Problem (CVPP)

Find a function π and an efficient algorithm CVPP such that $CVPP(\pi(\Lambda), \vec{t}) = CVP(\Lambda, \vec{t})$

- Only the running time of CVPP counts. The function π is arbitrary.
- Complexity
  - Still NP-hard [M'01]!
  - [LLS'93,AR'04] approximates within $n^{O(1)}$ in polynomial time
  - Polynomial time solutions require $|\pi(\Lambda)| \leq n^{O(1)}$
- Our work:
  - $CVPP(\pi(\Lambda), \vec{t})$ runs in $2^{O(n)}$ time
  - $\pi(\Lambda)$ has size $2^{O(n)}$
  - $\pi(\Lambda)$ can also be computed in time $2^{O(n)}$

Overview of CVP algorithm

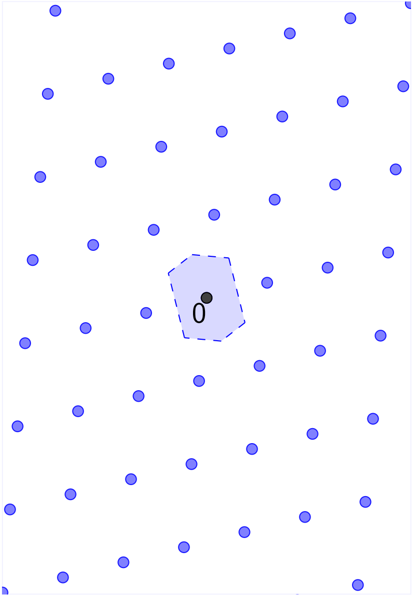
Building blocks:

- $\pi(\Lambda) = \mathcal{V}(\Lambda)$: Voronoi cell of the lattice
- Our approach: $CVP(\Lambda_n) \le CVPP(\mathcal{V}(\Lambda_n)) + \mathcal{V}(\Lambda_n)$
- $CVPP(\mathcal{V}(\Lambda_n))$ algorithm with running time $2^{O(n)}$
- Voronoi cell computation $\mathcal{V}(\Lambda_n) \le 2^{O(n)} CVP(\Lambda_n)$
- Dimension reduction $CVP(\Lambda_n) \le 2^{O(n)} \cdot CVP(\Lambda_{n-1})$

Computing the Voronoi cell of a lattice:

$$\begin{aligned}
\mathcal{V}(\Lambda_n) &\le 2^{O(n)} CVP(\Lambda_n) \\
&\le 2^{O(n)} \cdot 2^{O(n)} \cdot CVP(\Lambda_{n-1}) \\
&\le 2^{O(n)} \cdot 2^{O(n)} \cdot CVPP(\mathcal{V}(\Lambda_{n-1})) + \mathcal{V}(\Lambda_{n-1}) \\
&\le 2^{O(n)} 2^{O(n)} 2^{O(n)} + \mathcal{V}(\Lambda_{n-1}) \\
&= 2^{O(n)} + \mathcal{V}(\Lambda_{n-1}) \\
&\le 2^{O(n)} + 2^{O(n)} + \mathcal{V}(\Lambda_{n-2}) \le \ldots \le 2^{O(n)}
\end{aligned}$$

Voronoi Cell

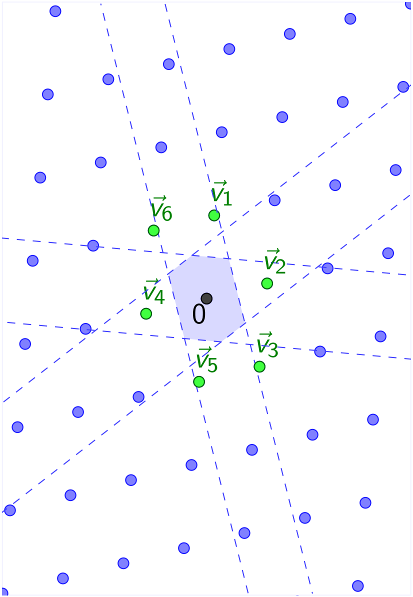


Definition (Voronoit Cell)

