Exercise 32.6

Let \( f \) be a bounded function on \([a, b]\). Suppose there exist sequences \((U_n)\) and \((L_n)\) of upper and lower Darboux sums for \( f \) such that \( \lim(U_n - L_n) = 0 \). Show \( f \) is integrable and \( \int_a^b f = \lim U_n = \lim L_n \).

**Solution:** We are given that \( U_n = U(f, P_n) \) and \( L_n = L(f, Q_n) \) for certain partitions \( P_n \) and \( Q_n \) of \([a, b]\). Since
\[
U(f, P_n \cup Q_n) - L(f, P_n \cup Q_n) \leq U_n - L_n \to 0
\]
it follows that for any \( \epsilon > 0 \), there is a partition \( P_\epsilon = P_n \cup Q_n \) for some \( n \) such that
\[
U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.
\]
Hence \( f \) is integrable on \([a, b]\).

From \( L_n \leq L(f) = \int_a^b f \leq U_n \) we have \( 0 \leq \int_a^b f - L_n \leq U_n - L_n \) and \( L_n - U_n \leq \int_a^b f \leq 0 \) and therefore \( \lim L_n = \lim U_n = \int_a^b f \).

Exercise 33.8

Let \( f \) and \( g \) be integrable functions on \([a, b]\)

(a) Show that \( fg \) is integrable on \([a, b]\).

**Solution:** Since \( 4fg = (f + g)^2 - (f - g)^2 \) and Exercise 33.7 states that the square of an integrable function is integrable, using linearity (Theorem 33.3) it follows that \( fg \) is integrable.

(b) Show that \( \max(f, g) \) and \( \min(f, g) \) are integrable on \([a, b]\).

**Solution:** Since \( \min(f, g) = (f + g)/2 - |f - g|/2 \) and \( \max(f, g) = -\max(-f, -g) \), using Theorem 33.5 and linearity (Theorem 33.3), it follows that \( \max(f, g) \) and \( \min(f, g) \) are integrable.

Exercise 34.2

Calculate

(a) \( \lim_{x \to 0} \frac{1}{x} \int_0^x e^{t^2} \, dt \)

**Solution:** Let \( F(x) = \int_0^x e^{t^2} \, dt \). Then \( \lim_{x \to 0} \frac{1}{x} \int_0^x e^{t^2} \, dt = \lim_{x \to 0} \frac{F(x)}{x} = F'(0) = e^{x^2} \big|_{x=0} = 1 \).

(b) \( \lim_{h \to 0} \frac{1}{h} \int_3^{3+h} e^{t^2} \, dt \)

**Solution:** Let \( g(x) = \int_3^x e^{t^2} \, dt \) so that \( g(3) = 0 \) and \( g'(x) = e^{x^2} \). Then \( \lim_{h \to 0} \frac{1}{h} \int_3^{3+h} e^{t^2} \, dt = \lim_{h \to 0} \frac{g(3+h)-g(3)}{h} = g'(3) = e^9 \).

Exercise 34.9

Use Example 3 to show \( \int_0^{1/2} \sin^{-1} x \, dx = \pi/12 + \sqrt{3}/2 - 1 \)

**Solution:** Take \( a = 0, b = \pi/6 \) and \( g(x) = \sin x \) in Example 3. Then
\[
\int_0^{\pi/6} \sin x \, dx + \int_0^{1/2} \sin^{-1} x \, dx = \frac{\pi}{6} \cdot \frac{1}{2} = \frac{\pi}{12}
\]
and
\[
\int_0^{\pi/6} \sin x \, dx = -\cos x \big|_0^{\pi/6} = -\frac{\sqrt{3}}{2} + 1.
\]