Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 5; January 29, 2007

Exercise 25.4, page 190

Let (f_n) be a sequence of functions on a set $S \subset \mathbf{R}$, and supporst that $f_n \to f$ uniformly on S. Prove that (f_n) is uniformly Cauchy on S.

Solution: For all $x \in S$ and for all m, n,

$$|f_n(x) - f_m(x)| \le |f_n(x) - f(x)| + |f(x) - f_m(x)|. \tag{1}$$

Let $\epsilon > 0$. There exists $N = N(\epsilon)$ such that

$$\sup_{x\in S} |f_n(x) - f(x)| < \epsilon/2 \text{ for all } n > N.$$

Thus, if $m > N$ and $m > N$, then by (1),

$$\sup_{x\in S} |f_n(x) - f_m(x)| < \epsilon.$$

□

Exercise 25.10, page 191

(a) Show that $\sum \frac{x^n}{1+x^n}$ converges for $x \in [0, 1)$.

Solution: Use the ratio test:

$$\frac{\frac{x^{n+1}}{1+x^{n+1}}}{\frac{x^n}{1+x^n}} = x\frac{1+x^n}{1+x^{n+1}} \to x.$$

The series therefore converges for $|x| < 1$, and in particular on $[0, 1)$.

(b) Show that the series converges uniformly on $[0, a)$ for each a, $0 < a < 1$.

Solution: For $0 \le x \le a < 1$,

$$\frac{x^n}{1+x^n} \le a^n.$$

Since $a < 1$, the geometric series $\sum a^n$ converges. By the Weierstrass M-test, the series converges uniformly on $[0, a)$.

(c) Does the series converge uniformly on $[0, 1)$? Explain.

Solution: NO. If the series converged uniformly on $[0, 1)$, then (see Example 5 on page 190) we would have

$$\sup_{x\in[0,1)} \frac{x^n}{1+x^n} \to 0 \text{ as } n \to \infty.$$

Let $\epsilon = 1/4$ and choose N such that

$$\sup_{x\in[0,1)} \frac{x^n}{1+x^n} < 1/4 \text{ for all } n > N.$$

In particular for every $x \in [0, 1)$, $\frac{x^{N+1}}{1+x^{N+1}} < \frac{1}{4}$, which is a contradiction to

$$\lim_{x\to 1} \frac{x^{N+1}}{1+x^{N+1}} = \frac{1}{2}.$$