Multilinear Maps From Ideal Lattices

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Outline

- Bilinear Maps: Recall and Applications
  - Motivating Multilinear maps
- Our Results
- Definitions of Multi-linear Maps
  - Classical Notion
  - Our Notion
- Our Construction
  - Security

Cryptographic Bilinear Maps

(Weil and Tate Pairings)

Recalling Bilinear Maps and its Applications: Motivating Multilinear Maps

Cryptographic Bilinear Maps

- Bilinear maps are extremely useful in cryptography
  - lots of applications
- As the name suggests allow pairing two things together

Bilinear Maps – Definitions

- Cryptographic bilinear map
  - Groups G_1 and G_2 of order p with generators $g_1, g_2 = e(g_1, g_1)$and a bilinear map $e : G_1 \times G_1 \rightarrow G_2$ such that

$$\forall\, a, b \in Z_p , \qquad e\left(g_1^a, g_1^b\right) = g_2^{ab}$$

- Instantiation: Weil or Tate pairings over elliptic curves.

CDH is hard

Given g_1^a, g_1^b hard to get g_1^{ab}

DDH is easy

Given g_1^a, g_1^b, T

$T \stackrel{?}{=} g_1^{ab}$

$e\left(g_1^a, g_1^b\right) = e(g_1, T)$

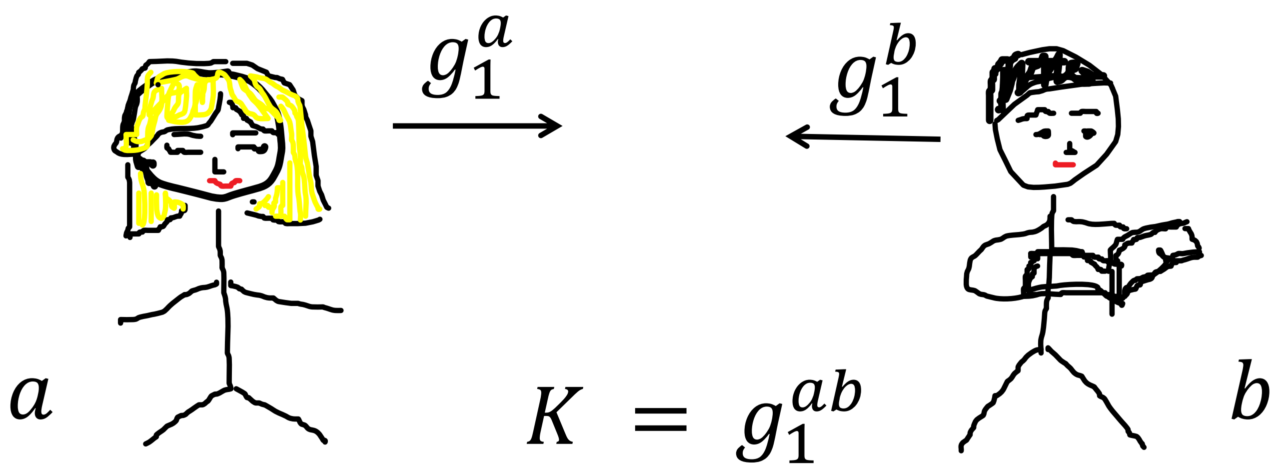
Bilinear Maps: ``Hard” Problem

- Bilinear Diffie-Hellman: Given

  $g_1, g_1^a, g_1^b, g_1^c \in G_1$ hard to distinguish $e(g_1, g_1^{abc}) = g_2^{abc}$ from Random

Application 1

Non-Interactive Key Agreement [DH76]



- Easy Application: Tri-partite key agreement [Joux00]:
  - Alice, Bob, Carol generate a, b, c and broadcast g_1^a, g_1^b, g_1^c.
  - They each separately compute the key $K = e(g_1, g_1)^{abc}$
- What if we have more than 3-parties? [BS03]

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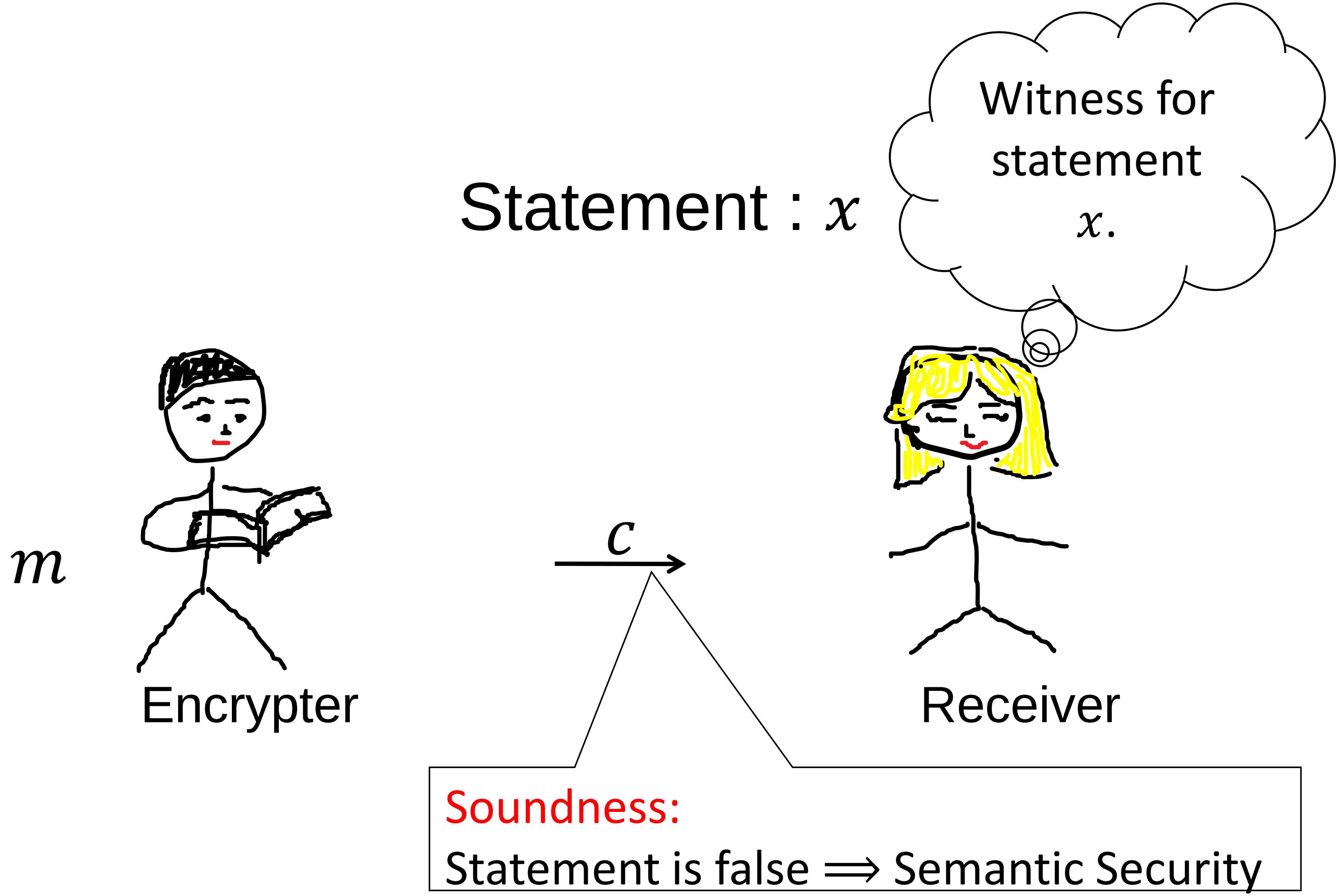
Our Results

