Post-quantum Key Exchange from LWE Problems

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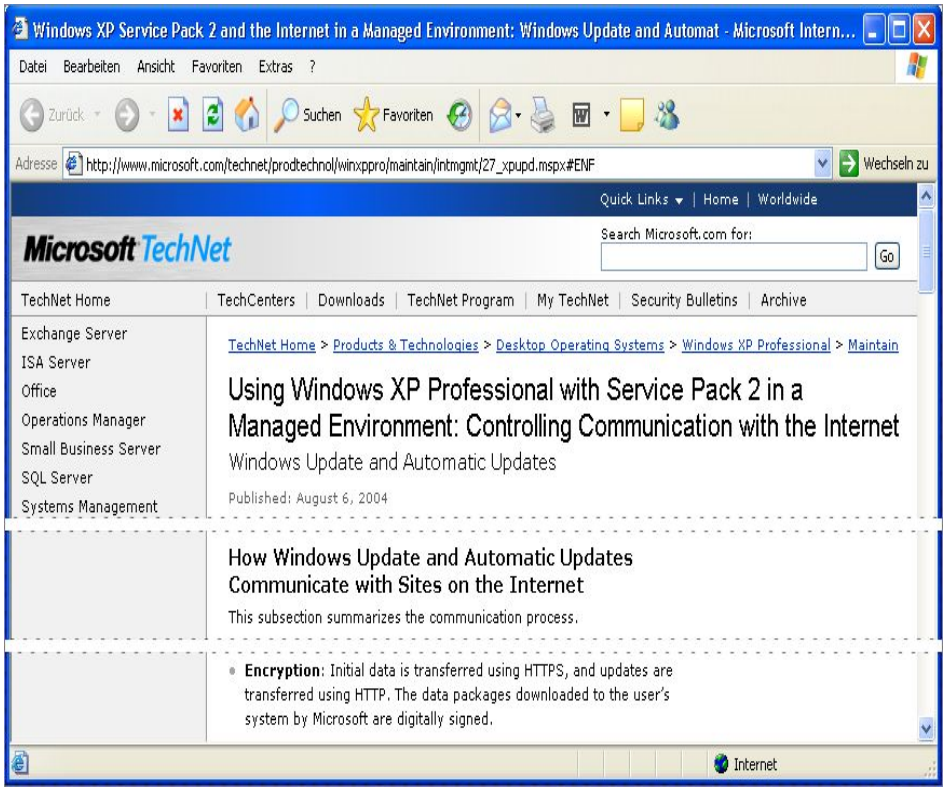
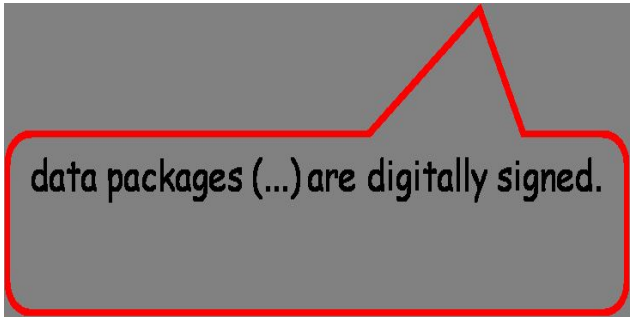
What is this number?

2133562529 1600027351 1427593551 9420913291 4767425698
0668648182 4528580269 7571587504 8271600387 9286718814
4217660057 9559348458 0081495826 8691260056 0376434697
9087161398 8653520618 5442348052 5894942341 3033375605
8732136514 8876038644 3075342912 0129705489 0001670606
7393246389 8375697515 1734774577 2076420507 4793016726
4791679237 3351492517 3209625562 4512058040 6546060184
8036703111 8237059907 4873628794 2617311911 1255520806
0025609009 0478884806 3977173442 6254325175 1228479981
6060960213 2860929278 0435354785 7716957089 8641110787
9876456259 1930871508 8016517131 0668371684 8928958136
1754587749 9229988091 2892709869 7538006934 6521176840
9897604596 0758751

RSA number for Microsoft update





PKC

- The number for Microsoft updates

PKC

- The number for Microsoft updates
- Digital signature based on RSA

PKC

- Mathematics behind: the hardness of integer factorization

$$n = pq.$$

$$15 = 3 \times 5.$$

PKC

- Mathematics behind: the hardness of integer factorization

$$n = pq.$$

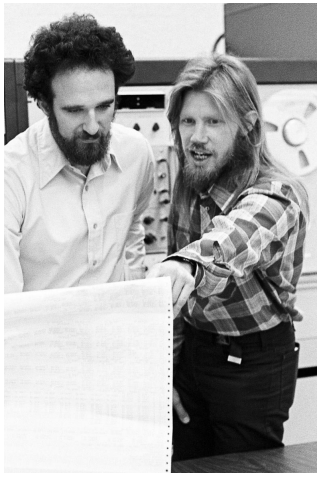
$$15 = 3 \times 5.$$

- The concept behind:

