Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 25; March 19, 2007

Exercise 34.1

Use Theorem 34.3 to prove Theorem 34.1 for the case that g' is continuous.

Solution: Theorem 34.1 says that if g' is integrable on $[a, b]$, then $\int_a^b g' = g(b) - g(a)$. Theorem 34.3 says that if f is integrable and $F(x)$ is defined by $F(x) = \int_a^x f$, then F is contnuous and at a point x where f is continuous, F is differentiable and $F'(x) = f(x)$.

Proof of Theorem 34.1: Since g' is continuous, if we define $F(x) = \int_a^x g'$, then Theorem 34.3 implies that $F'(x) = g'(x)$ for all x. Hence $F(x) = g(x) + c$ where c is a constant. To determine the value of c, note that $0 = F(a) = g(a) + c$, so that $c = -g(a)$ and $\int_a^x g' = g(x) - g(a)$.

Exercise 34.4

Let f be defined as follows: $f(t) = t$ for $t < 0$; $f(t) = t^2 + 1$ for $0 \le t \le 2$; $f(t) = 0$ for $t > 2$.

(a) Determine the function

$$F(x) = \int_0^x f(t)\,dt.$$

Solution:

$$F(x) = \begin{cases} x^2/2 & x < 0 \\ x^3/3 + x & 0 < x \le 2 \\ 14/3 & x > 2 \end{cases}$$

(b) Where is F continuous?

Solution: All $x \in \mathbf{R}$.

(c) Where is F differentiable? Calculate F' at the points of differentiability.

Solution: F is differentiable everywhere except at the two points $x = 0, 2$, and

$$F'(x) = \begin{cases} x & x < 0 \\ x^2 + 1 & 0 < x < 2 \\ 0 & x > 2 \end{cases}$$

Exercise 34.6

Let f be a continuous function on $\mathbf{R}$ and define

$$G(x) = \int_0^{\sin x} f(t)\,dt \text{ for } x \in \mathbf{R}$$

Show that G is differentiable on $\mathbf{R}$ and compute G'.

Solution: Write $G(x) = F(\sin x)$, where $F(u) = \int_0^u f$. Then by the chain rule and the fundamental theorem of calculus, $G'(x) = F'(\sin x)\cos x = f(\sin x)\cos x$.

Exercise 34.11

Suppose that f is a continuous real-valued function on $[a, b]$ and that $f(x) \ge 0$ for all $x \in [a, b]$. Show that if $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

Solution: See the text, Ross page 337.