- Candidate approximate constructions of multi-linear maps
- Lots of Applications:-
  - Witness Encryption
  - Indistinguishability Obfuscation

Application 2

Witness Encryption [GGSW13]

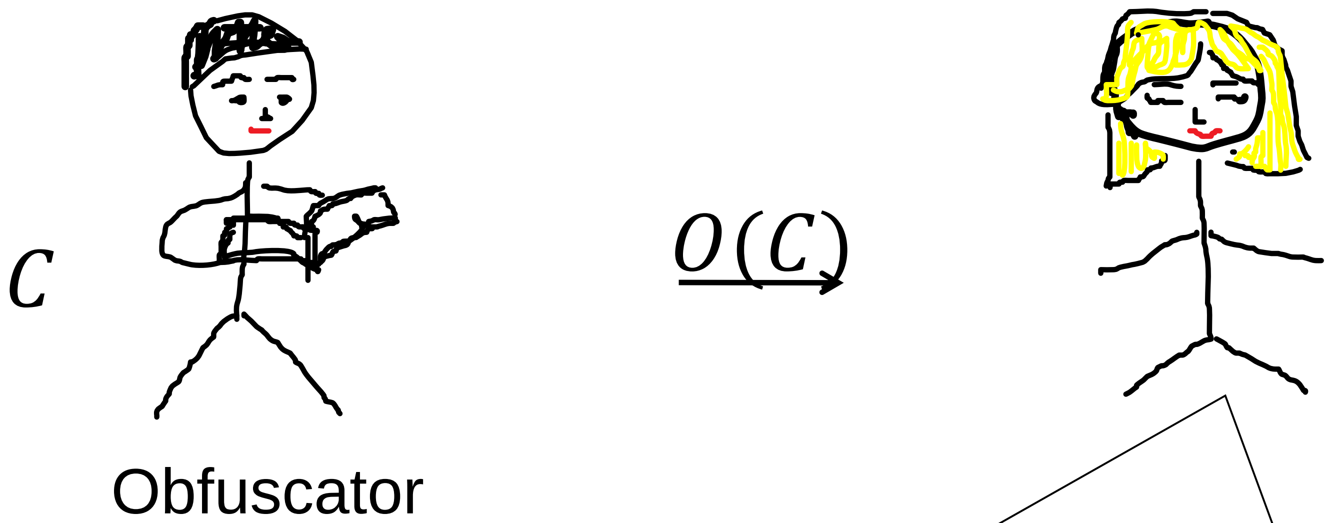
[TW87, Rudich89, IOS97, IS91, KMV07, CS02, CCKV08, GOVW12 …]



Application 3

Indistinguishability Obfuscation [GGHRSW13]

[Barak et al…]



Security : Can't tell if C = C_1 or C_2

As long as $\forall x\ C_1(x) = C_2(x)$ and $|C_1| = |C_2|$

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Cryptographic Multi-linear Maps

Definitions: Classical notion and our Approximate variant

Multilinear Maps: Classical Notion

- Cryptographic n-multilinear map (for groups)
  - Groups $G_1, \ldots, G_n$ of order p with generators $g_1, \ldots, g_n$
  - Family of maps:

    $e_{i,k}\colon G_i \times G_k \rightarrow G_{i+k}$ for $i + k \leq n$, where

    - $e_{i,k}\left(g_i^a, g_k^b\right) = g_{i+k}^{ab} \ \forall a, b \in Z_p$.

  - And at least the ``discrete log'' problems in each $G_i$ is ``hard''.
    - And hopefully the generalization of Bilinear DH

Getting to our Notion

Our visualization of (traditional) Bilinear Maps

→

Step by step I will make changes to get our notion of Bilinear Maps

→

At each step provide Extension to Multi-linear Maps

Bilinear Maps: Our visualization

Z_p	G_1	G_2
1	g_1^1	g_2^1
2	g_1^2	g_2^2
$\vdots$	$\vdots$	$\vdots$
p	g_1^p	g_2^p

Bilinear Maps: Our visualization Sampling

Z_p	G_1	G_2
1	g_1^1	g_2^1
2	g_1^2	g_2^2
$\vdots$	$\vdots$	$\vdots$
p	g_1^p	g_2^p

It was easy to sample uniformly from Z_p.

Bilinear Maps: Our visualization Equality Checking

Z_p	G_1	G_2
1	g_1^1	g_2^1
2	g_1^2	g_2^2
$\vdots$	$\vdots$	$\vdots$
p	g_1^p	g_2^p

Trivial to check if two terms are the same.

