Explicit points on Jacobians of superelliptic curves over global function fields

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This work was initiated at an AIM (American Institute of Mathematics) workshop on *Cohomological methods in abelian varieties* in Palo Alto, March 26–30, 2012.
Some Background

Ranks of elliptic curves over $\mathbb{F}_q(t)$ are unbounded.

- Tate & Shafarevich (1967): isotrivial
- Ulmer (2002): non-isotrivial
Theorem (Ulmer, 2007)

For all $g > 0$, all primes $p$, and all $R$, there exist absolutely simple non-isotrivial abelian varieties of dimension $g$ over $\mathbb{F}_p(t)$ with

$$\text{analytic rank} = \text{algebraic rank} > R.$$
If \( p \nmid (2g + 2)(2g + 1) \), then the Jacobian of

\[
y^2 = x^{2g+2} + x^{2g+1} + t^{p^n+1}
\]

over \( \mathbb{F}_p(t) \) is absolutely simple, non-isotrivial, and has rank \( \geq p^n/2n \).

In particular, the Jacobian of

\[
y^2 = x^{2g+2} + x^{2g+1} + t
\]

has unbounded rank in the tower \( \{\mathbb{F}_p(t^{1/d})\} \).
Some Background

Theorem (Ulmer (2007))

Roughly half the abelian varieties over $\mathbb{F}_q(t)$ have unbounded analytic rank in the tower

$$\{\mathbb{F}_q(t^{1/d})\}_{\gcd(d,q)=1}.$$

Ulmer’s results gave families of abelian varieties of fixed dimension and unbounded (analytic and algebraic) rank over $\mathbb{F}_q(t)$, but didn’t give explicit points.
In Ulmer’s preprint

*Explicit points on the Legendre curve*

he obtained high rank and explicit points on

\[ y^2 = x(x + 1)(x + t). \]
Our work

We generalize Ulmer’s work to Jacobians of superelliptic curves of higher genus.

For a certain family, our goal is to determine as much as we can about the arithmetic, such as rank, torsion, explicit points, and BSD.

In particular, we obtain large rank.
Consider

\[ C : y^r = x^{r-1}(x + 1)(x + t) \]

over \( \mathbb{F}_p(t) \) with \( p \) an odd prime, \( p \nmid r \) (\( r = 2 \) is Ulmer’s Legendre case). Then \( C \) has genus \( r - 1 \). Let \( J = \text{Jac}(C) \).

Note that

\[ \mathbb{Z}[\zeta_r] \hookrightarrow \text{End}(J) \]

via \( \zeta_r \mapsto [(x, y) \mapsto (x, \zeta_r y)] \).

**Theorem 1**

The full Conjecture of Birch and Swinnerton-Dyer holds for \( J \) over \( \mathbb{F}_q(t^{1/d}) \), for all \( d \) and for all \( q = p^a \).
For the remaining results, suppose that $r$ is a prime divisor of $d$ and $d = p^f + 1$ for some $f$. Take $q = p^a$ and let

$$K_d = \mathbb{F}_q(\mu_d, t^{1/d}).$$

**Theorem 2**

$$\text{rank}_{\mathbb{Z}} J(K_d) = (r - 1)(d - 2)$$
Let $u = t^{1/d}$, let $P_{0,0} := (u, u(u + 1)^{d/r})$, and more generally let

$$P_{i,j} := (\zeta_d^i u, \zeta_d^{i d/r + i} u(\zeta_d^i u + 1)^{d/r}) \in C(K_d)$$

with $i \mod d$ and $j \mod r$ (giving $dr$ points),

$$Q_\infty := \text{the point at infinity} \in C(K_d),$$

$$D_{i,j} := [P_{i,j}] - [Q_\infty] \in J(K_d).$$

**Theorem 3**

The $D_{i,j}$’s generate a subgroup of $J(K_d)$ of finite index.
Theorem 4

\[
\text{rank}_{\mathbb{Z}} J(\mathbb{F}_q(t^{1/d})) = (r - 1) \left[ \sum_{e \mid d} \frac{\varphi(e)}{o_q(e)} - \frac{2}{o_q(r)} \right]
\]

where

\[ o_q(e) := \text{the order of } q \text{ in } (\mathbb{Z}/e\mathbb{Z})^*. \]
Therefore over (fixed) $\mathbb{F}_q(u)$, Jacobians of curves of genus $r - 1$ have unbounded rank.

