Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 7; January 29, 2007

Exercise 26.2, page 199

(a) Observe that $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$.

Solution: By Example 1 on page 195, for $|x| < 1$,

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Therefore, for $|x| < 1$,

$$\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}.$$

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Solution:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2.$$

(c) Evaluate $\sum_{n=1}^{\infty} \frac{n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n}$.

Solution:

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{\frac{1}{3}}{(1-\frac{1}{3})^2} = \frac{3}{4}.$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} = \frac{-\frac{1}{3}}{(1+\frac{1}{3})^2} = -\frac{3}{16}.$$

Exercise 26.8, page 200

(a) Show that $\sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{1+x^2}$ for $x \in (-1, 1)$

Solution: For $|y| < 1$, $\sum_0^{\infty} y^n = 1/(1-y)$. Now let $y = -x^2$ with $|x| < 1$

(b) Show that $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for $x \in (-1, 1)$.

Solution:

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_0^{\infty}(-1)^n t^{2n}\, dt = \sum_0^{\infty}(-1)^n \int_0^x t^{2n}\, dt.$$

(c) Show that the equality in (b) also holds for $x = 1$. Use this to find a nice formula for π.

Solution: The series $\sum_0^{\infty} \frac{(-1)^n}{2n+1}$ converges by the alternating series test. By Abel's theorem and Theorem 24.3, $\sum_0^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ is a continuous function on $(-1, 1]$. Also, $\tan^{-1}$ is a continuous function on $(-1, 1]$. By (b), these two continuous functions agree on $(-1, 1)$. Thus

$$\frac{\pi}{4} = \tan^{-1}(1) = \lim_{x\to 1, x\neq 1} \tan^{-1}(x) = \lim_{x\to 1, x\neq 1} \sum_0^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \sum_0^{\infty} \frac{(-1)^n}{2n+1}.$$

(d) What happens at $x = -1$?

Solution: Same argument as in (c) shows that (b) holds for $x \in [-1, 1]$:

$$-\frac{\pi}{4} = \tan^{-1}(-1) = \lim_{x\to -1, x\neq -1} \tan^{-1}(x) = \lim_{x\to 1, x\neq 1} \sum_0^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \sum_0^{\infty} \frac{(-1)^{n+1}}{2n+1}.$$