EULER SQUARES.

By Harris F. MacNeish.

1. **Introduction.** Euler Squares were first considered in a paper, "Recherches sur une espèce de carrés magique," Commentationes Arithmeticæ Collectæ, 1849, vol. II, pp. 302–361. In this paper Euler proposed the following problem now well known as "The problem of the 36 officers."* Six officers of six different ranks are chosen from each of six different regiments. It is required to arrange them in a solid square so that no officer of the same rank or of the same regiment shall be in the same row or in the same column. The problem is equivalent to that of arranging 36 pairs of integers, each less than or equal to six, in a square array so that the first (or second) numbers of the pairs in any row or column are all distinct, and no two pairs are identical. Such a square array would be called an Euler Square of index 6, 2.

In this paper we shall be concerned with more general squares defined as follows. An Euler square of order n, degree k and index n, k is a square array of n^2 k-ads of numbers, $(a_{ij1}, a_{ij2}, \cdots, a_{ijk})$, where $a_{ijr} = 1, 2, \cdots, n$; $r = 1, 2, \cdots, k$; $i, j = 1, 2, \cdots, n$; $n > k$; $a_{ipr} \neq a_{iqr}$ and $a_{pjr} \neq a_{qjr}$ for $p \neq q$ and $a_{ijr}a_{ijs} \neq a_{pqr}a_{pqs}$ for $i \neq p$ and $j \neq q$.

The impossibility of constructing squares of index n, 2 for $n \equiv 2 \pmod 4$ was stated without proof by Euler in the paper referred to above. A very laborious proof for index 6, 2 obtained by combining two squares of index 6, 1 has been given by G. Tarry (Mathesis, vol. 20, July, 1901). A geometrical proof by methods of Analysis Situs has been given by J. Petersen (Annuaire des Mathématiciens, 1901–02, pp. 413–426). A third method is given for index n, 2, $n \equiv 2 \pmod 4$, by P. Wernicke, "Das Problem der 36 Offiziere," Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 19, 1910, p. 264. The method of Wernicke is proved to be incorrect in an article under the same title in the same journal, vol. 31, 1922, p. 151, by H. F. MacNeish. An Euler Square of degree one is called a Latin Square and of degree two a Græco-Latin Square.

We shall show how to construct Euler squares for the following cases: (A) Index p, $p - 1$ for p prime; (B) Index p^n, $p^n - 1$ for p prime; (C) Index n, k, where $n = 2^r p_1^{r_1} p_2^{r_2} \cdots$ for p_1, p_2, $\cdots$ distinct odd primes and where $k + 1$ equals the least of the numbers 2^r, $p_1^{r_1}$, $p_2^{r_2}$, $\cdots$. (The

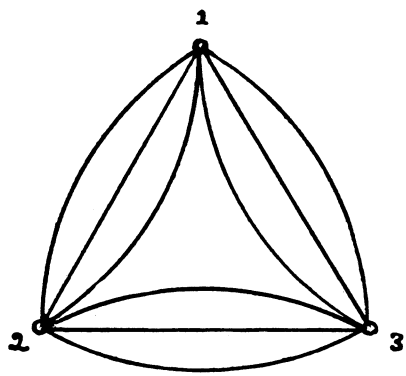
* Cf. Ahrens, Math. Unterhaltungen und Spiele. Leipzig, 1901, Chap. XIII. Encyc. des Sci. Math., Tome I, vol. 3, Fasc. I, p. 72.

proof that type (C) is impossible for degree greater than this value of k is a generalization of the Euler problem of the 36 officers which has not been proved. The simplest case would be to prove that the Euler Square of index 12, 3 is impossible.)

2. **A geometrical interpretation of the Euler Square.** For simplicity we consider first the Euler Square of index 3, 2,

1, 1	2, 2	3, 3,
2, 3	3, 1	1, 2,
3, 2	1, 3	2, 1.

The generalization to index n, 2 offers no difficulty. We shall consider the numbers 1, 2, 3 as representing points, and the first column omitting 1, 1 as representing the triangles 1, 2, 3 and 1, 3, 2 where the order of the numbers following 1 is the same as the order of the numbers in the number pairs in the Euler Square. Also in triangle 1, 2, 3 for instance 1 shall be called the first vertex, 2 the second vertex, 3 the third vertex; 1, 2 shall be called the first side, 2, 3 the second side and 3, 1 the third side. To make a diagram in a plane representing the six triangles of this Euler Square, the first sides shall be drawn as straight lines, the second sides as arcs bending outward, the third sides as arcs bending inward; giving the following figure in which segment ij is the same as segment ji only when they are both first sides, second sides, or third sides.



