Elementary Analysis
Math 140D—Fall 2007

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Contents

1 Friday September 28, 2007

1.1 Course information

- Mathematics 140D MWF 9:00–9:50 MSTB124
  Analysis in Several Variables

- Brief Description: Rigorous treatment of multivariable Riemann integration, differential forms and vector analysis (Theorems of Stokes, Green, Gauss).

  Good preparation for graduate work in mathematics. Grades will be based on homework problems only.

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- Discussion section: TuTh 9:00–9:50 HH156

- Teaching Assistant: Shaun Xue

- Text: R. C. Buck, Advanced Calculus

1.2 Outline of the course

- Riemann integration in $\mathbf{R}^n$ (Section 4.2 of Buck, Theorems 1,4)

- Jordan measurable sets (Section 4.2 of Buck, Theorems 2,3)

- Set Functions (Section 8.2 of Buck, Theorems 1,2)

- Change of variables in multiple integrals (Section 8.2 of Buck, Theorems 5,6)

- Curves and surfaces (Section 8.4 and 8.5 of Buck, Theorems 7,9,11,14)

- Differential forms (Section 9.2 of Buck, Theorem 2)

- Vector analysis (Section 9.3 of Buck, Theorems 3,4,5)

- Stokes' Theorem (Section 9.4 of Buck, Theorems 6,7,8,9)

1.3 The definition of the Riemann integral

If $f : [a, b] \to \mathbf{R}$ is a bounded function on a closed and bounded interval $I = [a, b] \subset \mathbf{R}$, its Riemann integral, if it exists, can be denoted in several ways, for example:

$$\int_a^b f(x)\,dx = \int_a^b f = \int_{[a,b]} f = \int_I f.$$

Similarly, if $f : [a,b] \times [c,d] \to \mathbf{R}$ is a bounded function on a closed and bounded rectangle $R = I \times J = [a,b] \times [c,d] \subset \mathbf{R}^2$, its Riemann (double) integral, if it exists, can be denoted in several ways, for example:

$$\int\int_{I\times J} f(x,y)\,dxdy = \int\int_R f = \int_R f.$$

We can even consider triple integrals: if $f : I \times J \times K \to \mathbf{R}$ is a bounded function on a closed and bounded box $B = I \times J \times K \subset \mathbf{R}^3$, its Riemann (triple) integral, if it exists, can be denoted in several ways, for example:

$$\int\int\int_{I\times J\times K} f(x,y,z)\,dxdydz = \int\int\int_B f = \int_B f.$$

Being foolish, we decide to consider the Riemann integral of a bounded function defined on an "n-box" in $\mathbf{R}^n$, for any $n \geq 1$. By an n-box we mean a product of n closed intervals: $B = I_1 \times I_2 \times \cdots \times I_n \subset \mathbf{R}^n$, where I_j is a closed and bounded interval in $\mathbf{R}$. Thus, an interval is a 1-box, a rectangle is a 2-box, and a box in $\mathbf{R}^3$ is a 3-box.

Let's get down to business. For simplicity, we start with (you guessed it) $n = 1$.

Let $I = [a,b]$ be a closed and bounded interval in $\mathbf{R}$. A *partition* of I is any finite subset of I which includes the endpoints a and b. We write the elements of P in increasing order: $P = \{a = x_0 < x_1 < \cdots < x_m = b\}$. Note that $m+1$ is the number of elements of P and m is the number of subintervals, that is

$$I_j = [x_{j-1}, x_j] \quad 1 \leq j \leq m \text{ and } I = \cup_{j=1}^m I_j.$$

Let $\mathcal{P}(I)$ denote the set of all partitions of I. This is a very large set consisting of all possible partitions with any number $m = 1, 2, \ldots$ of subintervals.

The *length* of I_j is $\ell(I_j) = x_j - x_{j-1}$. The *mesh* of the partition P is denoted by $d(P)$ and is defined by $d(P) = \max\{\ell(I_j) : 1 \leq j \leq m\}$. Next we need a choice of points $C = \{t_1, \ldots, t_m\}$ such that $t_j \in I_j$ for $1 \leq j \leq m$.

Now let f be a function defined on I. A *Riemann sum* of f with respect to a partition P and a choice C is defined by

$$S(f,P,C) = \sum_{j=1}^m f(t_j)\ell(I_j).$$

We can now state a fundamental theorem in the theory of Riemann integration.

