Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 14; February 26, 2007

Exercise 29.2, page 220

Prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbf{R}$.

Solution: For any $x \neq y$, by the Mean Value Theorem

$$\frac{\cos x - \cos y}{x - y} = \sin c \text{ for some } c \text{ between } x \text{ and } y.$$

Since $|\sin x| \leq 1$ for every x, we are done.

Exercise 29.4, page 221

Let f and g be differentiable functions on an open interval I. Suppose that $a, b \in I$ satisfy $a < b$ and $f(a) = f(b) = 0$. Show that $f'(x) + f(x)g'(x) = 0$ for some $x \in (a, b)$. Hint: Consider $h(x) = f(x)e^{g(x)}$.

Solution: $h(a) = h(b) = 0$ so by Rolle's theorem, $h'(x) = 0$ for some $x \in (a, b)$. But $h'(x) = f(x)e^{g(x)}g'(x) + f'(x)e^{g(x)} = e^{g(x)}[f(x)g'(x) + f'(x)]$ and $e^{g(x)} \neq 0$.

Exercise 29.6, page 221

Give the equation of the straight line used in the proof of the Mean Value Theorem 29.3.

Solution: $L(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x - a)$.

Exercise 29.10, page 221

Let $f(x) = x^2 \sin(\frac{1}{x}) + \frac{x}{2}$ for $x \neq 0$ and $f(0) = 0$.

(a) Show that $f'(0) > 0$

Solution:

$$\frac{f(x) - f(0)}{x - 0} = x \sin \frac{1}{x} + \frac{1}{2} \to \frac{1}{2} \text{ as } x \to 0.$$

(b) Show that f is not increasing on any open interval containing 0.

Solution: Proof by contradiction. Suppose a function f is differentiable and increasing on some open interval Then its derivative is non-negative on that interval. (This is like a converse to Corollary 29.7(iii)). (PROOF: If a and $a+h$ belong to the interval with $h > 0$, then $[f(a + h) - f(a)]/h \geq 0$ so $f'(a) \geq 0$.)

For our function f, for $x \neq 0$, $f'(x) = -\cos \frac{1}{x} + 2x \sin \frac{1}{x} + \frac{1}{2}$. Given any open interval containing 0, choose x_0 in that interval such that $\cos \frac{1}{x_0} = 1$ and $|2x_0| < 1/4$. Thus $|f'(x_0) + \frac{1}{2}| = |f'(x_0) + \cos \frac{1}{x_0} - \frac{1}{2}| = |2x_0 \sin \frac{1}{x_0}| < \frac{1}{4}$, showing that $f'(x_0) \leq -1/4$, contradicting the previous paragraph.

(c) Compare this example with Corollary 29.7(i).

Solution: Corollary 29.7(i) requires a positive derivative <u>throughout</u> the interval.