(This is for $r$ and $p$ such that $r \mid (p^f + 1)$ for some $f$, i.e., such that the order of $p$ mod $r$ is even, i.e., $-1 \in \langle p \rangle \subset (\mathbb{Z}/r\mathbb{Z})^*$.

For each $r$, we get at least half the primes $p$. Later we’ll see a different example without this restriction.)
**Theorem 5**

As $\mathbb{Z}[\zeta_r]$-modules,

$$J(K_d)_{\text{tors}} \cong \mathbb{Z}[\zeta_r]/(\zeta_r - 1) \times \mathbb{Z}[\zeta_r]/(\zeta_r - 1)^2.$$ 

So as abelian groups,

$$J(K_d)_{\text{tors}} \cong \begin{cases} (\mathbb{Z}/r\mathbb{Z})^3 & \text{if } r > 2, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} & \text{if } r = 2. \end{cases}$$
Decomposition and Endomorphisms

**Theorem 6**

If $r > 2$ then

$$J \sim B^2$$

where $B$ is an $(r - 1)/2$-dimensional absolutely simple abelian variety with real multiplication by $\mathbb{Q}(\zeta_r)^+$. 

**Theorem 7**

If $r > 2$ then

$$\text{End}(J) \otimes_{\mathbb{Z}} \mathbb{Q} \cong M_2(\mathbb{Q}(\zeta_r)^+)$$

$$\text{End}_{\bar{\mathbb{F}}_q(t)}(J) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}(\zeta_r).$$
Theorem 8

\[ L(J/\mathbb{F}_q(t), s) = 1, \]
\[ L(J/K_d, s) = (1 - q_1^{1-s})(r-1)(d-2) \]

where \( q_1 = |\mathbb{F}_q(\mu_d)| \), i.e., with \( T = q_1^{-s} \) we have

\[ L(T, J/K_d) = (1 - q_1 T)^{(r-1)(d-2)} \in \mathbb{Z}[T] \]
The case $r = 2$ gives the Legendre curve. For that case, all these results were shown earlier by Ulmer.

We generalize his statements and proofs.
Conjecture of Birch & Swinnerton-Dyer

Conjecture (BSD I)

\[ \text{rank}_{\mathbb{Z}} J(K) = \text{ord}_{s=1} L(J, s) \]

Conjecture (BSD II)

As \( s \to 1 \),

\[ L(J, s) \sim \frac{R|III|\tau}{|J(K)|^{2_{\text{tors}}}^{2}} (s - 1)^{r} \]

where \( r \) is the analytic rank, \( R \) is the regulator, \( III \) is the Tate-Shafarevich group, and \( \tau \) is the Tamagawa number.

For function fields it’s known that

\[ \text{rank}_{\mathbb{Z}} J(K) \leq \text{ord}_{s=1} L(J, s) \]

and that BSD I \( \iff \) BSD II, by Tate, Milne, . . . , Kato, Trihan.
The full Conjecture of Birch and Swinnerton-Dyer holds for $J$ over $\mathbb{F}_q(t^{1/d})$, for all $d$ and for all $q = p^a$.

\[ C \cong y^r = \frac{(x + 1)(x + t)}{x} \]

Let

\begin{align*}
C_d &: \beta^d = \alpha^r - 1, \\
D_d &: \delta^{-d} = \gamma^r - 1.
\end{align*}

Then

\[ C_d \times D_d \rightarrow y^r = \frac{(x + 1)(x + u^d)}{x} \]

\[(\alpha, \beta, \gamma, \delta) \mapsto (x, y, u) = (\alpha^r - 1, \alpha \gamma, \beta / \delta)\]
Thus, the surface $y^r = (x + 1)(x + u^d)/x$ over $\mathbb{F}_q$ is dominated by a product of curves $C_d \times D_d$.

