Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 11; February 19, 2007

Define f as follows: $f(x) = \exp(-1/x^2)$ for $x \neq 0$, and $f(0) = 0$. Show that

(a) f is continuous for all x.

Solution: Since f is differentiable everywhere (by (b), which doesn't use (a) in its proof), f is continuous.

(b) The n-th derivative of f is continuous for all x and vanishes at $x = 0$, for all $n \geq 1$.

Solution: The only statement that needs proof is $f^{(k)}(0) = 0$,

NOTE: This solution is taken from R. P. Boas, *A Primer of Real Functions, p. 154*. Another solution is given as the solution to Exercise 31.5 in the book by Ross. (I prefer the solution in Boas.)

1. It is easy to see by induction that for $x \neq 0$, $f^{(k)}(x) = R(x)e^{-1/x^2}$ where R is a rational function, $R(x) = P(x)/Q(x)$ for polynomials P and Q.
2. For any integer n, positive, negative, or zero, $x^n e^{-1/x^2} \to 0$ as $x \to 0$. Indeed, for $n = 0$, $e^{-1/x^2} \to 0$; for $n > 0$, $x^n \to 0$ and $e^{-1/x^2} \leq 1$; and for $n < 0$, let $t = 1/x$ to get $x^n e^{-1/x^2} = t^{-n}/e^{t^2} \to 0$ as $t \to \infty$ by L'Hopital's rule.
3. LEMMA: If $\lim_{x\to y} f'(x)$ exists as a finite real number, then $f'(y)$ exists and equals this limit.
   PROOF: By the mean value theorem,
   $$\frac{f(x) - f(y)}{x - y} = f'(t) \text{ for some } t \text{ between } x \text{ and } y. \tag{1}$$
   As x approaches y, so does t, so the right side of (1) approaches a limit. By the left side of (1), this limit is $f'(y)$.
4. By 1. and 2. , $f^{(k)}(x) \to 0$ as $x \to 0$, hence by 3. , $f^{(k)}(0) = 0$. To prove that $f^{(k)}(x) \to 0$, write
   $$f^{(k)}(x) = \frac{P(x)}{Q(x)}\, e^{-1/x^2} = \frac{a_n x^m + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}\; e^{-1/x^2}.$$
   If $b_0 \neq 0$, then
   $$\lim_{x\to 0} f^{(k)}(x) = \frac{\lim_{x\to 0}(a_n x^m + a_{n-1}x^{n-1} + \cdots + a_0)e^{-1/x^2}}{\lim_{x\to 0}(b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0)} = \frac{0}{b_0} = 0.$$
   If $b_0 = 0$, write $Q(x) = x^k(b_m x^{m-k} + \cdots + b_k)$, where $b_k \neq 0$. Then
   $$\lim_{x\to 0} f^{(k)}(x) = \frac{\lim_{x\to 0} x^{-k}(a_n x^m + a_{n-1}x^{n-1} + \cdots + a_0)e^{-1/x^2}}{\lim_{x\to 0}(b_m x^{m-k} + b_{m-1}x^{m-1-k} + \cdots + b_k)} = \frac{0}{b_k} = 0.$$