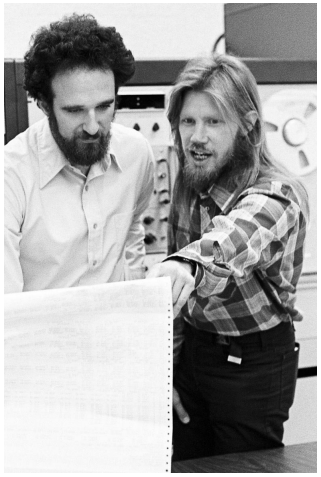
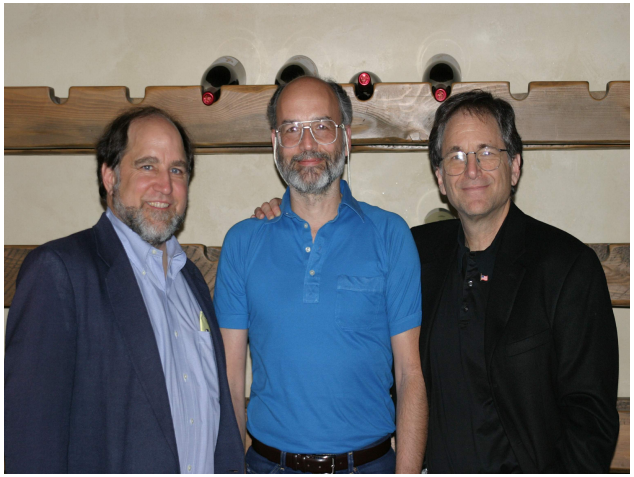
Public key Cryptography

IN (the god of) MATH WE TRUST!

PKC

- Diffie-Hellman  The inventors of the idea of PKC – Turing Prize 2016

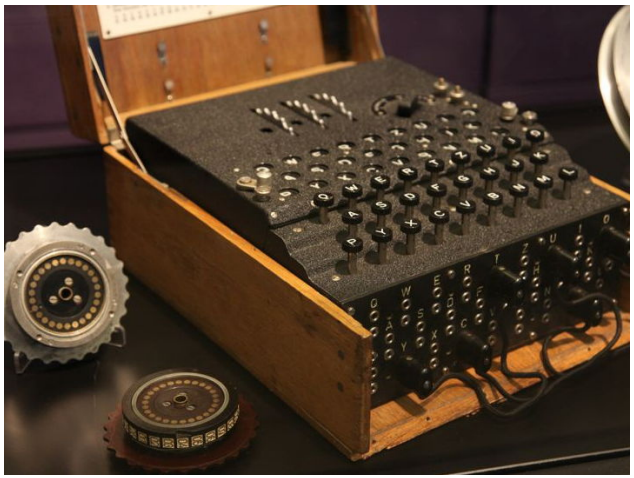
PKC

- Diffie-Hellman  The inventors of the idea of PKC – Turing Prize 2016
-  RSA – 2003 Turing prize

Symmetric system

Traditional symmetric systems

- The sender and receiver have the same keys:
  Ceasar Cipher, Enigma machines, DES, AES (Advanced Encryption Standards)



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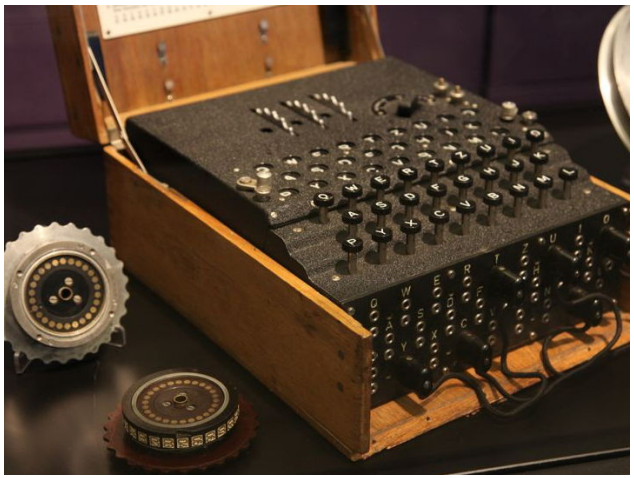

- The two parties must have a prior secure key exchange.

Why PKC

- Large computer networks

Why PKC

- Large computer networks
- Cost of key agreement

PKC

- The idea was proposed in 1970s by Diffie-Hellmann

PKC

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- Traditionally the information is symmetric. PKC is asymmetric.

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- The idea was proposed in 1970s by Diffie-Hellmann
- Traditionally the information is symmetric. PKC is asymmetric.
- There are two sets of keys:
  one public (N, r): $\gcd(r, (p-1)(q-1)) = 1$
  one private
  (d, p, q): $d \times r = 1 \mod (p-1)(q-1)$

PKC

- Encryption
  The public key is for encryption:

$$e = m^r \mod N$$

  The private key for decryption:

$$a = e^d \mod N.$$

- RSA: n is public and p,q is private.
- One knows how to factor n, one can defeat RSA.
  One can calculate d from $(p-1)(q-1)$.