The Tate Conjecture for the surface then follows, and this in turn implies (full) BSD for the Jacobian of the curve $y^r = (x + 1)(x + u^d)/x$ over $\mathbb{F}_q(u)$, for all $q = p^a$.

This gives (full) BSD for $J$ over $\mathbb{F}_q(t^{1/d})$, for all $q = p^a$ and all $d$. 

Sketch of Proof of BSD
Recall: $r$ is prime, $r \mid d = p^f + 1$ for some $f$, $q = p^a$, $K_d = \mathbb{F}_q(\mu_d, t^{1/d})$, $u = t^{1/d}$.

**Theorem 2**

$$\text{rank}_{\mathbb{Z}} J(K_d) = (r - 1)(d - 2)$$

**Theorem 3**

The $D_{i,j}$’s generate a subgroup of $J(K_d)$ of finite index, where

$$D_{i,j} := [P_{i,j}] - [Q_\infty] \in J(K_d)$$

with

$$P_{i,j} := (\zeta_d^i u, \zeta_d^{jd/r+i} u (\zeta_d^i u + 1)^{d/r}) \in C(K_d)$$

($i \mod d$ and $j \mod r$).
Sketch of Proof of Theorems 2 and 3

We compute the dimension of the image of \( \langle D_{i,j} \rangle \) under the \((\zeta_r - 1)\)-descent map (Poonen-Schaefer’s \((x - T)\) map):

\[
(x - T) : J(K_d)/(\zeta_r - 1)J(K_d) \hookrightarrow H^1(K_d, J[\zeta_r - 1]) \cong ((K_d[T]/(T(T + 1)(T + t)))^*)/(\ldots)^r \cong (K_d^*/(K_d^*)^r)^3
\]

\((x, y) \in C(K_d) \mapsto (x, x + 1, x + t)\)

where \([\cdot]_1\) denotes the kernel of the weighted norm map

\[
(x, y, z) \mapsto x^{r-1}yz = yz/x \in K_d^*/(K_d^*)^r.
\]
Sketch of Proof of Theorems 2 and 3

\[ \text{rank}_{\mathbb{Z}[\zeta_r]} J(K_d) = \]
\[ \dim_{\mathbb{F}_r} J(K_d)/(\zeta_r - 1) - \dim_{\mathbb{F}_r} J(K_d)_{\text{tors}}/(\zeta_r - 1) \]
\[ \geq \dim_{\mathbb{F}_r} ((x - T)(\langle D_{i,j} \rangle)) - 2 = d - 2 \]

giving

\[ \text{rank}_{\mathbb{Z}} J(K_d) = (r - 1) \text{rank}_{\mathbb{Z}[\zeta_r]} J(K_d) \geq (r - 1)(d - 2). \]
Then

\[(r - 1)(d - 2) \leq \text{rank} \leq (= \text{with BSD}) \text{ analytic rank} \leq \text{degree of } L\text{-function} = (r - 1)(d - 2)\]

giving a different proof of BSD.

A sketch of the proof of the last equality is as follows:
Combining work of Ulmer, Milne, and others, one gets that the degree of the $L$-function is

$$-4 \dim(J) + \deg(\text{cond}(J[\ell]))$$

for any prime $\ell \nmid 2pr$, and

$$\text{cond}(J[\ell]) = \sum_{x \in \mathbb{P}^1} (t_x + 2u_x)[x]$$

where $t_x$ is the dimension of the toric part of the special fiber (over $x$) of the Néron model of $J$, and $u_x$ is the dimension of the unipotent part.
Sketch of Proof of Theorems 2 and 3