In a more complicated figure the second sides instead of bending outward might be represented by red lines or dotted lines, and the third sides by blue lines or dashed lines.

Evidently then each segment has precisely two regions abutting upon it, for ij is an rth side in but one triangle and ji is an rth side in but one triangle and the two triangles are distinct.

We shall also consider any segment as positively or negatively related to a triangle which it abuts according as the numbers specifying that side in the notation for the triangle occur in the cyclic order (123) or the cyclic order (132); and a point as positively or negatively related to a segment which it terminates according as it is the first or the second point in the notation for the segment as chosen above.

This Euler Square therefore represents a closed two-sided two-dimensional complex (see "Manifolds of n dimensions," O. Veblen and J. W. Alexander, Annals of Math., vol. 14, p. 164) and the two matrices $A_{0,1}$ and $A_{1,2}$ defining it are as follows, where 1 indicates incidence and positive relation, -1 indicates incidence and negative relation and 0 indicates non-incidence:

		Lines as first sides			Lines as second sides			Lines as third sides		
	Points	1, 2	2, 3	3, 1	1, 2	2, 3	3, 1	1, 2	2, 3	3, 1
	1	1	0	−1	1	0	−1	1	0	−1
$A_{0,1}$:	2	−1	1	0	−1	1	0	−1	1	0
	3	0	−1	1	0	−1	1	0	−1	1

	Triangles	123	132	231	213	312	321
	First Sides						
	12	1	0	0	−1	0	0
	23	0	0	1	0	0	−1
	31	0	−1	0	0	1	0
	Second Sides						
$A_{1,2}$:	12	0	0	0	0	1	−1
	23	1	−1	0	0	0	0
	31	0	0	1	−1	0	0
	Third Sides						
	12	0	−1	1	0	0	0
	23	0	0	0	−1	1	0
	31	1	0	0	0	0	−1

The Euler Square specifies all of the incidence relations of the configuration given by these two matrices in a more compact form.

3. **The Euler Square of index n, 2 for $n \equiv 2 \pmod 4$ is impossible.** From paragraph 2 in the general case the Euler Square of index n, 2 represents a closed two-sided two-dimensional complex with n points, $3n(n-1)/2$ segments and $n(n-1)$ triangular regions.

If the complex is a single two-dimensional circuit (loc. cit., Veblen and Alexander, p. 166), the configuration is a polyhedral region and the $a_0 = n$

points, $a_1 = 3n(n-1)/2$ segments and $a_2 = n(n-1)$ regions satisfy the relation

$$a_0 - a_1 + a_2 = 2 - 2p \tag{1}$$

for some positive integral value of p, in which case p represents the genus of the surface of the polyhedral region. (See Veblen and Young, "Projective Geometry," vol. II, § 188.)

We shall consider the values of p for various Euler squares. If p is not a positive integer no configuration exists of the above type, hence no Euler square exists. If the square of index n, 2 does not exist, then the square of index n, k for $k > 2$ cannot exist; hence we shall first consider squares of index n, 2.

(A) For an Euler Square of index n, 2, if the configuration is a single two-dimensional circuit,

$$a_0 = n, \qquad a_1 = \frac{3}{2}n(n-1), \qquad a_2 = n(n-1).$$

From (1)

$$n - \frac{3}{2}n(n-1) + n(n-1) = 2 - 2p.$$

Then

$$p = 1 + \frac{1}{4}n(n-3).$$

Therefore n must have the form $4k$ or $4k+3$.

(B) If the configuration is separable into m two-dimensional subcircuits, each of the n vertices must occur in the same number m' of circuits. For one of these circuits $a_0 = n_i$, $a_1 = 3k_i/2$, $a_2 = k_i$, therefore

$$n_i - k_i/2 = 2 - 2g_i.$$

Taking the sum of the m equations of this type,

$$m'n - n(n-1)/2 = 2m - 2\sum g_i,$$

or

$$n(2m' - n + 1) = 4(m - \sum g_i).$$

Therefore n must be a multiple of 4 or $2m' - n + 1$ must be a multiple of 4, in which latter case n must be an odd integer.

In neither case (A) nor (B) can n have the form $4k + 2$, therefore the Euler Square is impossible for order $n \equiv 2 \pmod 4$.

