Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 15; February 27, 2007

Exercise 29.8, page 221

Prove (ii)-(iv) of Corollary 29.7

Solution:

(ii) If $a < x_1 < x_2 < b$, then $[f(x_2) - f(x_1)]/(x_2 - x_1) = f'(c) < 0$ for some $c \in (x_1, x_2)$. Therefore $x_1 < x_2 \Rightarrow x_2 - x_1 > 0 \Rightarrow f(x_2) - f(x_1) < 0 \Rightarrow f(x_1) > f(x_2)$.

(iii) If $a < x_1 < x_2 < b$, then $[f(x_2) - f(x_1)]/(x_2 - x_1) = f'(c) \geq 0$ for some $c \in (x_1, x_2)$. Therefore $x_1 < x_2 \Rightarrow x_2 - x_1 > 0 \Rightarrow f(x_2) - f(x_1) \geq 0 \Rightarrow f(x_1) \leq f(x_2)$.

(iv) If $a < x_1 < x_2 < b$, then $[f(x_2) - f(x_1)]/(x_2 - x_1) = f'(c) \leq 0$ for some $c \in (x_1, x_2)$. Therefore $x_1 < x_2 \Rightarrow x_2 - x_1 > 0 \Rightarrow f(x_2) - f(x_1) \leq 0 \Rightarrow f(x_1) \geq f(x_2)$.

Exercise 29.12, page 221

(a) Show that $x < \tan x$ for all $x \in (0, \pi/2)$.

Solution: Let $f(x) = \tan x - x$. Then $f'(x) = \sec^2 x - 1 > 0$ for all $x \in (0, \pi/2)$. Therefore f is strictly increasing on $(0, \pi/2)$, that is $f(x_1) < f(x_2)$ whenever $0 < x_1 < x_2 < \pi/2$. Now let $x_1 \to 0$. Since $f(x_1)$ is decreasing as $x_1 \to 0$, $0 = f(0) = \lim_{x_1 \to 0} f(x_1) < f(x_2)$, that is $f(x) > 0$ for all $x \in (0, \pi/2)$. This is the same as $x < \tan x$.

(b) Show that $x/\sin x$ is a strictly increasing function on $(0, \pi/2)$.

Solution: If $f(x) = x/\sin x$, then $f'(x) = (\sin x - x\cos x)/\sin^2 x$. By part (a) $\sin x > x \cos x$ so $f'(x) > 0$. Therefore f is strictly increasing on $(0, \pi/2)$.

(c) Show that $x \leq \frac{\pi}{2} \sin x$ for $x \in [0, \pi/2]$.

Solution: Equality holds at the endpoints $0, \pi/2$ and $x/\sin x$ is (strictly) increasing on $(0, \pi/2)$ by part (b). Hence, if $0 < x < y < \pi/2$, we have

$$\frac{x}{\sin x} < \frac{y}{\sin y} \text{ and } \frac{x}{\sin x} < \lim_{y \to \pi/2} \frac{y}{\sin y} = \frac{\pi/2}{1} = \pi/2.$$

Exercise 29.14, page 221

Suppose that f is differentiable on $\mathbf{R}$, that $1 \leq f'(x) \leq 2$ for $x \in \mathbf{R}$, and that $f(0) = 0$. Prove that $x \leq f(x) \leq 2x$ for all $x \geq 0$.

Solution: Let $g(x) = 2x - f(x)$, so that $g'(x) = 2 - f'(x) \geq 0$ and therefore g is increasing on $\mathbf{R}$. Since $g(0) = 0$, $g(x) \geq 0$ for $x \geq 0$, thus $f(x) \leq 2x$ for all $x \geq 0$.

Let $hx) = f(x) - x$, so that $h'(x) = f'(x) - 1 \geq 0$ and therefore h is increasing on $\mathbf{R}$. Since $h(0) = 0$, $h(x) \geq 0$ for $x \geq 0$, thus $x \leq f(x)$ for all $x \geq 0$.

Exercise 29.18, page 222

Let f be differentiable on $\mathbf{R}$ with $a = \sup\{|f'(x)| : x \in \mathbf{R}\} < 1$. Select $s_0 \in \mathbf{R}$ and define $s_n = f(s_{n-1})$ for $n \geq 1$. Prove that (s_n) is a convergence (sic) sequence.

Solution: $s_{n+1} - s_n = f(s_n) - f(s_{n-1}) = (s_n - s_{n-1})f'(c_n)$ so that $|s_{n+1} - s_n| \leq |s_n - s_{n-1}|a$. Continuing you get $|s_{n+1} - s_n| \leq |s_1 - s_0|a^n$. Next, you get $|s_{n+m} - s_n| \leq |s_1 - s_0| \sum_{k=n}^{n+m-1} a^k = |s_1 - s_0| \frac{a^n(1-a^m)}{1-a} \leq a^n/(1-a)$ for every $m \geq 1$. Since $a^n \to 0$, (s_n) is a Cauchy sequence, hence convergent.