We compute that the reduction of $J$ at $u = 0$ and $u = \infty$ is totally multiplicative and the reduction at the $d$ places $u^d = 1$ is half good and half additive. Thus,

$$
\deg(\text{cond}(J[\ell])) = \sum_{x \in \mathbb{P}^1} (t_x + 2u_x) = (r - 1) + (r - 1) + d \cdot 2 \cdot \frac{r - 1}{2} = (r - 1)(d + 2)
$$

so

$$
\deg(L\text{-function}) = -4 \dim(J) + \deg(\text{cond}(J[\ell])) = -4(r - 1) + (r - 1)(d + 2) = (r - 1)(d' - 2).
$$
Recall other Ranks Theorem

**Theorem 4**

\[
\text{rank}_\mathbb{Z} J(\mathbb{F}_q(t^{1/d})) = (r - 1) \left[ \sum_{e \mid d} \frac{\varphi(e)}{o_q(e)} - \frac{2}{o_q(r)} \right]
\]

where \(o_q(e)\) is the order of \(q\) in \((\mathbb{Z}/e\mathbb{Z})^*\).

**Sketch of Proof:** We know how \(\text{Gal}(K_d/\mathbb{F}_q(t^{1/d}))\) acts on the \(P_{i,j}\)'s and we know all the relations among the \(P_{i,j}\)'s (from our rank calculations), so we can compute

\[
\langle P_{i,j} \rangle_{\text{Gal}(K_d/\mathbb{F}_q(t^{1/d}))}.
\]
Recall Theorem on Torsion

**Theorem 5**

\[ J(K_d)_{\text{tors}} \cong \mathbb{Z}[\zeta_r]/(\zeta_r - 1) \times \mathbb{Z}[\zeta_r]/(\zeta_r - 1)^2 \]

In particular,

\[ J(K_d)_{\text{tors}} = J(K_d)[r^{\infty}] \]

and

\[ J(K_d)[\ell] = 0 \]

for all primes \( \ell \neq r \).
Let $Q_0 = (0, 0), Q_1 = (-1, 0), Q_t = (-t, 0)$. Then $[Q_i] - [Q_\infty]$ are $(\zeta_r - 1)$-torsion points for $i = 0, 1, t$.

We found a divisor $D \in \langle D_{i,j} \rangle$ such that

$$(\zeta_r - 1)D \sim [Q_0] - [Q_\infty].$$

We show that the $\mathbb{F}_r$-dimension of the image of the known $(\zeta_r - 1)\infty$-torsion under the $(\zeta_r - 1)$-descent map is 2; this shows we have all of it.

It’s generated over $\mathbb{Z}[\zeta_r]$ by $[Q_1] - [Q_\infty]$ and $D$, so

$$J(K_d)[r] \cong \mathbb{Z}[\zeta_r]/(\zeta_r - 1) \times \mathbb{Z}[\zeta_r]/(\zeta_r - 1)^2.$$
To show

\[ J(K_d)[\ell] = 0 \]

for all \( \ell \nmid 2pr \):

Use the geometry of the Néron model and group theory to understand the image of the mod \( \ell \) representation

\[ \text{Gal}(\overline{F}_q(t)(J[\ell])/\overline{F}_q(t)) \hookrightarrow \text{GL}_{2(r-1)}(\overline{F}_\ell). \]

We show \( J(L)[\ell] = 0 \) for all solvable extensions \( L \) of \( \overline{F}_q(t) \).
To show

\[ J(K_d)[p] = 0 : \]

We show that \( J \) is ordinary, i.e.,

\[ \#J(\overline{F_q(t)})[p] = p^{r-1}, \]

and calculate the Kodaira-Spencer map to show that

\[ J(\overline{F_p(t)}^{sep})[p] = 0. \]
To show

\[ J(K_d)[2] = 0 : \]

Use that \( C \) is isomorphic to the hyperelliptic curve

\[
y^2 = x^{2r} - 2(t + 1)x^r + t^2 - 2t + 1
\]

\[
= (x^r - (u^{d/2} + 1)^2)(x^r - (u^{d/2} - 1)^2)
\]

and apply a 2001 paper of Cornelissen, *Two-torsion in the Jacobian of hyperelliptic curves over finite fields.*
Decomposition and Endomorphisms

For $r > 2$:

**Theorem 6**

$J \sim B^2$ where $B$ is an $(r - 1)/2$-dimensional absolutely simple abelian variety with real multiplication by $\mathbb{Q}(\zeta_r)^+$. 