Secure Communications

- To establish a shared keys
- the fast communication is done using the shared keys with symmetric systems – AES
- SSL, TLS, Online shopping (sending credit card securely on the internet)

Authentications

- To authenticate a message or a transaction
- The public key is to verify and anyone can verify.
- The private key is used to sign
- Software update, legal document, voting

Bitcoin-Blockchain

- An open ledger
- Decentralized
  Anyone can participate and can verify
  Good privacy and highly efficient in time
- Used as cryptocurrency
  ECDSA, Hash functions and Elliptic curve signature
  Public key (or address: the hash of public key) is to receive bitcoins
  Private key is used to send money, while the transaction cab be verified by the public key
  Hash functions for address and POW – for synchronization

PKC and Quantum computer

- Quantum computer: quantum mechanics for computations

R. Feynman

PKC and Quantum computer

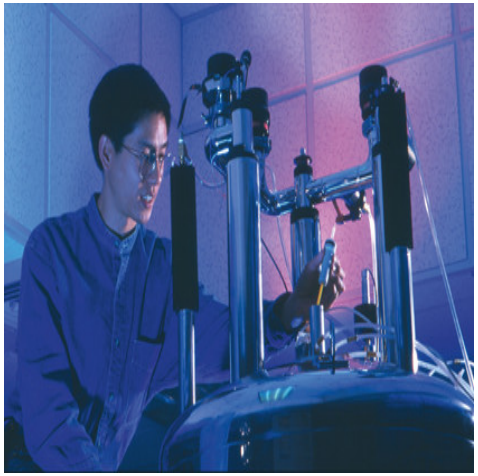
- Quantum computer: quantum mechanics for computations

R. Feynman

- In 1995, Quantum algorithm for factoring, discrete logarithm.

P. Shor

PKC and Quantum computer

- Can quantum computer really work?
-  Isaac Chuang
  15 million dollars to show that