Bilinear Maps: Our visualization Addition

Z_p	G_1	G_2
1	g_1^1	g_2^1
2	g_1^2	g_2^2
$\vdots$	$\vdots$	$\vdots$
p	g_1^p	g_2^p

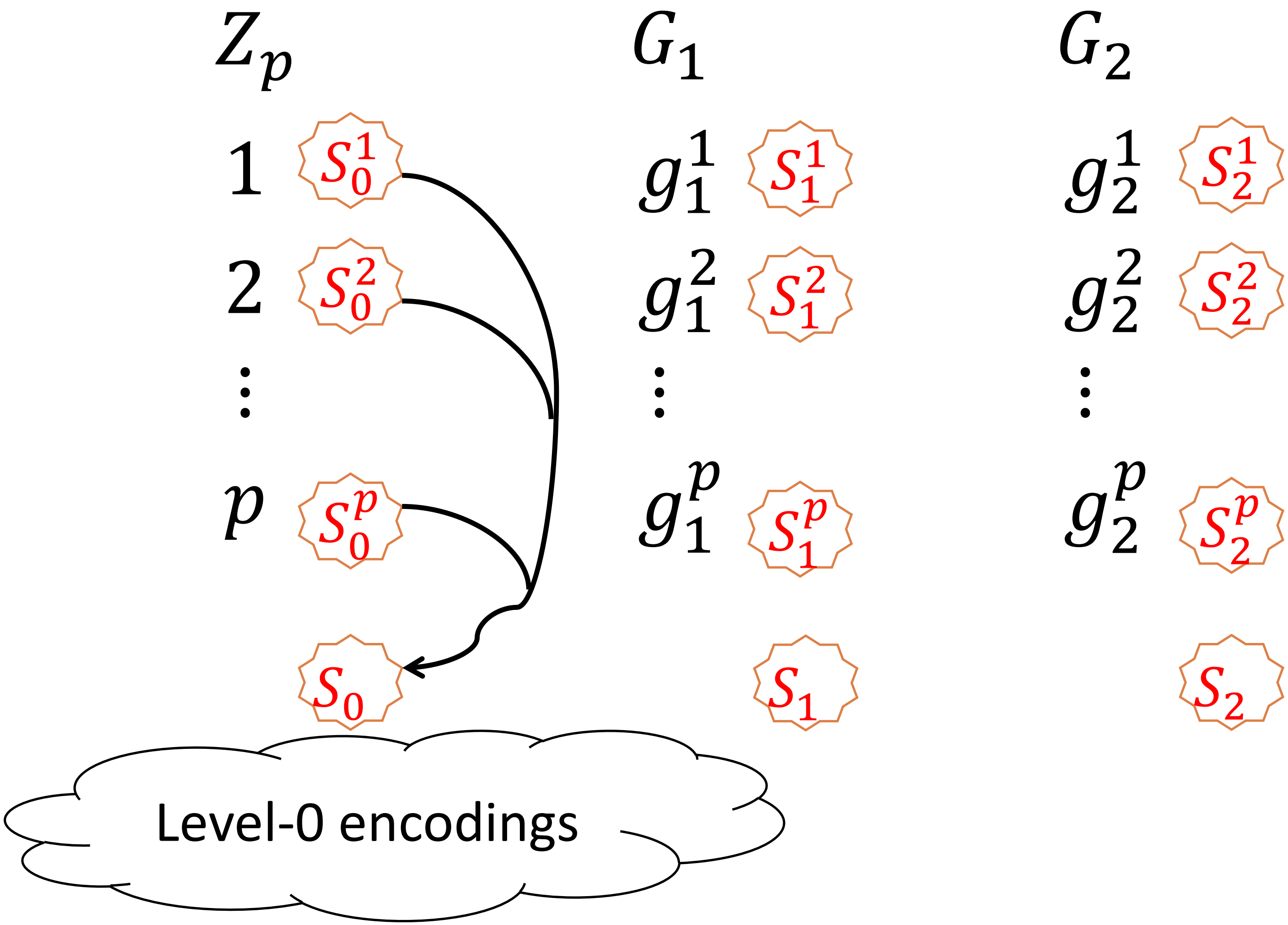
g_1^3

Bilinear Maps: Our visualization Multiplication

Z_p	G_1	G_2
1	g_1^1	g_2^1
2	g_1^2	g_2^2
$\vdots$	$\vdots$	$\vdots$
p	g_1^p	g_2^p

Bilinear Maps: Sets

(Our Notion)

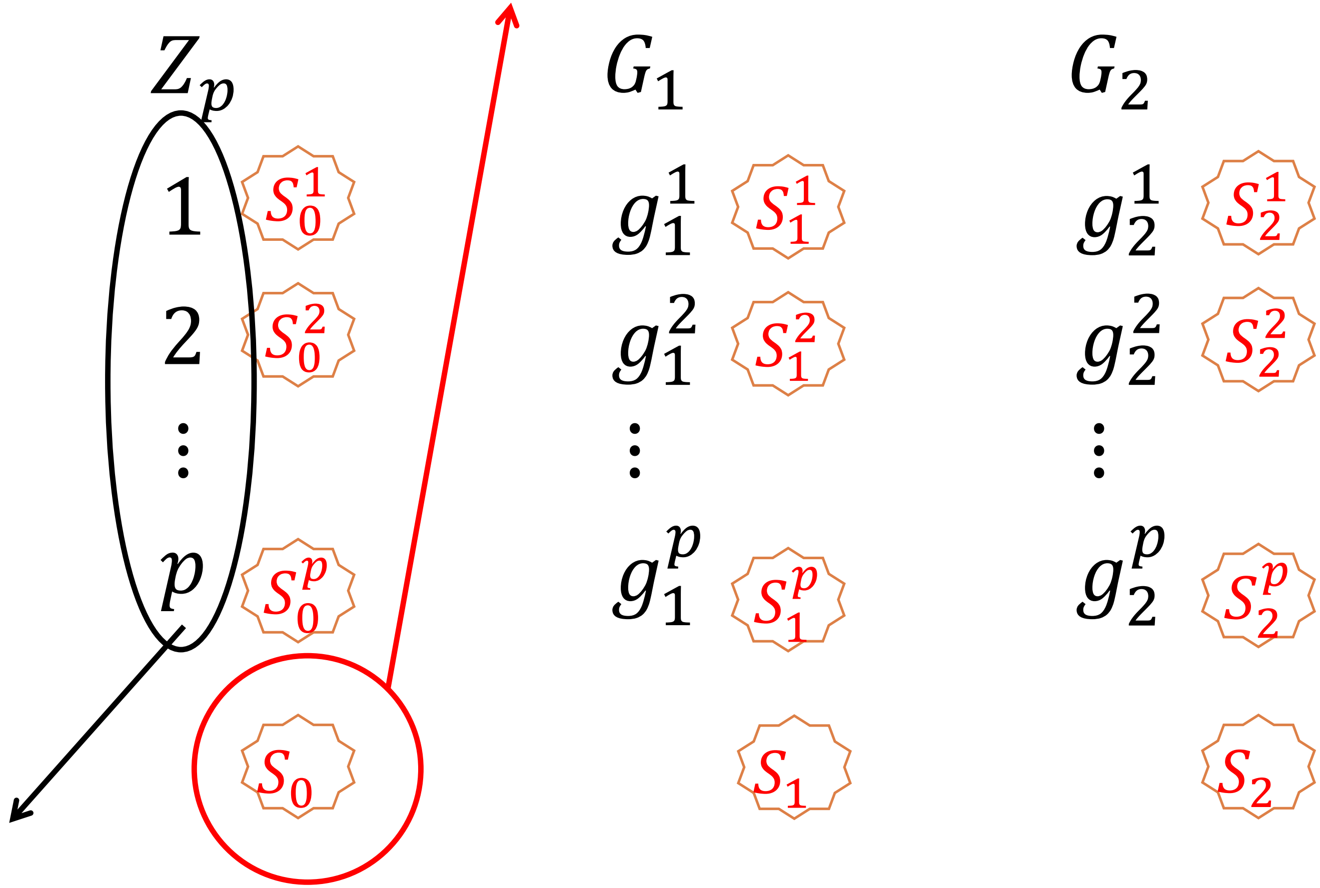


Multilinear Maps: Our Notion

- Finite ring R and sets S_i $\forall i \in [n]$: ``level-i encodings”
- Each set S_i is partitioned into S_i^a for each $a \in R$: ``level-i encodings of a”.

Bilinear Maps: Sampling

(Our Notion) I should be efficient to sample $\alpha \leftarrow S_0$ such that $\alpha \in S_0^a$ for a uniform a. It may not be uniform in S_0 or S_0^a.



