Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 2; January 22, 2007

Exercise 24.4, page 183

For $x \in [0, \infty)$, let $f_n(x) = \frac{x^n}{1+x^n}$.

(a) Find $f(x) = \lim_n f_n(x)$

Solution:

- $f(0) = 0$ and $f(1) = 1/2$
- For $0 < x < 1$, $x^n \to 0$, so $f(x) = 0$.
- For $1 < x$, $x^n \to \infty$ so
$$f_n(x) = \frac{1}{\frac{1}{x^n} + 1} \to 1.$$
Thus
$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1/2 & x = 1 \\ 1 & 1 < x \end{cases}$$

(b) Determine whether $f_n \to f$ uniformly on $[0, 1]$.

Solution: The answer is no, by Theorem 24.3, since f is not continuous on $[0, 1]$.

(c) Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

Solution: The answer is no, by Theorem 24.3, since f is not continuous on $[0, \infty]$.

Important additional problem: Is the convergence uniform on

- $[0, 1)$?
- $[0, \alpha]$, with $0 < \alpha < 1$?
- $(1, \infty)$?
- $[\beta, \infty)$, with $1 < \beta$?

Exercise 24.6, page 183

Let $f_n(x) = (x - \frac{1}{n})^2$ for $x \in [0, 1]$.

(a) Does the sequence (f_n) converge pointwise on the set $[0, 1]$? If so, give the limit function.

Solution: YES; $f(x) = x^2$ (LOL)

(b) Does (f_n) converge uniformly on $[0, 1]$? Prove your assertion.

Solution: YES;
$$\left|(x - \frac{1}{n})^2 - x^2\right| = \left|-\frac{2x}{n} + \frac{1}{n^2}\right| \le \frac{2}{n} + \frac{1}{n^2}.$$
By the first domination principle (see the minutes of January 12), the convergence is uniform.