$$15 = 3 \times 5.$$

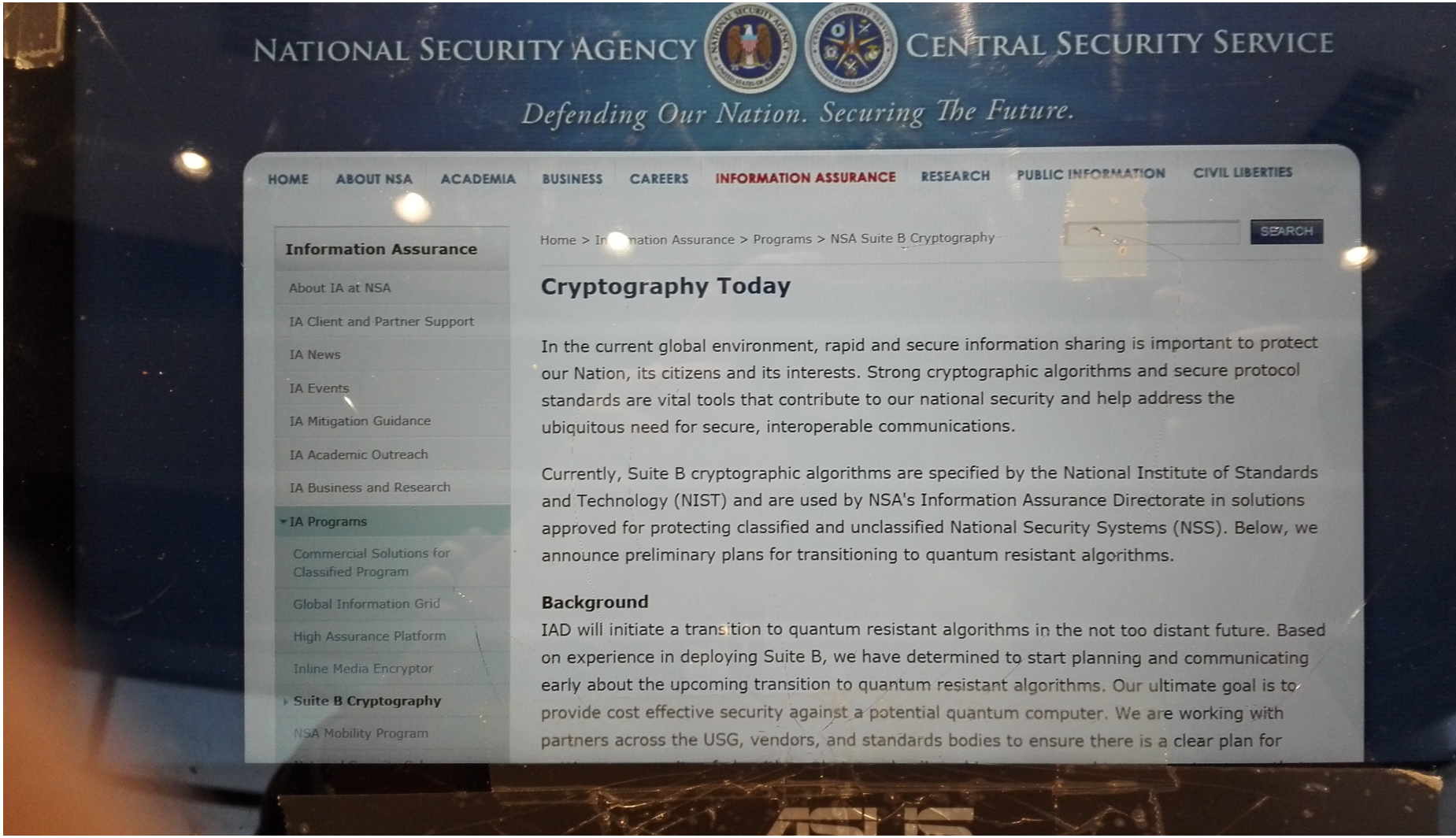
- The problem of scaling

The context of our work - PQC

- Shor's quantum algorithm
- **Post-quantum cryptography**

  Develop public key cryptosystems that could resist future quantum computer attacks

A commercial for PQC from NSA



Post Quantum Needs – Functionality

- Key Exchange – for secure communications
- Signatures – for Authentication

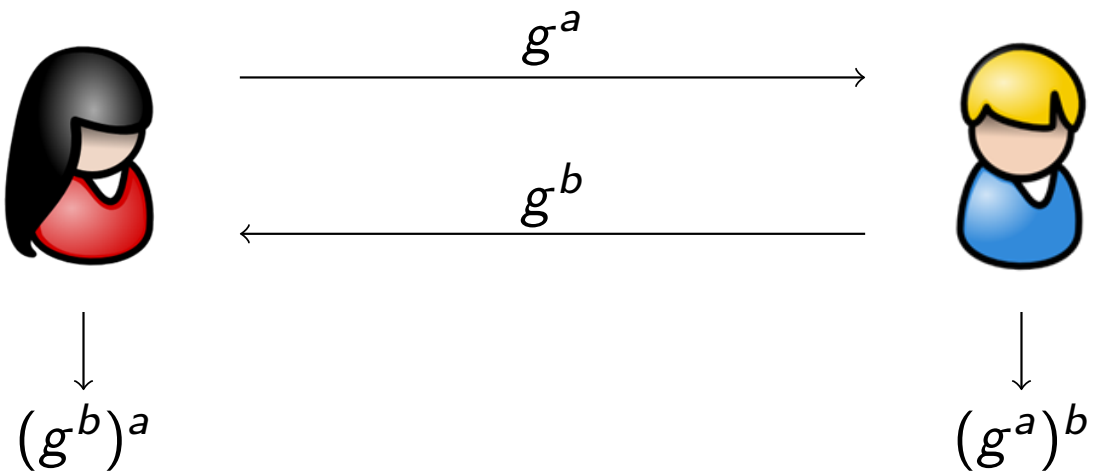
Key Exchange Applications — SSL/TLS

- RSA
- Diffie–Hellman
- Our goal – replacements for post quantum world

Forward Security

- RSA does not offer forward security since compromise of static private key allows decrypting the session keys.
- Possible to achieve forward security with RSA with ephemeral keys but expensive.
- Diffie Hellman offers forward security.
  Forward security: If static keys compromised, previous session keys remain secure.

