Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 21; March 18, 2007

Show that the Taylor series for $f(x) = \log x + \log(4-x)$ in powers of $x-2$ converges only at the point $x = 2$. The statement is actually false. The power series converges for $|x-2| < 2$, that is $0 < x < 4$.

Solution: For $n \geq 1$,

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n} - (n-1)!(4-x)^{-n}$$

so that $f^{(n)}(2) = [(-1)^{n-1} - 1](n-1)!2^{-n}$. The formal Taylor series for f is therefore

$$f(x) \stackrel{?}{=} 2\log 2 + \sum_{k=1}^{\infty} a_k (x-2)^k$$

where $a_k = f^{(k)}(2)/k! = [(-1)^{k-1} - 1]/k2^k$. To find the radius of convergence, note that

$$|a_k|^{1/k} = \begin{cases} 0 & k \text{ even} \\ 1/k^{1/k}2^{(k-1)/k} & k \text{ odd.} \end{cases},$$

so that $\limsup |a_k|^{1/k} = 1/2$ and the radius of convergence is 2. (See Theorem 23.1 and Example 7 in section 23 of Ross)

Just for the record, here is a possible alternate interpretation of what I meant to say in this assignment. If you expand $\log x$ in powers of $x-1$ and $\log(4-x)$ in powers of $x-3$, then the sum of the two series converges only at $x = 2$ (since $\{2\} = (0,2] \cap [2,4)$). While correct, this is not a very interesting result.