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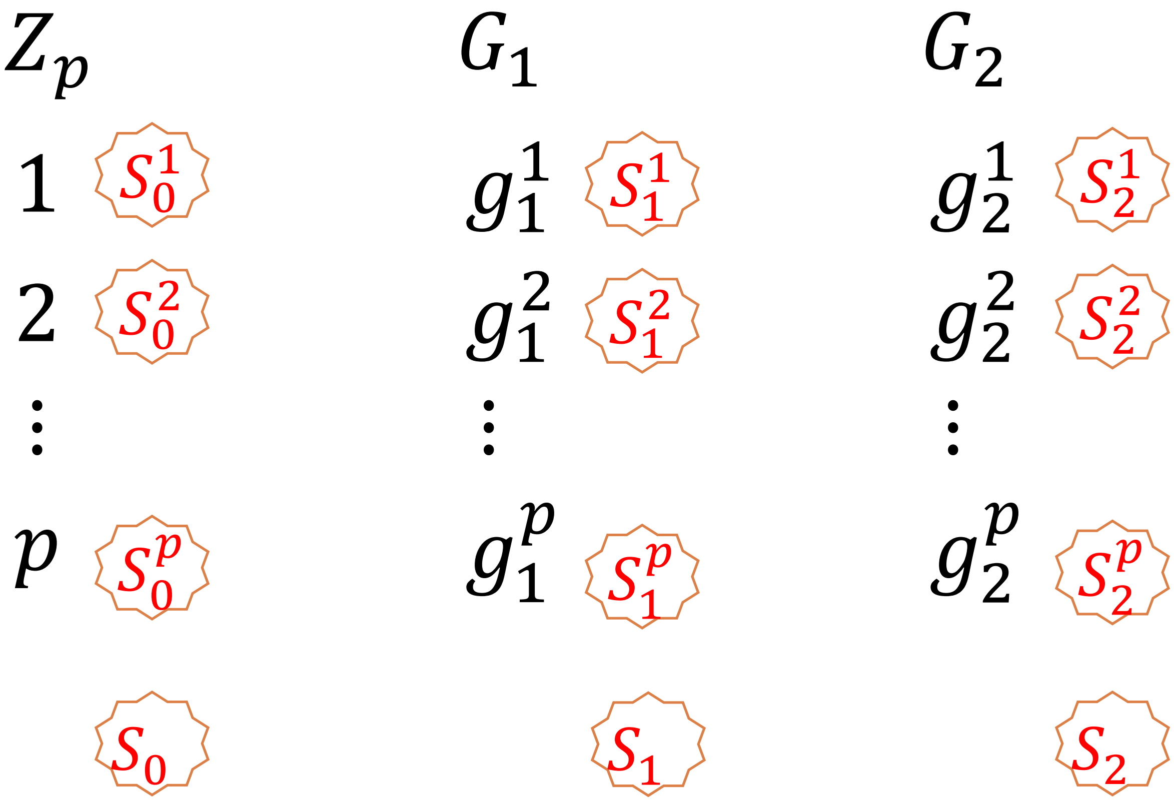
- Sampling: Output α such that $\alpha \in S_0^a$ for a unifrom a

Bilinear Maps: Equality Checking

(Our Notion)



Check if two values come from the same set.

It was trivial to check if two terms are the same.

Multilinear Maps: Our Notion

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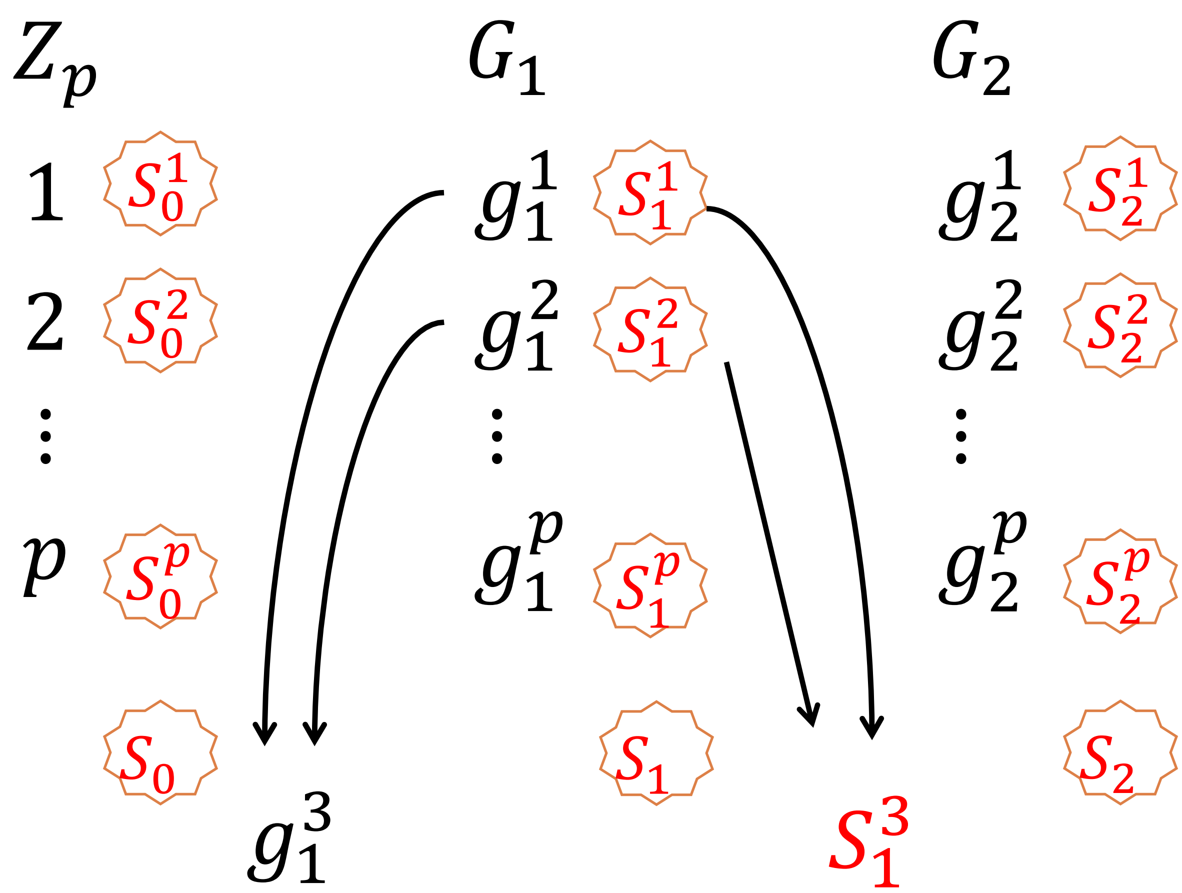
- Each set S_i is partitioned into S_i^a for each $a \in R$: ``level-i encodings of a”.
- Sampling: Output α such that $\alpha \in S_0^a$ for a random a
- Equality testing(α, β, i): Output 1 *iff* $\exists a$ such that $\alpha, \beta \in S_i^a$

Bilinear Maps: Addition

(Our Notion)