Diffie-Hellman Key Exchange



Generalizing DH

- DH works because maps $f(x) = x^a$ and $h(x) = x^b$ commute

$$f \circ h = h \circ f,$$

$\circ$ – composition
Nonlinearity

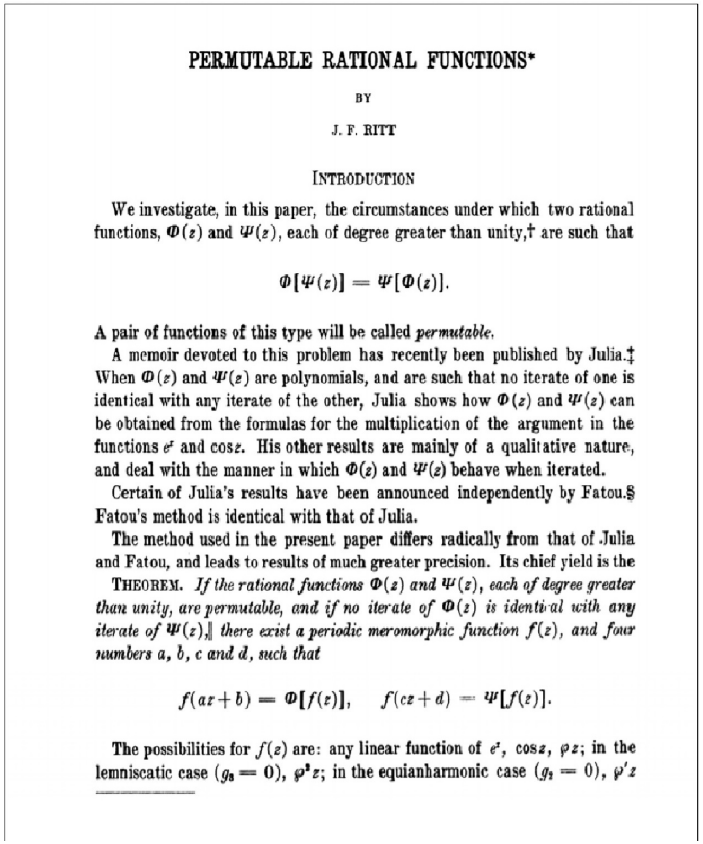
- Many attempts – Braid group etc

Generalizing DH

- When do we have commuting *nonlinear* maps?
  - Powers of x (normal DH)
  - Iterates of a polynomial
  - J. Ritt (1923) – Power polynomials, Chebychev polynomials. Elliptic curve

Who is J. Ritt: 1893-1951

Who is J. Ritt: 1923: PERMUTABLE RATIONAL FUNCTIONS



PERMUTABLE RATIONAL FUNCTIONS*

BY

J. F. RITT

INTRODUCTION

We investigate, in this paper, the circumstances under which two rational functions, $\Phi(z)$ and $\Psi(z)$, each of degree greater than unity,† are such that

$$\Phi[\Psi(z)] = \Psi[\Phi(z)].$$

A pair of functions of this type will be called *permutable*.

A memoir devoted to this problem has recently been published by Julia.‡ When $\Phi(z)$ and $\Psi(z)$ are polynomials, and are such that no iterate of one is identical with any iterate of the other, Julia shows how $\Phi(z)$ and $\Psi(z)$ can be obtained from the formulas for the multiplication of the argument in the functions e^z and $\cos z$. His other results are mainly of a qualitative nature, and deal with the manner in which $\Phi(z)$ and $\Psi(z)$ behave when iterated.

Certain of Julia's results have been announced independently by Fatou.§ Fatou's method is identical with that of Julia.

The method used in the present paper differs radically from that of Julia and Fatou, and leads to results of much greater precision. Its chief yield is the

THEOREM. *If the rational functions $\Phi(z)$ and $\Psi(z)$, each of degree greater than unity, are permutable, and if no iterate of $\Phi(z)$ is identical with any iterate of $\Psi(z)$,‖ there exist a periodic meromorphic function $f(z)$, and four numbers a, b, c and d, such that*

$$f(az+b) = \Phi[f(z)], \qquad f(cz+d) = \Psi[f(z)].$$

The possibilities for $f(z)$ are: any linear function of e^z, $\cos z$, $\wp z$; in the lemniscatic case ($g_3 = 0$), $\wp^2 z$; in the equianharmonic case ($g_2 = 0$), $\wp' z$

J. Ritt (1923) – Power polynomials, Chebychev polynomials. Elleptic curve

Generalizing DH

Our basic idea — adding "small" **noise or perturbation**:

- (Ring) LWE approximately commutes—use to build DH generalization
  From
  $$(s_1 \times a) \times s_2 = s_1 \times (a \times s_2)$$
  to
  $$(as_1 + e_1)s_2 \approx s_1 a s_2 \approx (as_2 + e_2)s_1.$$

A historical Note

Our basic idea — adding "small" **noise or perturbation** is not new!!!