If a configuration representing a single circuit determined as above by an Euler Square of index n, 2 be projected on a surface of the same genus so that none of its segments intersect, since at each vertex the same number of segments $3(n-1)$ and the same number of regions $n-1$ meet, there is determined a regular reticulation of the surface. H. S. White has considered regular reticulations for surfaces of genus $p = 2, 3, \cdots, 9$. Whenever the genus determined by an Euler Square lies in that interval, the corresponding reticulation appears in his list. (H. S. White, "Numerically Regular Reticulations upon Surfaces of Deficiency Higher than One," Bull. Amer. Math. Soc., vol. 3, p. 116, vol. 4, p. 376.)

The following is a table of the genus of the surfaces upon which Euler Squares of order $n = 3, 4, \cdots, 12$ may be developed:

Index	3, 2	4, 2	5, 2	7, 2	7, 2	8, 2	9, 2	11, 2	11, 2	12, 2
Genus	1	2	1 2 circuits	8	1 3 circuits	11	1 4 circuits	23	1 5 circuits	28

4. **Methods of constructing Euler squares.** As the members of the first row are arbitrary subject to the restrictions of the definition, the numbers of the ith k-ad of the first row may all be taken equal to i merely by proper choice of notation. Also since the rows may be permuted the initial members of the first column are taken in the numerical order $1, 2, 3, \cdots$. Furthermore the second k-ad of the first column may be taken in the numerical order $2, 3, 4, \cdots$ since the same permutation may evidently be applied to all the k-ads of an Euler Square.

(A) Suppose $n = p$, p a prime > 2. Call G_1 the cyclic group of powers of the substitution $S_1 = (1, 2, 3, \cdots, n)$, and G_2 the cyclic group of the powers of a substitution S_2 of the numbers $2, 3, \cdots, n$, omitting 1, so chosen that it does not send any two numbers to the same two numbers as any substitution of G_1. For $n = 3$ or $n = 5$ there is only one choice for S_2, for $n = 7$ there are 7 choices and the number of choices increases rapidly with n. To construct the Euler Square of index n, $n - 1$ apply the substitutions of G_2 to the $(n - 1)$-ad $2, 3, 4, \cdots, n$ which was chosen as the second member of the first column, to obtain the remaining members of the first column, then apply the substitutions of G_1 to the first column to obtain the other columns.

S_1 and S_2 generate a group G of degree n and order $n(n - 1)$ called the group of the Euler Square. All of the $n(n - 1)$ members of the Euler Square omitting the first row may be obtained by applying the substitutions of G to the $(n - 1)$-ad $2, 3, \cdots, n$. For example for $n = 5$, the Euler Square of index 5, 4 is obtained from $S_1 = (1, 2, 3, 4, 5)$ and $S_2 = (2, 3, 5, 4)$ as follows:

1, 1, 1, 1	2, 2, 2, 2	3, 3, 3, 3	4, 4, 4, 4	5, 5, 5, 5
2, 3, 4, 5	3, 4, 5, 1	4, 5, 1, 2	5, 1, 2, 3	1, 2, 3, 4
3, 5, 2, 4	4, 1, 3, 5	5, 2, 4, 1	1, 3, 5, 2	2, 4, 1, 3
4, 2, 5, 3	5, 3, 1, 4	1, 4, 2, 5	2, 5, 3, 1	3, 1, 4, 2
5, 4, 3, 2	1, 5, 4, 3	2, 1, 5, 4	3, 2, 1, 5	4, 3, 2, 1

In a similar manner an Euler Square can be constructed of index p, $p - 1$ for any prime p.

Remark. A cyclic group of even order has a subgroup of order 2. Therefore any Euler Square of order $2k + 1$ is separable into k Euler Rectangles, because G_2 is a cyclic group of order $2k$ and hence has a subgroup G_3 of order 2. Each Euler Rectangle will give a separate circuit in the configuration, hence an Euler Square of order $2k + 1$ represents k circuits on a surface of genus 1, for

$$a_0 = 2k + 1, \qquad a_1 = 3k(2k + 1), \qquad a_2 = 2k(2k + 1).$$

Therefore from (1) $p = 1$.

For instance, in the square of index 5, 4 above, the first, second and fifth rows form one Euler Rectangle and the first, third and fourth another. In the ith column of an Euler Rectangle the numbers except i occur a number of times equal to the order of the sub-group G_3, hence each number does not appear in every position of the k-ads of a column as is the case in an Euler Square.