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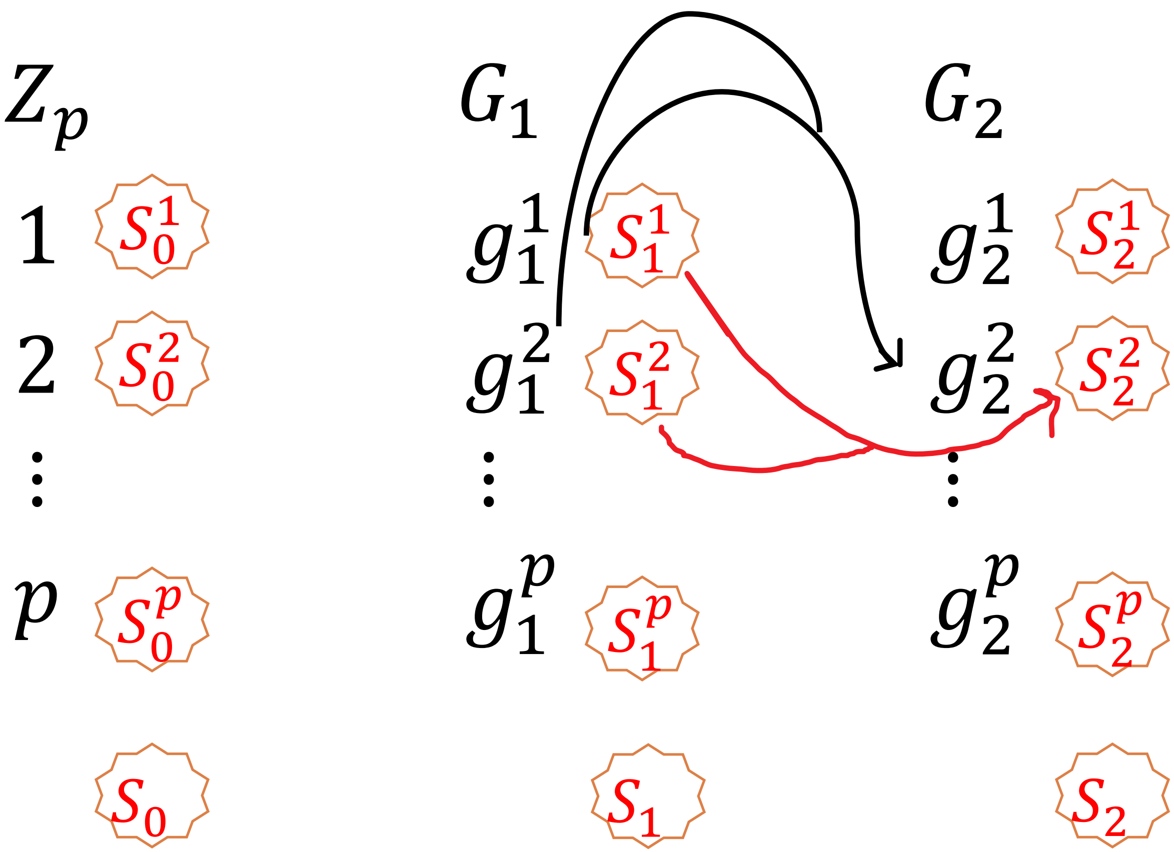
- Sampling: Output α such that $\alpha \in S_0^a$ for a random a
- Equality testing(α, β, i): Output 1 *iff* $\exists a$ such that $\alpha, \beta \in S_i^a$
- Addition/Subtraction: There are ops $+$ and – such that:
  - $\forall i \in [n], a, b \in R, \alpha \in S_i^a, \beta \in S_i^b$:
  - We have $\alpha + \beta \in S_i^{a+b}$ and $\alpha - \beta \in S_i^{a-b}$.

Bilinear Maps: Multiplication

(Our Notion)



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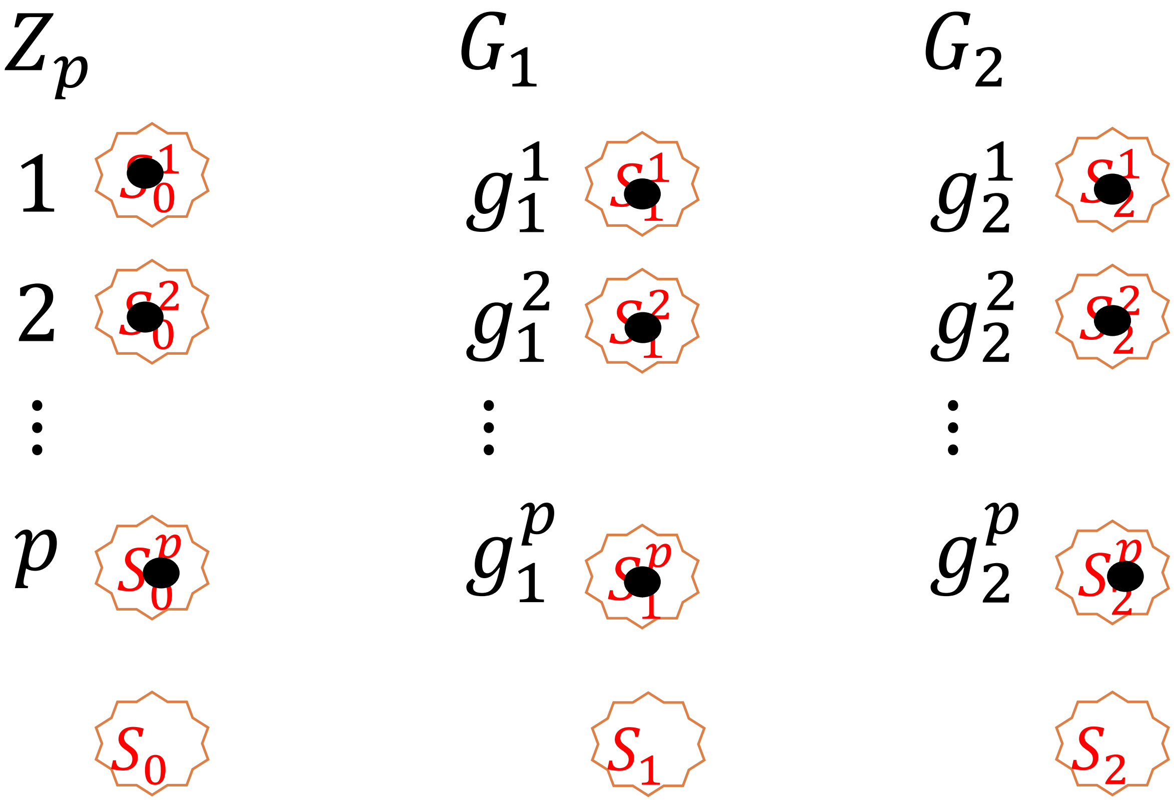
- Sampling: Output α such that $\alpha \in S_0^a$ for a random a
- Equality testing(α, β, i): Output 1 *iff* $\exists a$ such that $\alpha, \beta \in S_i^a$
- Addition/Subtraction: There are ops + and – such that:
- Multiplication: There is an op $\times$ such that:
  - $\forall i, k$ such that $i + k \leq n, \forall a, b \in R, \alpha \in S_i^a, \beta \in S_k^b$:
  - We have $\alpha \times \beta \in S_{i+k}^{ab}$.