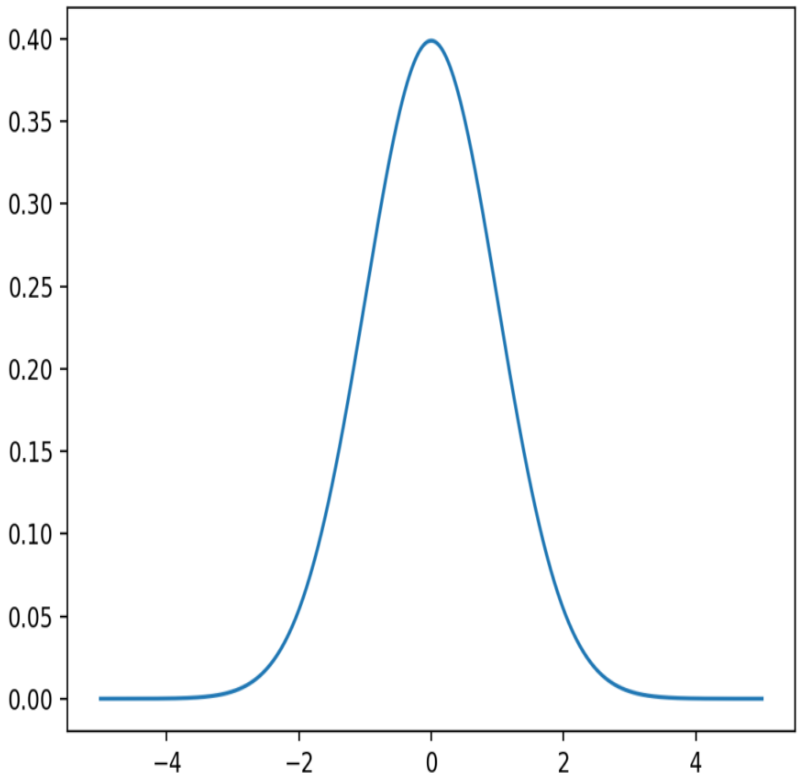
- GCHQ – Communications-Electronics Security Group(CESG) – James Elias – "Invention of non-secret encryption" 1969 Clifford Cocks – RSA, Malcolm Williamson – DH, 1973
- The forgotten inspiration of J. Ellis – "Ellis said that the idea first occurred to him after reading a paper from World War II by someone at Bell Labs describing a way to protect voice communications by the receiver adding (and then later subtracting) random noise (possibly this 1944 paper[4] or the 1945 paper co-authored by Claude Shannon)" – Wikipedia

Learning with Errors [2006, Regev]

$$\underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\vec{b}} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}}_{\vec{s}} + \underbrace{\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix}}_{\vec{e}}$$

- Approximate system over $\mathbb{Z}_q$
- Hard to find $\vec{s}$ from $A, \vec{b}$.
- Hard to tell if $\vec{s}$ even exists
- Reduction to lattice approximation problems

Discrete Gaussian



Ring LWE

Definition

Let n be a power of 2, $q \equiv 1 \pmod{2n}$ prime. Define the ring

$$R_q = \frac{\mathbb{Z}_q[x]}{(x^n + 1)}.$$

- Again, $b = as + e$ hard to find s
- Hard to distinguish from uniform b
- Approximation problems on *ideal* lattices
- More efficient than standard LWE

Diffie-Hellman from Ideal Lattices

$$\xrightarrow{\quad p_A = as_A + 2e_A \quad}$$

$$\xleftarrow{\quad p_B = as_B + 2e_B \quad}$$

- Public $a \in R_q$. Acts like generator g in DH.

Diffie-Hellman from Ideal Lattices

$$\xrightarrow{\quad p_A = as_A + 2e_A \quad}$$

$$\xleftarrow{\quad p_B = as_B + 2e_B \quad}$$

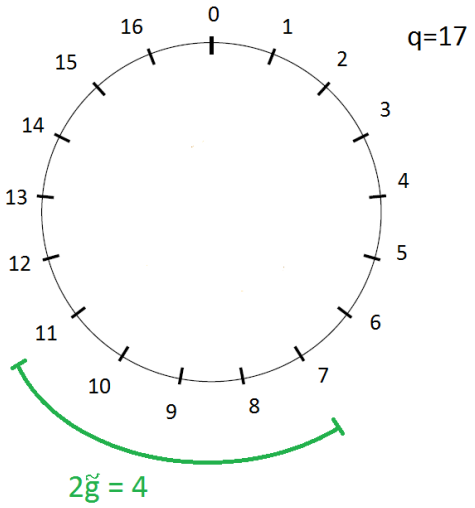
$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$k_A = s_A p_B + 2e'_A \qquad \approx \qquad k_B = p_A s_B + 2e'_B$$

$$= aS_A S_B + 2S_A e_B + 2e'_A \qquad\qquad = aS_A S_B + 2S_B e_A + 2e'_B$$

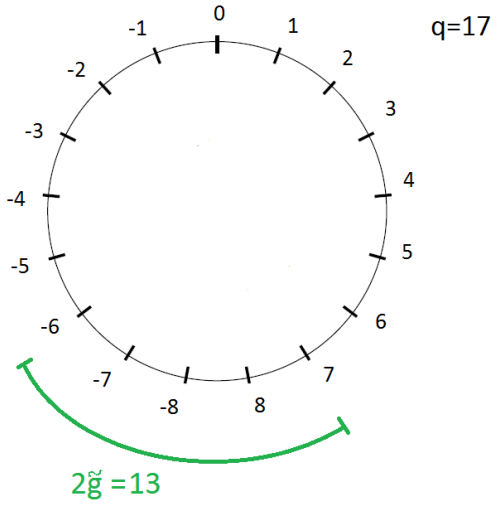
- Public $a \in R_q$. Acts like generator g in DH.
- Each side's key is only *approximately* equal to the other.
- Difference is even—same low bits.
- No authentication—MitM

Wrap-around Illustrated



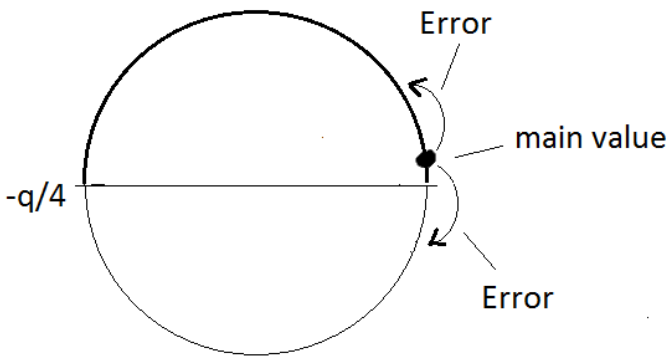
- Difference 4, both odd.