(B) Suppose $n = p^r$, p a prime. In this case G_1 cannot be chosen as a cyclic group, but may be chosen as a group of substitutions which are products of p^{r-1} cycles of p numbers each; while G_2 may be chosen as a cyclic group fulfilling the same conditions as in (A), i.e., its substitutions must not transform any two numbers to the same two numbers as any substitution of G_1.

For example, for $n = 2^3$ let G_1 consist of the identity and the following substitutions:

$$\begin{aligned} A &= (12)(34)(56)(78), & B &= (13)(24)(57)(68), \\ C &= (14)(23)(58)(67), & D &= (15)(26)(37)(48), \\ E &= (16)(25)(38)(47), & F &= (17)(28)(35)(46), \\ H &= (18)(27)(36)(45), \end{aligned}$$

and let G_2 be the cyclic group of powers of the substitution $S_2 = (2354786)$; several other choices for S_2 are possible. G_1 and G_2 determine the group of the Euler Square of index 8, 7 by the method given in (A). As a second example, for $n = 3^2$ let G_1 consist of the identity and the following substitutions:

$$\begin{aligned} A &= (123)(468)(597), & B &= (132)(486)(579), \\ C &= (145)(269)(387), & D &= (154)(296)(378), \\ E &= (167)(285)(349), & F &= (176)(258)(394), \\ H &= (189)(247)(365), & J &= (198)(274)(356), \end{aligned}$$

and let G_2 be the cyclic group of powers of the substitution $S_2 = (24693578)$; several other choices for S_2 are possible. G_1 and G_2 generate the Euler Square of index 9, 8.

By the method illustrated in these two examples an Euler Square can be constructed of index p^r, $p^r - 1$ for p any prime.

(*C*) Let $n = 2^r p_1^{r_1} p_2^{r_2} \cdots$; $r, r_1, r_2, \cdots$ positive integers, $r \neq 1$ and $p_1, p_2, \cdots$ distinct odd prime numbers.

Jordan has proved the following theorem (Recherches sur les Substitutions, Liouville Jr. de Math., vol. XVII, 1873, p. 355): "A transitive group of degree n and order $n(n-1)$ whose operations other than the identity displace all or all but one of the symbols can exist only when n is a power of a prime." From this theorem the method used in (*A*) and (*B*) cannot be extended to case (*C*).

For this case we shall use the following method, which is an extension of the method used by G. Tarry (Ahrens, loc. cit.) for degree 2, by combining two Euler Squares of orders a and b to obtain one of order ab; which is similar to the method used for combining two magic squares.

The method may be illustrated as follows, using Euler Squares of indices 5, 3 and 4, 3 to obtain a square of index 20, 3. Given the Euler Square of index 5, 3 as follows:

1, 1, 1	2, 2, 2	3, 3, 3	4, 4, 4	5, 5, 5
2, 3, 4	3, 4, 5	4, 5, 1	5, 1, 2	1, 2, 3
3, 5, 2	4, 1, 3	5, 2, 4	1, 3, 5	2, 4, 1
4, 2, 5	5, 3, 1	1, 4, 2	2, 5, 3	3, 1, 4
5, 4, 3	1, 5, 4	2, 1, 5	3, 2, 1	4, 3, 2

decrease by one all of the numbers of the Euler Square of index 4, 3 given in paragraph 1, giving the following square array:

0, 0, 0	1, 1, 1	2, 2, 2	3, 3, 3
1, 2, 3	0, 3, 2	3, 0, 1	2, 1, 0
2, 3, 1	3, 2, 0	0, 1, 3	1, 0, 2
3, 1, 2	2, 0, 3	1, 3, 0	0, 2, 1,

then replace each triple i, j, k of this array by an entire Euler Square of index 5, 3 obtained from the above Euler Square of index 5, 3 by adding to each of its 25 number triples the numbers $5i$, $5j$, $5k$ respectively. In general by this method we will obtain an Euler Square of index n, k where $k + 1$ is the least of the numbers $2^r, p_1^{r_1}, p_2^{r_2}, \cdots$.

The Euler Square of index n, k gives a schedule for a contest between k teams of n members each, where each member is to meet each member of the other teams precisely once, and each member is to participate but once at each field (table, court, etc.) (see E. H. Moore, "Tactical Memoranda, III," Amer. Jr. of Math., vol. XVIII, 1896, p. 264).