Exercise 25.4, page 190

Let \((f_n)\) be a sequence of functions on a set \(S \subset \mathbb{R}\), and supporst that \(f_n \to f\) uniformly on \(S\). Prove that \((f_n)\) is uniformly Cauchy on \(S\).

**Solution:** For all \(x \in S\) and for all \(m, n\),

\[
|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)|. \tag{1}
\]

Let \(\epsilon > 0\). There exists \(N = N(\epsilon)\) such that

\[
\sup_{x \in S} |f_n(x) - f(x)| < \epsilon/2 \text{ for all } n > N.
\]

Thus, if \(m > N\) and \(m > N\), then by (1),

\[
\sup_{x \in S} |f_n(x) - f_m(x)| < \epsilon. \quad \square
\]

Exercise 25.10, page 191

(a) Show that \(\sum x^n / (1 + x^n)\) converges for \(x \in [0, 1)\).

**Solution:** Use the ratio test:

\[
\frac{x^{n+1} / (1 + x^{n+1})}{x^n / (1 + x^n)} = x \frac{1 + x^n}{1 + x^{n+1}} \to x.
\]

The series therefore converges for \(|x| < 1\), and in particular on \([0, 1)\).

(b) Show that the series converges uniformly on \([0, a)\) for each \(a, 0 < a < 1\).

**Solution:** For \(0 \leq x \leq a < 1\),

\[
x^n \frac{1}{1 + x^n} \leq a^n.
\]

Since \(a < 1\), the geometric series \(\sum a^n\) converges. By the Weierstrass \(M\)-test, the series converges uniformly on \([0, a)\).

(c) Does the series converge uniformly on \([0, 1)\)? Explain.

**Solution:** NO. If the series converged uniformly on \([0, 1)\), then (see Example 5 on page 190) we would have

\[
\sup_{x \in [0,1)} \frac{x^n}{1 + x^n} \to 0 \text{ as } n \to \infty.
\]

Let \(\epsilon = 1/4\) and choose \(N\) such that

\[
\sup_{x \in [0,1)} \frac{x^n}{1 + x^n} < 1/4 \text{ for all } n > N.
\]

In particular for every \(x \in [0, 1)\), \(\frac{x^{N+1}}{1 + x^{N+1}} < \frac{1}{4}\), which is a contradiction to

\[
\lim_{x \to 1} \frac{x^{N+1}}{1 + x^{N+1}} = \frac{1}{2}.
\]