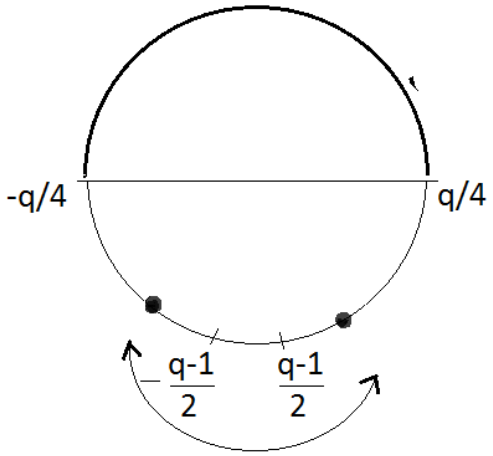
Wrap-around Illustrated



- But wait! If $q = 17$,
  $\mathbb{Z}_q = \{-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$.
- 11 becomes -6, now parities disagree!

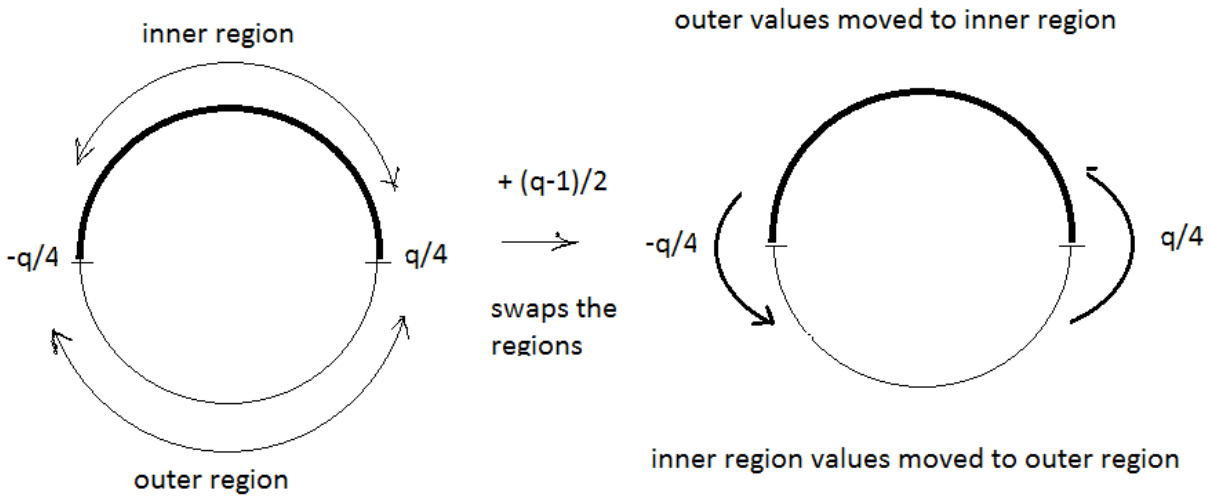
Rounding Intuition – Outer Region problem



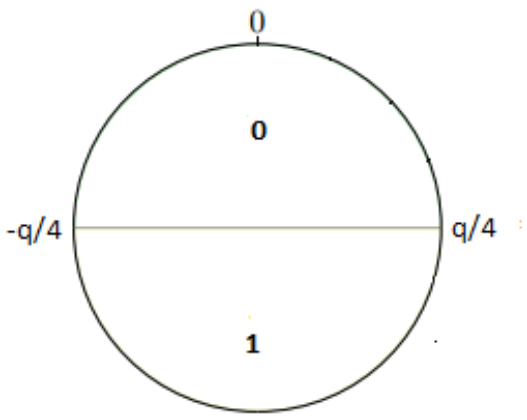


Additional modular operation

Rounding Intuition



Rounding Intuition – Region Division



Signal function Sig(.)

Compensating for Wrap-Around

- $g = 2S_A e_B - 2S_B e_A + 2e'_A - 2e'_B$.
- Recall: $|g^{(j)}| < \frac{q}{8}$
- Define $E = \{-\lfloor \frac{q}{4} \rfloor, \ldots, \lfloor \frac{q}{4} \rceil\}$. Middle half of $\mathbb{Z}_q$.
- If $k_B^{(j)} \in E$, no wrap-around occurs; $k_A^{(j)} \equiv k_B^{(j)}$.
- If $k_B^{(j)} \notin E$, then $k_B^{(j)} + \frac{q-1}{2} \in E$
- If $k_B^{(j)} \notin E$, $k_A^{(j)} + \frac{q-1}{2} \equiv k_B^{(j)} + \frac{q-1}{2}$.