Bilinear Maps: Noisy

(Our Notion)



All operations are required to work as long as ``noise'' level remains small.

Multilinear Maps: Our Notion

- **Discrete Log**: Given level-j encoding of a, hard to compute level-$(j\text{-}1)$ encoding of a.

- **n-Multilinear DDH:** Given level-1 encodings of $1, a_1, \dots, a_{n+1}$ and a level-n encoding T distinguish whether T encodes $a_1 \cdots a_{n+1}$ or not.

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``Noisy” Multilinear Maps

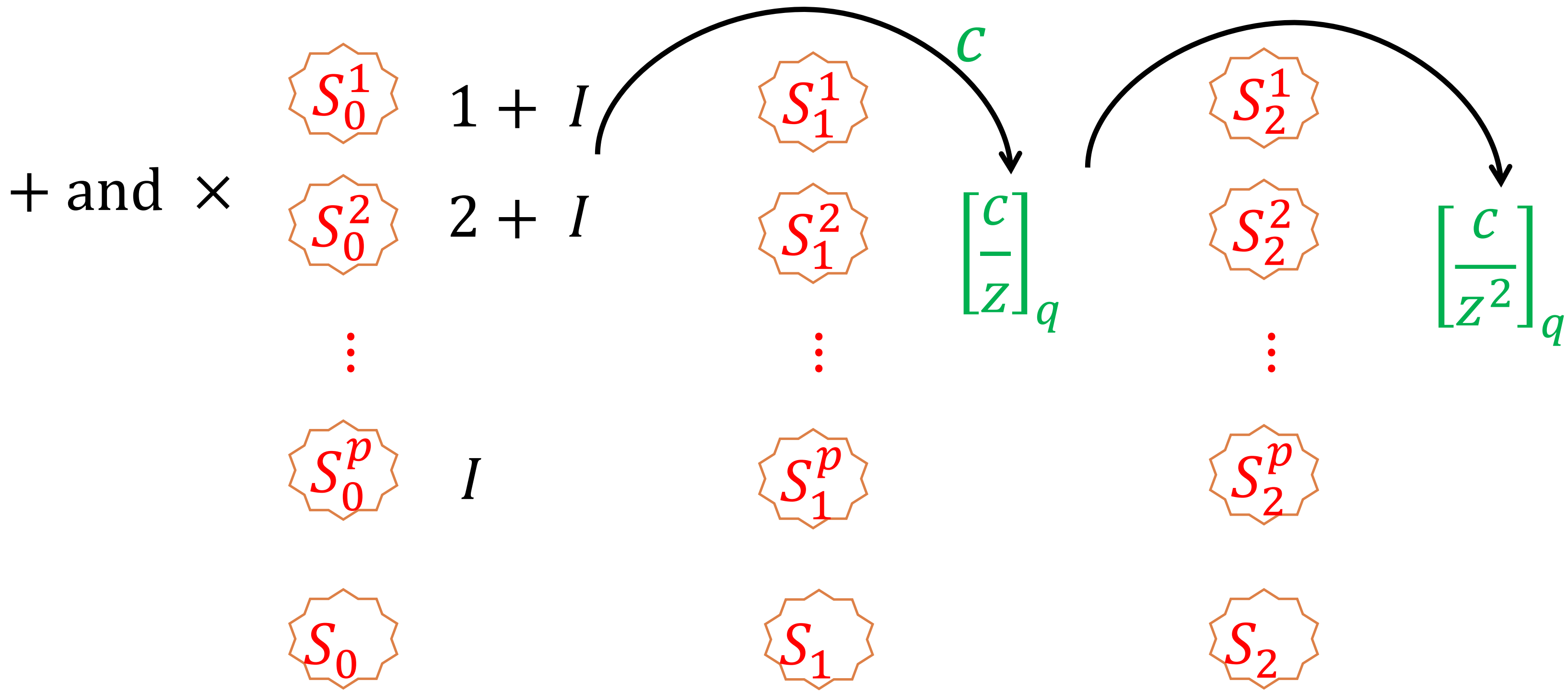
(Kind of like NTRU-Based FHE, but with Equality Testing)

Our Construction

- We work in polynomial ring $R = Z[x]/f(x)$
  - E.g., $f(x) = x^n + 1$ (n is a power of two)
  - Also use $R_q = R/qR = Z[x]/(f(x), q)$
- Public parameters hide a small $g \in R_q$ and a random (large) $z \in R_q$
  - g defines a principal ideal $I = (g)$ over R
  - The ``scalars" that we encode are cosets of I (i.e., elements in the quotient ring R/I)
    - e.g., if $|R/I| = p$ is a prime, then we can represent these cosets using the integers $1,2 \ldots, p$

Our Construction

- $R = Z[x]/f(x)$ and $R_q = R/qR$
- Small $g \in R_q$ defines a principal ideal $I = (g)$ over R



- A random (large) $z \in R_q$

c should have small coefficients

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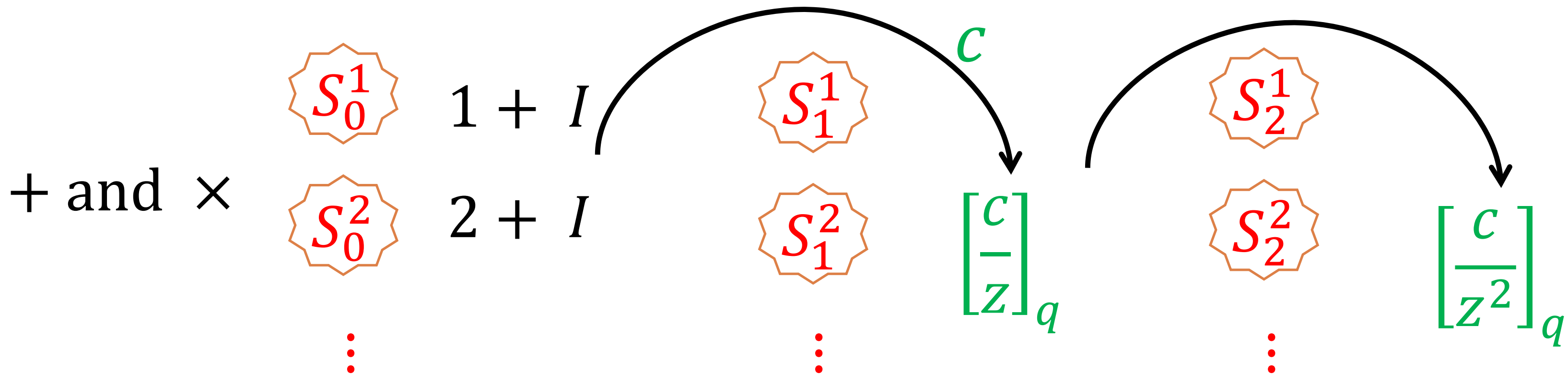