Set of points in $\mathbb{R}^n$ closer to 0 than to any other lattice point

$$\mathcal{V}(\Lambda) = \{\vec{x} \colon \forall \vec{v} \in \Lambda, \|\vec{x}\| \leq \|\vec{x} - \vec{v}\|\}$$

Representing the Voronoi cell



- Each $\vec{v} \in \Lambda$ defines

$$\mathcal{H}_{\vec{v}} = \{\vec{x} \colon \|\vec{x}\| \leq \|\vec{x} - \vec{v}\|\}$$

- $\mathcal{V}$ is the intersection

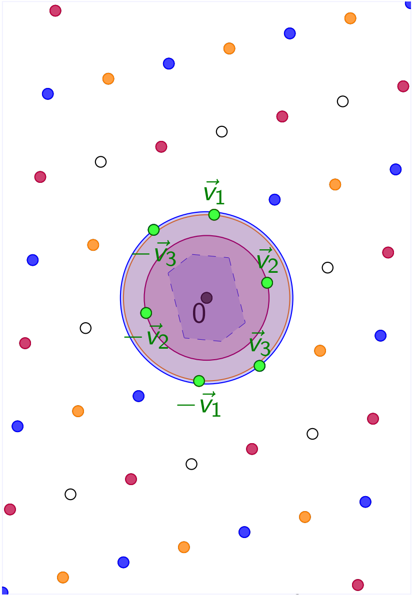
$$\mathcal{V} = \bigcap_{\vec{v} \in \Lambda R} \mathcal{H}_{\vec{v}}, R \subset \Lambda$$

- Not all $\vec{v} \in \Lambda$ are needed

Theorem (Voronoi)

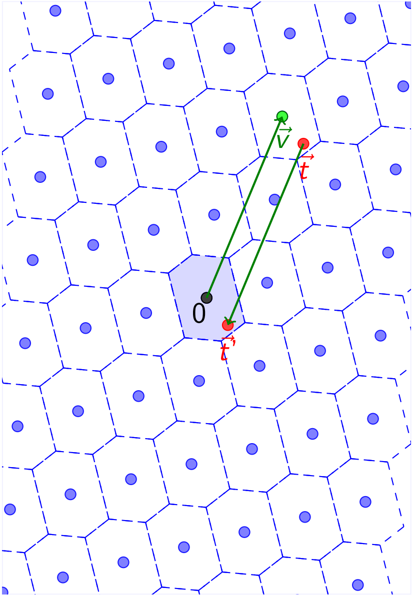
The numer of relevant points is at most $|R| \leq 2 \cdot (2^n - 1)$

Computing $\mathcal{V}(\Lambda_n) \le 2^n CVP(\Lambda_n)$



- Why is $|R| \le 2 \cdot (2^n - 1)$?
- Partition Λ into cosets modulo 2Λ
- There are $2^n - 1$ nonzero cosets
- From each coset, select the pair $\vec{v}, -\vec{v}$ closest to $\vec{0}$
- R is the set of all such pairs
- Each pair is found by a CVP computation in lattice 2Λ
- $CVP(2\Lambda)$ is equivalent to $CVP(\Lambda)$

CVP and Voronoi cell



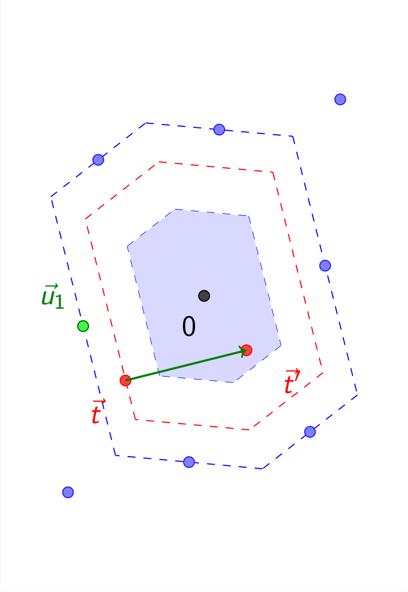
Definition (CVP)