**Theorem 7**

\[
\text{End}(J) \otimes \mathbb{Q} \cong M_2(\mathbb{Q}(\zeta_r)^+) \\
\text{End}_{\overline{\mathbb{F}_q}(t)}(J) \otimes \mathbb{Q} \cong \mathbb{Q}(\zeta_r).
\]
We decompose

\[ J \sim \ker(\sigma - 1) \times \text{im}(\sigma - 1) \sim B^2 \]

for the involution

\[ \sigma : (x, y) \mapsto (-1 - \frac{v^r}{x + 1}, \frac{v^2}{y}) \]

where \( v^r = t - 1 \), and use group theory to show that \( B \) is absolutely simple and has endomorphism algebra \( \mathbb{Q}(\zeta_r)^+ \).

The isogeny, and all endomorphisms, are defined over

\[ \mathbb{F}_q(\mu_r, v) = \mathbb{F}_q(\zeta_r, (t - 1)^{1/r}) \].
\( L \)-function

**Theorem 8**

\[
L(J/\mathbb{F}_q(t), s) = 1,
\]

\[
L(T, J/K_d) = (1 - q_1 T)^{(r-1)(d-2)} \in \mathbb{Z}[T],
\]

\[
L(J/K_d, s) = (1 - q_1^{1-s})^{(r-1)(d-2)}
\]

*where* \( q_1 = |\mathbb{F}_q(\mu_d)| \).

This follows since we showed that

\[
\deg(L(T, J/K_d)) = (r - 1)(d - 2)
\]

\[
= \text{rank}_{\mathbb{Z}} J(K_d) = \text{analytic rank}
\]

and we can similarly show that \( \deg(L(T, J/\mathbb{F}_q(t))) = 0 \).
Another Example

We started with:

\[ y^2 = x \prod_{i=1}^{g} (x + a_i)(a_i x + t) \]

or more generally

\[ C : y^2 = x h(x) x^g h(t/x) \]

where the genus \( g \) is odd, \( h(x) \in \mathbb{F}_q[x] \) has degree \( g \) and distinct roots, \( h(0) \neq 0 \), \( q = p^a \) with \( p \) an odd prime. Let

\[ J = \text{Jac}(C). \]
If

- $d = p^f + 1$ for some $f$,
- $h$ splits completely over $\mathbb{F}_q(\mu_d, t^{1/d})$, and
- $u^d = t$,

then

$$\left(\zeta_d^i u, \left(\zeta_d^i u\right)^{g+1 \over 2} h(\zeta_d^i u)^{d/2}\right) \in C(K_d).$$
We computed the image of the points we know, under the 2-descent map:

\[ J(K_d)/2J(K_d) \hookrightarrow H^1(K_d, J[2]) \cong \left[ (K_d[T]/(h(T)h(t/T)T^{g+1}))^* / (\ldots)^2 \right]_1 \]

\[ D \mapsto [\sigma \mapsto \sigma(D_1) - D_1] \text{ where } 2D_1 \sim D \]

\[ \dim_{\mathbb{F}_2}(\text{image}) \leq \dim_{\mathbb{F}_2}(J(K_d)/2J(K_d)) \]

\[ = \text{rank } J(K_d) + \dim_{\mathbb{F}_2}(J(K_d)[2]) \]

We showed

\[ \text{rank } J(K_d) \geq d - 2. \]
This gives unbounded rank over $\mathbb{F}_p(t)$ for all primes $p$, for dimension $g$ abelian varieties.

But the degree of the $L$-function is large compared to $d - 2$, so the rank might be a lot larger than $d - 2$.

That’s why we decided to consider a different example.
Possible Future Work

- Compute the regulator of the subgroup generated by the known points.
- Compute the index (in the full Mordell-Weil group) of the points we know.
- Compute other BSD data, such as $\zeta_r - 1$. (A full $(\zeta_r - 1)$-descent would give $|\text{III}[\zeta_r - 1]|$).
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