Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 17; February 27, 2007

(a) Let f be a pure jump function with jumps $j_n = 1/2^n$ at $x_n = 1/3^n$. Show that $f(0) = 0$. Then assuming that the right hand derivative of f at zero, namely

$$f'_+(0) := \lim_{h \to 0+} \frac{f(h) - f(0)}{h}$$

exists, show that it is equal to $+\infty$.

Solution: Recall that a pure jump function is a function of the form $f = \sum_{n=1}^{\infty} f_n$, where $f_n(x) = j_n$ for $x \geq x_n$ and $f_n(x) = 0$ for $x < x_n$. Here, (x_n) is a sequence of real numbers and (j_n) is a sequence of positive numbers with $\sum_n j_n < \infty$.

It is obvious that $f(0) = 0$ since $f_n(0) = 0$ for all $n \geq 1$. Now, taking a sequence $h_m \to 0+$, where $h_m \in (3^{-(m+1)}, 3^{-m})$, you have $f(h_m) = \sum_{k=m+1}^{\infty} 2^{-k} = 2^{-m}$ so that $f(h_m)/h_m = 2^{-m}/h_m > 2^{-m}3^m = (3/2)^m \to +\infty$.

(b) Let g be a pure jump function with jumps $j_n = 1/3^n$ at $x_n = 1/2^n$. Show that $g(0) = 0$. Then assuming that the right hand derivative of g at zero, namely

$$g'_+(0) := \lim_{h \to 0+} \frac{g(h) - g(0)}{h}$$

exists, show that it is equal to 0.

Solution: As in part (a), it is obvious that $g(0) = 0$. Now, taking a sequence $h_m \to 0+$, where $h_m \in (2^{-(m+1)}, 2^{-m})$, you have $g(h_m) = \sum_{k=m+1}^{\infty} 3^{-k} = 3^{-m}$ so that $g(h_m)/h_m = 3^{-m}/h_m < 3^{-m}2^m = (2/3)^m \to 0$.