Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 4; January 22, 2007

Exercise 24.14, page 183

Let $f_n(x) = \frac{nx}{1+n^2x^2}$ for $x \in \mathbf{R}$.

(a) Show that $f_n \to 0$ pointwise on $\mathbf{R}$.

Solution: For any n, $f_n(0) = 0$ so that if f denotes the pointwise limit function (assuming it exists), then $f(0) = 0$. On the other hand for all $x \in \mathbf{R}$,

$$f_n(x) = \frac{x/n}{\frac{1}{n^2} + x^2} \to 0.$$

Thus f exists and $f(x) = 0$ for all $x \in \mathbf{R}$.

(b) Does (f_n) converge uniformly on $[0,1]$? Justify.

Solution: NO; by using calculus:

$$f_n'(x) = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2}$$

so the critical points of f_n in $[0,1]$ are at $0, 1/n, 1$. Then

$$\sup\{f_n(x) : x \in [0,1]\} = \max\{f_n(0), f_n(1/n), f_n(1)\} = \max\{0, \frac{1}{2}, \frac{n}{n^2+1}\} = \frac{1}{2}$$

By Remark 24.4, (f_n) does not converge to 0 uniformly on $[0,1]$.

(c) Does (f_n) converge uniformly on $[1,\infty)$? Justify.

Solution: YES; if $x \geq 1$,

$$f_n(x) = \frac{\frac{n}{x}}{\frac{1}{x^2} + n^2} \leq \frac{n}{n^2} = \frac{1}{n} \to 0.$$

so the convergence is uniform by the first domination principle (see the minutes of January 12).

Exercise 24.17, page 184

Let (f_n) is a sequence of continuous functions on an interval $[a,b]$ that converges uniformly to a function f on $[a,b]$. Show that if (x_n) is a sequence in $[a,b]$ and if $x_n \to x$, then $\lim_{n\to\infty} f_n(x_n) = f(x)$.

Solution: For $n \geq 1$ and $m \geq 1$,

$$|f(x) - f_n(x_n)| \leq |f(x) - f_m(x)| + |f_m(x) - f_m(x_n)| + |f_m(x_n) - f_n(x_n)|. \tag{1}$$

By the uniform convergence, given $\epsilon > 0$, choose $N_1 = N_1(\epsilon)$ such that

$$\sup\{|f(t) - f_m(t)| : t \in [a,b]\} < \epsilon/3$$

for $m > N_1$. Thus the first term on the right side of (1) is less than $\epsilon/3$ if $m > N_1$.

By Exercise 25.4, choose $N_2 = N_2(\epsilon)$ such that

$$\sup\{|f_m(t) - f_n(t)| : t \in [a,b]\} < \epsilon/3$$

for $m, n > N_2$. Thus the third term on the right side of (1) is less than $\epsilon/3$ if $m, n > N_2$.

By the continuity of f_m, choose $N_3 = N_3(\epsilon, x, m)$ such that $|f_m(x) - f_m(x_n)| < \epsilon/3$ if $n < N_3$.

Finally, for $n > \max\{N_1, N_2, N_3\}$, it follows from (1) that $|f(x) - f_n(x_n)| \leq \epsilon$. □