Addition

If $c \in s + I, d \in t + I$, are both short then,

$\left[\frac{c}{z} + \frac{d}{z}\right]_q$ has the form $\left[\frac{c+d}{z}\right]_q$,

where $c + d$ is still short and $c + d \in s + t + I$

- A random (large) $z \in R_q$

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Our Construction

- $R = Z[x]/f(x)$ and $R_q = R/qR$
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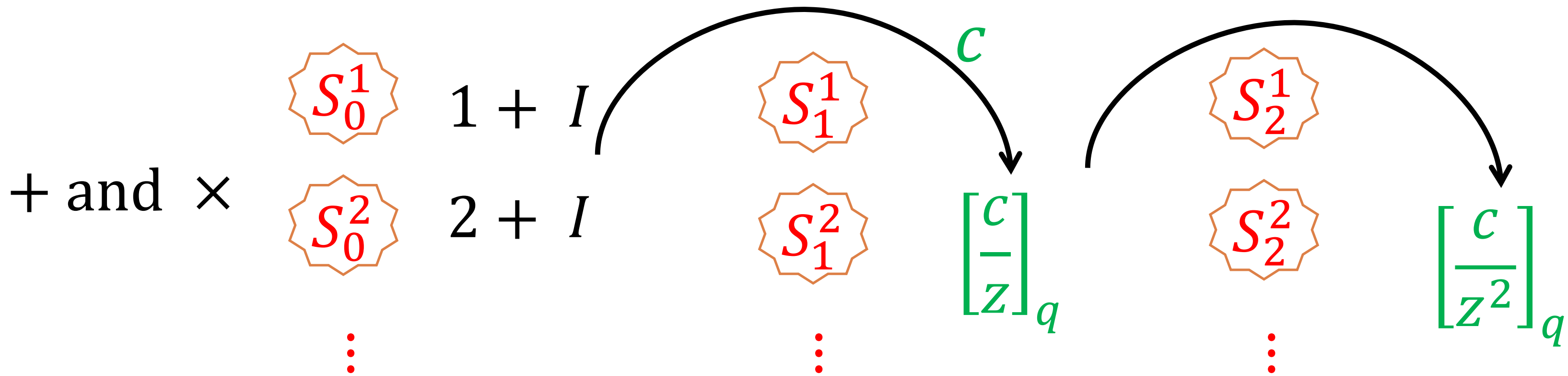


Multiplication

If $c \in s + I, d \in t + I$, are both short then,

$\left[\frac{c}{z} \times \frac{d}{z}\right]_q$ has the form $\left[\frac{c \times d}{z^2}\right]_q$,

where $c \times d$ is still short and $c \times d \in s \cdot t + I$

- A random (large) $z \in R_q$

c should have small coefficients

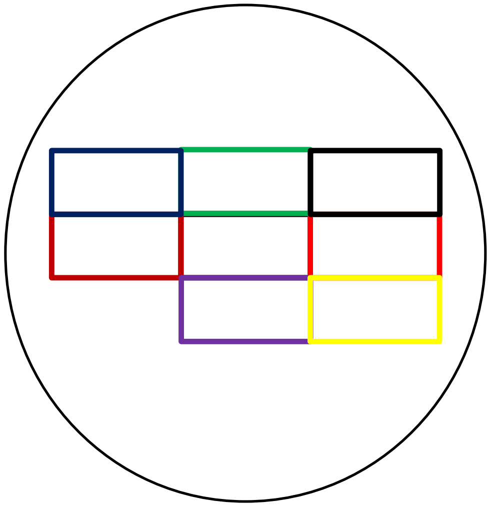
Our Construction (in general)

- In general, ``level-k encoding” of a coset $s + I$ has the form $\left[\frac{c}{z^k}\right]_q$ for a short $c \in s + I$
- Addition: Add encodings $u_i = \left[\frac{c_i}{z^j}\right]_q$
  - as long as $|\sum_i c\_i \,| \ll q$
- Multi-linear: Multiply encodings $u_i = \left[\frac{c_i}{z^{j_i}}\right]_q$
  - to get an encoding of the product at level $\sum_i j_i$
  - as long as $|\prod_i c_i| \ll q$
- ``Somewhat homomorphic” encoding

Sampling and equality check?

Sampling

- Sampling: If $c \leftarrow DiscreteGaussian(Z^n)$ (wider than smoothing parameter [MR05] of g but still smaller than q), then c encodes a random coset.
  - Why should this work?
  - Recall $I = (g)$ -- vector with tiny coefficients



Encoding this random coset

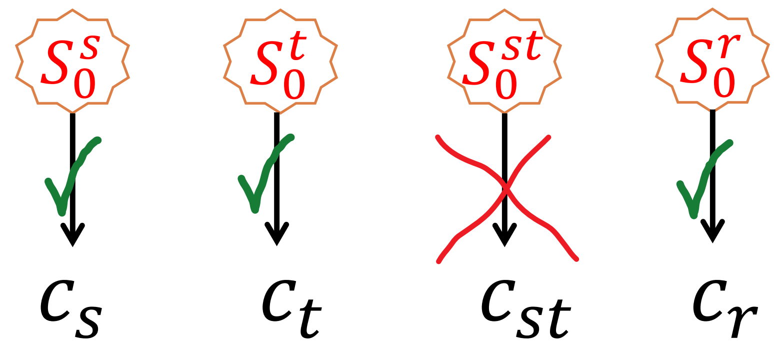
- Publish an encoding of 1:
  - $y = [a/z]_q$
- Sampling: If $c \leftarrow DiscreteGaussian(Z^n)$ (wide enough), then c encodes a random coset.
  - Don't know how to encode specific elements

- Given this short c, set $u = [c \cdot y]_q$
  - u is a valid level-1 encoding of the coset $c + I$
- Translating from level i to $i + 1$: $u_{i+1} = [u_i \cdot y]_q$

Equality Checking

- Do u, u' encode the same coset?
  - Suffices to check -$[u - u']_q$ encodes 0.
- Publish a (level-*k*) zero-testing param
$$v_k = [hz^k/g]_q$$
  - *h* is ``somewhat short" (e.g. of size $\sqrt{q}$)
- To test, if $u = [c/z^k]_q$ encodes 0, compute
- $w = [u \cdot v_k]_q = \left[\frac{c}{z^k} \cdot \frac{hz^k}{g}\right]_q = \left[\frac{ch}{g}\right]_q$
  - Which is small if $c \in I$ (or, $c = c'g$)