Theorem 1.1 (Theorem 1 on page 169 of Buck ($n = 1$)) *If f is a continuous function on a closed bounded interval $I \subset \mathbf{R}$, then there is a unique real number v (depending on f and I) with the following property:*

> *For every $\epsilon > 0$, there exists $\delta > 0$ such that for all partitions P with $d(P) < \delta$ and for every choice C, we have $|S(f,P,C) - v| < \epsilon$.*

We now consider double integrals. Let $R = I \times J$ be a closed rectangle in $\mathbf{R}^2$. A *grid* of R a set of the form $N = P \times Q$, where P is a partition of I and Q is a partition of J. We can write $P = \{a = x_0 < x_1 < \cdots < x_m = b\}$ and $Q = \{c = y_0 < y_1 < \cdots < y_r = d\}$. Note that there are mr subrectangles $R_{ij} = I_i \times J_j$ of R, where

$$I_i = [x_{i-1}, x_i] \quad 1 \le i \le m \text{ and } J_j = [y_{j-1}, y_j] \quad 1 \le j \le r.$$

Moreover $R = \cup_{j=1}^r \cup_{i=1}^m R_{ij}$. Let $\mathcal{N}(R)$ denote the set of all grids of R.

The *area* of R_{ij} is $A(R_{ij}) = \ell(I_i)\ell(J_j)$. The *mesh* of the grid N is denoted by $d(N)$ and is defined by

$$d(N) = \max_{1\le i\le m, 1\le j\le r} |(x_i, y_j) - (x_{i-1}, y_{j-1})| = \max_{1\le i\le m, 1\le j\le r} \{[(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2]^{1/2}\}.$$

Next we need a choice of points $C = \{p_{ij} : 1 \le i \le m, 1 \le j \le r\}$ such that $p_{ij} \in R_{ij}$ for $1 \le i \le m, 1 \le j \le r$.

Now let f be a function defined on R. A *Riemann sum* of f with respect to a grid N and a choice C is defined by

$$S(f, N, C) = \sum_{j=1}^{r} \sum_{i=1}^{m} f(p_{ij}) A(R_{ij}).$$

We can now restate a fundamental theorem in the theory of Riemann integration, this time for $n = 2$

Theorem 1.2 (Theorem 1 on page 169 of Buck ($n = 2$)) *If f is a continuous function on a closed bounded rectangle $R \subset \mathbf{R}^2$, then there is a unique real number v (depending on f and R) with the following property:*

> *For every $\epsilon > 0$, there exists $\delta > 0$ such that for all grids N with $d(N) < \delta$ and for every choice C, we have $|S(f, N, C) - v| < \epsilon$.*

Assignment 1 (Due October 5) Prove the uniqueness of v in Theorem 1.1 or in Theorem 1.2.

Definition 1.3 *The number v whose existence is guaranteed by Theorem 1.1 is denoted by $\int_I f$. The number v whose existence is guaranteed by Theorem 1.2 is denoted by $\int_R f$.*

2 Monday October 1, 2007

We begin the proof of Theorem 1.2. We are given the "data" f, R and we shall start with the statement of three lemmas. In the first two, it is only required that f be a bounded function. This will be important for later when you need to study integration of discontinuous functions[1]. Only in the third lemma will the continuity of f be needed. Of course, this continuity and the compactness of R implies that f is bounded, and moreover, perhaps more importantly, that f is uniformly continuous[2].

[1] this will be 99% of the time!, real life is not continuous

[2] remember, this is supposed to be an application of uniform continuity

2.1 Three lemmas

Let's get down to business. Suppose f is a bounded function on the closed and bounded rectangle $R = I \times J$, and let N be a grid of R. Thus $N = P \times Q$, where P is a partition of I and Q is a partition of J, say $P = \{a = x_0 < x_1 < \cdots < x_m = b\}$ and $Q = \{c = y_0 < y_1 < \cdots < y_r = d\}$. Recall that there are mr subrectangles $R_{ij} = I_i \times J_j$ of R, where

$$I_i = [x_{i-1}, x_i] \;\; 1 \le i \le m \text{ and } J_j = [y_{j-1}, y_j] \;\; 1 \le j \le r.$$

Moreover $R = \cup_{j=1}^{r} \cup_{i=1}^{m} R_{ij}$.

Since f is bounded on R, it is also bounded on each subrectangle R_{ij} and we can define

$$M_{ij} = \sup_{p \in R_{ij}} f(p) \text{ and } m_{ij} = \inf_{p \in R_{ij}} f(p).$$

Notice that for continuous f, by the extreme values theorem, there will exist points $x_{ij}, y_{ij} \in R_{ij}$ such that $f(x_{ij}) = m_{ij}$ and $f(y_{ij}) = M_{ij}$. We shall use this fact in the third lemma below but for the first two lemmas, only the numbers m_{ij}, M_{ij} are needed.

