Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 12; February 26, 2007

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be the sum of a power series with radius of convergence 1. Suppose that f vanishes on an infinite sequence t_n in $(-1, 1)$ which converges to a point in $t_0 \in (-1, 1)$. Show that f is identically zero in $(-1, 1)$.

Solution: Since f is continuous on $(-1, 1)$, $f(t_0) = \lim_{n\to\infty} f(t_n) = 0$. We may assume, by passing to a subsequence if necessary, that $t_n \neq t_0$ for all n.

We know from properties of convergent power series that $a_k = f^{(k)}(0)/k!$ for $k = 0, 1, 2, \ldots$. Moreover by the integral form of Taylor's theorem (Theorem 31.5), the remainder

$$R_n(x) = \int_0^x \frac{(x-t)^{n-1} f^{(n)}(t)}{(n-1)!}\, dt \to 0 \text{ as } n \to \infty \text{ for all } |x| < 1.$$

From the theorem proved in class today (analog of Theorem 31.5 with the origin replaced by any other number, in this case t_0)

$$f(x) = f(t_0) + f'(t_0)(x - t_0) + \cdots + \frac{f^{(n-1)}(t_0)}{(n-1)!}(x - t_0)^{n-1} + R_n(x, t_0),$$

where

$$R_n(x, t_0) = \int_{t_0}^x \frac{(x-t)^{n-1} f^{(n)}(t)}{(n-1)!}\, dt \to 0 \text{ for all } |x| < 1.$$

By a property of Riemann integrals, $R_n(x, t_0) = R_n(x) - R_n(t_0) \to 0$ for all $|x| < 1$ and $|t_0| < 1$. Therefore

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(t_0)}{k!}(x - t_0)^k \text{ for all } |x| < 1 \text{ and } |t_0| < 1.$$

Writing b_k for $\frac{f^{(k)}(t_0)}{k!}$, either $b_k = 0$ for all $k \geq 0$, or there is a smallest m such that $b_m \neq 0$. In the latter case, we may therefore write

$$f(x) = (x - t_0)^m g(x),$$

where $g(x) = b_m + b_{m+1}(x - t_0) + \cdots$ and $g(t_0) = b_m \neq 0$. Since g is continuous at t_0, there exists $\delta > 0$ such that $g(x) \neq 0$ whenever $|x - t_0| < \delta$. For this δ there exists N such that $|t_n - t_0| < \delta$ for $n \geq N$. Thus, for $n \geq N$, we have the contradiction

$$0 = f(t_n) = (t_n - t_0)^m g(t_n) \neq 0.$$

Because of this contradiction, we must have $b_k = 0$ for all $k \geq 0$, and therefore $f(x) = 0$ for all $|x| < 1$. □