Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 23; March 19, 2007

Exercise 33.10

Let $f(x) = \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Show that f is integrable on $[-1, 1]$.

Solution: Let $\epsilon > 0$. Since f is continuous on $[\epsilon, 1]$, it is integrable there and so there is a partition P_1 of $[\epsilon, 1]$ such that

$$U_\epsilon^1(f, P_1) - L_\epsilon^1(f, P_1) < \epsilon.$$

Similarly, there is a partition P_2 of $[-1, -\epsilon]$ such that

$$U_{-1}^{-\epsilon}(f, P_2) - L_{-1}^{-\epsilon}(f, P_2) < \epsilon.$$

The union $P_1 \cup P_2$ is a partition of $[-1, 1]$ and on the interval $[-\epsilon, \epsilon]$ we have $M(f, [-\epsilon, \epsilon]) = 1$ and $m(f, [-\epsilon, \epsilon]) = -1$. Therefore

$$\begin{aligned} U(f, P_1 \cup P_2) - L(f, P_1 \cup P_2) &= U_{-1}^{-\epsilon}(f, P_2) + 1 \cdot 2\epsilon + U_\epsilon^1(f, P_1) \\ &- [L_{-1}^{-\epsilon}(f, P_2) + (-1) \cdot 2\epsilon + L_\epsilon^1(f, P_1)] \\ &= [U_{-1}^{-\epsilon}(f, P_2) - L_{-1}^{-\epsilon}(f, P_2)] \\ &+ [U_\epsilon^1(f, P_1) - L_\epsilon^1(f, P_1)] + 4\epsilon \\ &< 6\epsilon. \end{aligned}$$

By Theorem 32.5, f is integrable on $[-1, 1]$.

Exercise 33.12

Let f be the function described in Exercise 17.14. That is, for each rational number $x = p/q \neq 0$ where p and q are integers with no common factors and $q > 0$, define $f(x) = 1/q$. Also, define $f(x) = 0$ if $x = 0$ or x is irrational. (By Exercise 17.14, f is discontinuous at each non-zero rational number.)

(a) Show that f is not piecewise continuous or piecewise monotonic on any interval $[a, b]$.

Solution: Since f is discontinuous at every non-zero rational number, it cannot be continuous (much less uniformly continuous) on any open interval. Hence continuous on any closed interval. This is the only easy part of this exercise.

NOTE: The rest of (a) and most of (b) is very tedius and complicated, so LET'S AGREE TO IGNORE THIS FUNCTION!

(b) Show f is integrable on every interval $[a, b]$ and that $\int_a^b f = 0$.

Solution: TO BE IGNORED! You can get an idea of its complexity by looking at the solution provided by the TA K. Lam in the case $[a, b] = [0, 1]$. To repeat, for purposes of the final examination, you may ignore the function f.