Given Λ and $\vec{t}$, find $\vec{v} \in \Lambda$ such that $\vec{t} \in \vec{v} + \mathcal{V}$

- $\vec{t} \in \vec{v} + \mathcal{V} \equiv \vec{t} - \vec{v} \in \mathcal{V}$
- CVP goal: bring $\vec{t}$ inside $\mathcal{V}$ by shifting it by $\vec{v} \in \Lambda$
- Algorithm [SFS'09]:
  - While $\vec{t} \notin \mathcal{V}$:
  - Select $\vec{v} \in R$. $\vec{t} \notin \mathcal{H}_{\vec{v}}$
  - size reduce $\vec{t}$ using $\vec{v}$

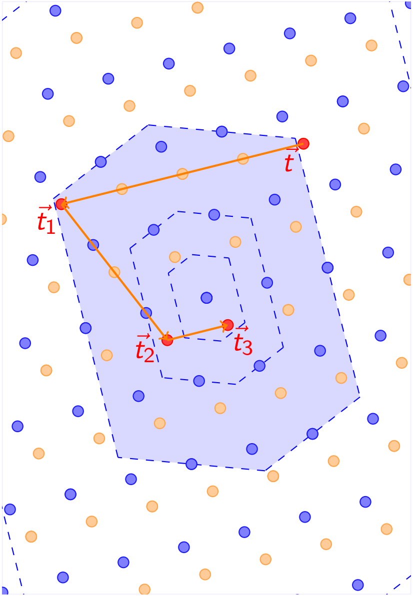
[SFS'09] only proves termination
Question: What is a good selection strategy for $\vec{v} \in R$?

Our selection strategy



- Assume $\vec{t} \in 2\mathcal{V}$
- Goal: find $\vec{t}' \in \vec{t} - \Lambda \cap \mathcal{V}$:
- Strategy:
  - Compute smallest $k \in \mathbb{R}$ such that $\vec{t} \in k\mathcal{V}$
  - Subtract the relevant vector associated to corresponding facet
- Why does it work?
  - The new vector $\vec{t}'$ is shorter than $\vec{t}$
  - still $\vec{t}' \in 2\mathcal{V}$
  - $|(\vec{t} - \Lambda) \cap 2\mathcal{V}| \leq 2^n$

Doubling the Voronoi Cell



Solve CVP for any $\vec{t}$:

- Find $\vec{k} \in \mathbb{Z}$ such that $\vec{t} \in 2^k \mathcal{V}$
- Use $\mathrm{CVP}_{2\mathcal{V}}$ to go from $2^k \mathcal{V}$ to $2^{k-1}\mathcal{V}$

Summary

- CVP can be solved deterministically in time $2^{c \cdot n}$
- Algorithms for SVP, SIVP and many other problems follow by reduction
- Question: what is the best possible c?
  - Under ETH, $c = \Omega(1)$
  - In this talk, we didn't optimize c
  - With some more work, we can reduce $c = 2$
- SVP: improves previous $c < 2.5$, deterministically!
- CVP: First $2^{O(n)}$ time algorithm, and first asymptotic improvement since [K'87]

Open Problems

- Reduce space complexity to polynomial
  - Closely related to the problem of compressing the description of the Voronoi cell of a lattice
- Faster CVPP solutions
  - Can the number of iterations in our algorithm be bounded by $O(n)$?
  - Can CVPP be approximated in polynomial time using approximate Voronoi cell?
- Extend algorithms to other norms (e.g., ℓ_∞)
  - Useful in cryptanalysis, integer programming, optimization, etc.
  - Is the number of ℓ_∞-relevant points still bounded by $2^{O(n)}$
- Better algorithms for special classes lattices (e.g., ideals of the ring of integers of algebraic number fields)
  - Small improvements can be obtained using symmetries
  - No NP-hardness results, so subexponential algorithms may be possible
  - Important for cryptographic applications