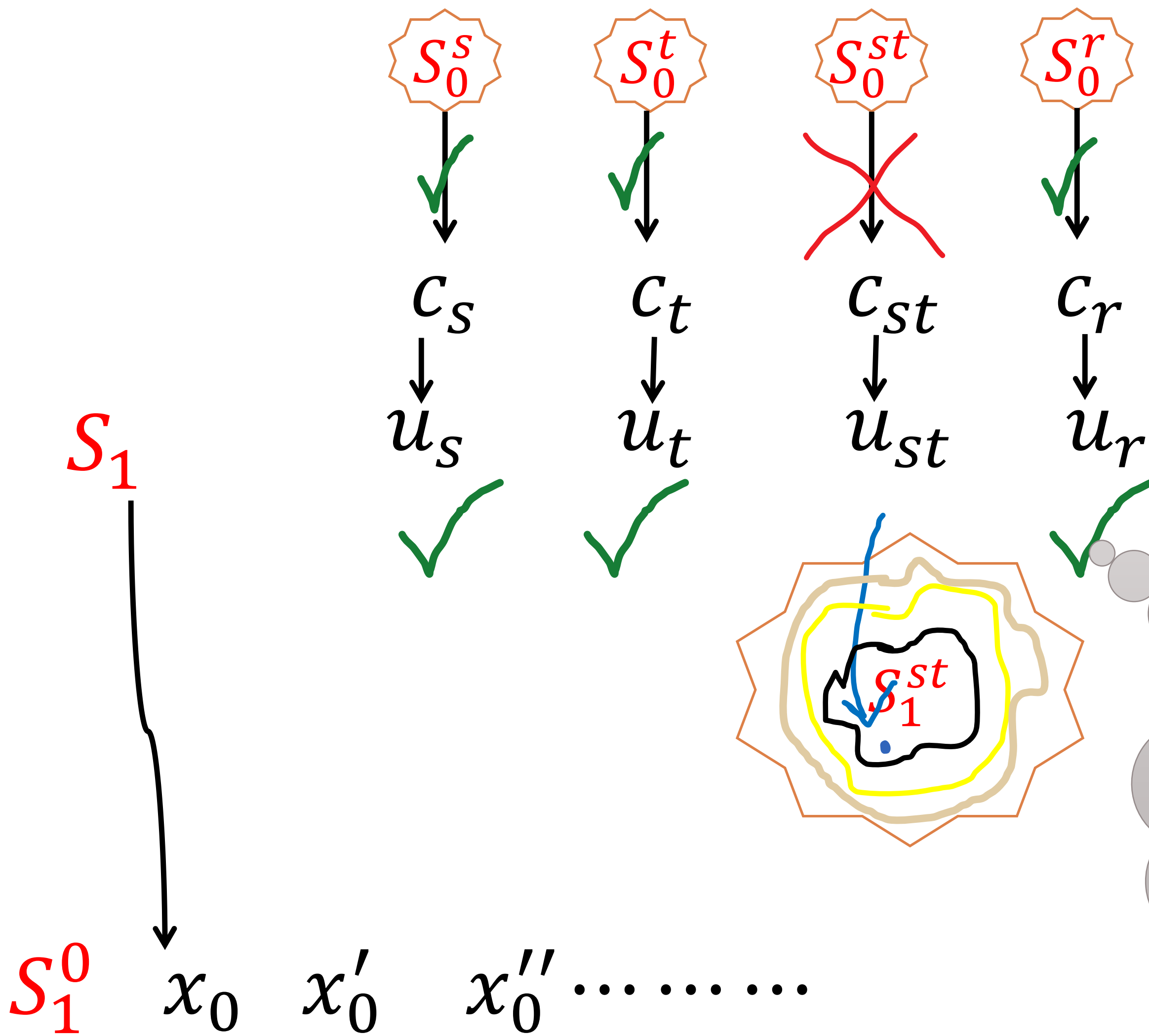
Re-randomizaton



- Compute $c_{st} = c_s c_t$
- And encode $u_s = [c_s y]_q, u_t = [c_t y]_q, u_{st} = [c_{st} y]_q$
  - But then $u_{st} = \frac{u_s u_t}{y}$
- We need to re-randomize the encoding, to break these simple algebraic relations

Re-randomizaton

This re-randomization gets us statistically close to the actual distribution [AGHS12].



Need to re-randomize this as well.

The Complete Encoding Scheme

- Parameters:

$$y = \left[\frac{a}{z}\right]_q, \left\{x_i = \left[\frac{b_i}{z}\right]_q\right\}_i, \text{ and } v_k = \left[\frac{hz^k}{g}\right]_q$$

- Encode a random element:
  - Sample c and set $u = [cy + \sum_i \rho_i x_i]_q$
  - $\rho_i \leftarrow DiscreteGaussian_s(Z)$
- Re-randomize *u* (at level 1):
  - $u' = [u + \sum_i \rho_i x_i]_q$
- Zero Test:
  - Map to level k (by multiplying by y^j for appropriate *j*)
  - Check if $[u \cdot v_k]_q$ is small

Variants

- Asymmetric variants (many z_i's), XDH analog

$$y_i = \left[\frac{a_i}{z_i}\right]_q, \left\{x_{i,j} = \left[\frac{b_{i,j}}{z_i}\right]_q\right\}_{i,j}, v_k = \left[\frac{h \prod_i z_i}{g}\right]_q$$

- Partially symmetric and partially asymmetric

Security: Cryptanalysis

Assumptions

$$y_0 = \left[\frac{a_0}{z}\right]_q, \dots\ y_k = \left[\frac{a_k}{z}\right]_q \text{ and } v_k = \left[\frac{hz^k}{g}\right]_q$$

- Goal: Distinguish
  - $\left[\frac{\prod a_i}{z^k}\right]_q$ from $\left[\frac{r}{z^k}\right]_q$
- Easy
  - $\left\{x_i = \left[\frac{b_i}{z}\right]_q\right\}_i$
  - General computation and not just multilinear
- Difficult
  - $y_0 = \left[\frac{a_0}{z_1}\right]_q, \dots\ y_k = \left[\frac{a_k}{z_k}\right]_q$ and $v_k = \left[\frac{h\prod z_i}{g}\right]_q$

Attacks

$$y = \left[\frac{a}{z}\right]_q, \left\{x_i = \left[\frac{b_i}{z}\right]_q\right\}_i, \text{ and } v_k = \left[\frac{hz^k}{g}\right]_q$$

- Goal: To find z or g
- Covering the basics (Not ``Trivially'' broken)
  - Adversary that only (iteratively) adds, subtracts, multiplies, or divides pairs of elements that it has already computed cannot break the scheme
  - Similar in spirit to Generic Group model
- Without the v_k- essentially the NTRU problem

Some attacks

$$y = \left[\frac{a}{z}\right]_q, \left\{x_i = \left[\frac{b_i}{z}\right]_q\right\}_i, \text{ and } v_k = \left[\frac{hz^k}{g}\right]_q$$

- Goal: To find z or g
- Can easily find ideal for $\langle h \rangle, \langle h \cdot g \rangle$ and $\langle g \rangle$

- Can not hope to hide $I = \langle g \rangle$ itself
  - But not small
  - This is the basis for conjectured hardness

Summary

- Presented ``noisy" cryptographic multilinear map.
- Construction is similar to NTRU-based homomorphic encryption, but with an equality-testing parameter.
- Security is based on somewhat stronger computational assumptions than NTRU.
- But more cryptanalysis needs to be done!

Thank You! Questions?