Wrap-around Defeated

Define $Sig(v) = \begin{cases} 0 & v \in E, \\ 1 & v \notin E. \end{cases}$

and, $w_B^{(j)} = Sig(k_B^{(j)})$

Then $k_B^{(j)} + w_B^{(j)} \frac{q-1}{2} \in E$.

Also, $k_B^{(j)} + w_B^{(j)} \frac{q-1}{2} \equiv k_A^{(j)} + w_B^{(j)} \frac{q-1}{2} \pmod 2$.

- $k_B^{(j)} + w_B^{(j)} \frac{q-1}{2} \bmod q \mod 2 = k_A^{(j)} + w_B^{(j)} \frac{q-1}{2} \bmod q \bmod 2$.
- Wrap-around correction $w_B = (w_B^{(0)}, w_B^{(1)}, \ldots, w_B^{(n-1)})$
- $\sigma_B = k_B + w_B \frac{q-1}{2} \bmod 2$.
- $\sigma_A = k_A + w_B \frac{q-1}{2} \bmod 2$.

Key Derivation

Obtaining shared secret from approximate shared secret:

$$
\begin{aligned}
k_A &= (k_A^{(0)}, k_A^{(1)}, \ldots, k_A^{(n-1)}) \\
k_B &= (k_B^{(0)}, k_B^{(1)}, \ldots, k_B^{(n-1)}) \\
\tilde{g} &= (g^{(0)}, g^{(1)}, \ldots, g^{(n-1)}) \\
k_A - k_B &= 2\tilde{g} \\
k_A &\equiv k_B \pmod 2
\end{aligned}
$$

Key Derivation

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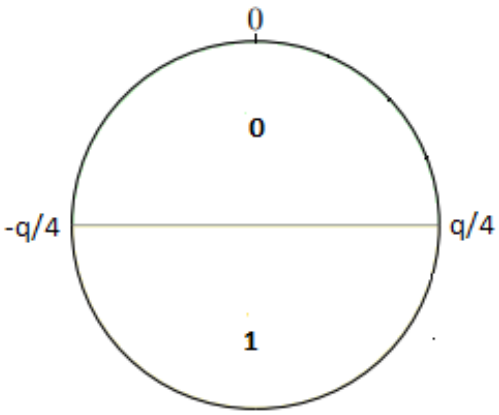
k_A - k_B &= 2\tilde{g} \\
k_A &\equiv k_B \pmod 2
\end{aligned}
$$

- Each $k_A^{(j)} = k_B^{(j)} + 2g^{(j)}$.
- Each $g^{(j)}$ is small ($|g^{(j)}| < \frac{q}{8}$).
- Matching coefficients differ by small multiple of 2
- Take each coefficient mod 2, get n bit secret

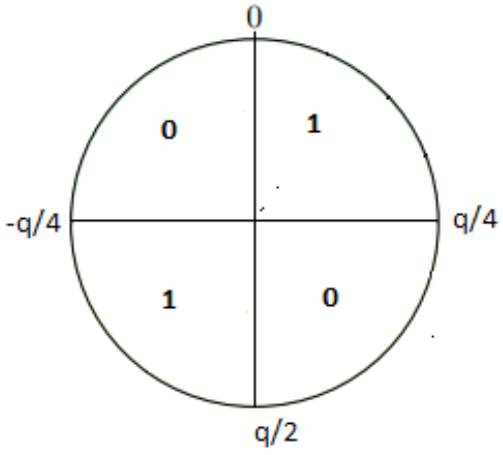
LWE KE

- Ding's paper:
  Cryptology ePrint Archive: Report 2012/688
  20121210:115748 (posted 10-Dec-2012 11:57:48 UTC)
- Cryptology ePrint Archive: Report 2014/070, C. Peikert
  Cryptology ePrint Archive: Report 2014/599, Joppe W. Bos and Craig Costello and Michael Naehrig and Douglas Stebila
  Cryptology ePrint Archive: Report 2015/1092, Erdem Alkim and Leo Ducas and Thomas Poppelmann and Peter Schwabe

Comparison of Signal



Signal function Sig(.)



cross rounding b= $\langle . \rangle_2$

Proof Games

Proof proceeds by series of games:

- Begin with simulated protocol
- Adversary cannot distinguish from previous game
- Eventually, if original protocol can be distinguished from random, rLWE can be broken
- Important step to post quantum key exchange.
- Authenticated key exchange in Eurocrypt 2015.

Thank You

Thank Air Force and NSF for support.

Questions to jintai.ding@gmail.com