**Elementary Analysis Math 140B—Winter 2007**
**Homework answers—Assignment 9; February 7, 2007**

**Exercise 27.2, page 204**

Show that if $f$ is continuous on $\mathbf{R}$, then there exists a sequence $(p_n)$ of polynomials such that $p_n \to f$ uniformly on each bounded subset of $\mathbf{R}$.

**Solution:** By the Weierstrass approximation Theorem (Theorem 27.5, page 203), there is for each $n \geq 1$ a polynomial $p_n$ such that

$$|f(x) - p_n(x)| < \frac{1}{n} \text{ for all } x \in [-n, n]. \tag{1}$$

Now let $S$ be any bounded set. Choose $N$ such that $S \subset [-N, N]$. Then $S \subset [-n, n]$ for all $n \geq N$. For $n \geq 1$, let $q_n = P_{n+N}$. From (1), we have

$$|q_n(x) - f(x)| < \frac{1}{n+N} \text{ for all } n \geq 1 \text{ and for all } x \in S.$$

Therefore, $q_n \to f$ uniformly on $S$.

**Exercise 27.4, page 204**

Let $f$ be a continuous function on $[a, b]$. Show that there exists a dequence $(p_n)$ of polynomials such that $p_n \to f$ uniformly on $[a, b]$, and such that $p_n(a) = f(a)$ for all $n$.

**Solution:** Choose polynomials $p_n$ such that $p_n \to f$ uniformly on $[a, b]$. Define $q_n(x) = p_n(x) + f(a) - p_n(a)$. Then $q_n(a) = f(a)$ and

$$|f(x) - q_n(x)| = |f(x) - p_n(x) - f(a) + p_n(a)| \leq |f(x) - p_n(x)| + |p_n(a) - f(a)|,$$

which shows that $q_n \to f$ uniformly on $[a, b]$.