**Elementary Analysis Math 140B—Winter 2007**
**Homework answers—Assignment 3; January 22, 2007**

**Exercise 24.13, page 183**

Prove that if $(f_n)$ is a sequence of uniformly continuous functions on an interval $(a, b)$, and if $f_n \to f$ uniformly on $(a, b)$, then $f$ is also uniformly continuous on $(a, b)$.

**Solution:** For $x, y \in (a, b)$ and $n \geq 1$,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|. \tag{1}$$

By the uniform convergence, given $\epsilon > 0$, choose $N = N(\epsilon)$ such that

$$\sup\{|f(t) - f_n(t)| : t \in (a, b)\} < \epsilon/3$$

for $n > N$. Thus the first and third terms on the right side of (1) are each less than $\epsilon/3$. By the uniform continuity of $f_{N+1}$, choose $\delta = \delta(\epsilon, N) > 0$ such that $|f_{N+1}(s) - f_{N+1}(t)| < \epsilon/3$ whenever $|t - s| < \delta$. Thus, if $n = N + 1$, then the middle term on the right side of (1) is less than $\epsilon/3$ if $|x - y| < \delta$.

Finally, if $|x - y| < \delta$, then by (1), $|f(x) - f(y)| < \epsilon$. □

**Exercise 24.16, page 184**

Let $f_n(x) = \frac{nx}{1+nx^2}$ for $x \in [0, \infty)$.

**(a)** Find $f(x) = \lim f_n(x)$.

**Solution:**

$$f_n(x) = \frac{x}{\frac{1}{n} + x^2} \to \frac{1}{x}$$

if $x \neq 0$. Therefore

$$f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & x > 0 \end{cases}$$

**(b)** Does $(f_n)$ converge uniformly on $[0, 1]$? Justify.

**Solution:** NO; $f$ is not continuous on $[0, 1]$.

**(c)** Does $(f_n)$ converge uniformly on $[1, \infty)$? Justify.

**Solution:** YES; if $x \geq 1$,

$$\left|\frac{1}{x} - \frac{nx}{1 + nx^2}\right| = \frac{\frac{1}{n}}{x\left(\frac{1}{n} + x^2\right)} \leq \frac{1}{n} \to 0,$$

so the convergence is uniform by the first domination principle (see the minutes of January 12).