We now define the upper and lower Riemann sums corresponding to a grid, namely,

$$\overline{S}(N) := \sum_{j=1}^{r} \sum_{i=1}^{m} M_{ij} A(R_{ij}) \qquad \text{(upper Riemann sum)}$$

and

$$\underline{S}(N) := \sum_{j=1}^{r} \sum_{i=1}^{m} m_{ij} A(R_{ij}) \qquad \text{(lower Riemann sum)}$$

Since $m_{ij} \le f(p) \le M_{ij}$ for every $p \in R_{ij}$, and $A(R_{ij}) > 0$, for every grid N and every choice C, we have

$$\underline{S}(N) \le S(f, N, C) \le \overline{S}(N). \tag{1}$$

We are now ready to state the three lemmas.

Lemma 2.1 (Lemma 1 on page 170 of Buck) *Let f be a bounded function on a closed and bounded rectangle $R \subset \mathbf{R}^2$. Let N and $\tilde{N}$ be grids of R and suppose $N \subset \tilde{N}$. Then*

(a) $\underline{S}(N) \le \underline{S}(\tilde{N})$

(b) $\overline{S}(N) \ge \overline{S}(\tilde{N})$

Lemma 2.2 (Lemma 2 on page 170 of Buck) *Let f be a bounded function on a closed and bounded rectangle $R \subset \mathbf{R}^2$.*

(a) *The following two subsets[3] of $\mathbf{R}$ are bounded sets:*

$$\{\underline{S}(N) : N \in \mathcal{N}(R)\} \text{ and } \{\overline{S}(N) : N \in \mathcal{N}(R)\}.$$

[3]recall that $\mathcal{N}(R)$ denotes the set of all grids of R

(b) *Let*

$$s := \sup\{\underline{S}(N) : N \in \mathcal{N}(R)\} \text{ and } S = \inf\{\overline{S}(N) : N \in \mathcal{N}(R)\}.$$

Then

- $s \leq S$
- *for every grid* N, $S - s \leq \overline{S}(N) - \underline{S}(N)$

Lemma 2.3 (Lemma 3 on page 171 of Buck) *Let f be a continuous function on a closed and bounded rectangle $R \subset \mathbf{R}^2$. For every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\overline{S}(N) - \underline{S}(N) < \epsilon \text{ for every grid } N \text{ with mesh } d(N) < \delta.$$

We shall prove these three lemmas, one after the other. But first, we use them to give a proof of Theorem 1.2.

2.2 Proof of Theorem 1.2

Proof of Theorem 1.2: From Lemmas 2.2 and 2.3, $0 \leq S - s < \epsilon$ for every $\epsilon > 0$, that is, $S - s = 0$. Let v denote the common value of s and S.

The following sequence of statements will complete the proof. Be sure you can supply the justification for each statement.

- $S - \underline{S}(N) \leq \overline{S}(N) - \underline{S}(N)$ for every grid N
- $\overline{S}(N) - s \leq \overline{S}(N) - \underline{S}(N)$ for every grid N
- $0 < v - \underline{S}(N) < \epsilon$ if $d(N) < \delta$
- $0 < \overline{S}(N) - v < \epsilon$ if $d(N) < \delta$
- $v - S(f, N, C) \leq v - \underline{S}(N) < \epsilon$ if $d(N) < \delta$ and C is any choice of points
- $S(f, N, C) - v \leq \overline{S}(N) - v < \epsilon$ if $d(N) < \delta$ and C is any choice of points
- $|S(f, N, C) - v| < \epsilon$ if $d(N) < \delta$ and C is any choice of points.

This completes the proof of Theorem 1.2.

We now turn to the proofs of the three lemmas.

2.3 Proof of the first lemma

Proof of Lemma 2.1: There are two parts to this proof. We first prove the lemma in the special case where $\tilde{N}$ is obtained from $N = P \times Q$ by adding a single point to either P or Q. Then we prove the general case easily from this.

Step 1: We start with the following simple observation:

Let $\phi : D \to \mathbf{R}$ be a bounded function on a set $D \subset \mathbf{R}^n$ and suppose that $A \subset D$. Then

$$\sup_{p\in A} \phi(p) \le \sup_{p\in D} \phi(p) \text{ and } \inf_{p\in A} \phi(p) \ge \inf_{p\in D} \phi(p).$$

We now assume that $\tilde{N} = (P \cup \{u\}) \times Q$ and define i_0 by $x_{i_0-1} < u < x_{i_0}$. We have $R_{i_0 j} = R'_{i_0 j} \cup R''_{i_0 j}$, where $R'_{i_0 j} = [x_{i_0-1}, u] \times [y_{j-1}, y_j]$ and $R''_{i_0 j} = [u, x_{i_0}] \times [y_{j-1}, y_j]$. Then

$$\underline{S}(N) = \sum_{j=1}^{r}\sum_{i=1}^{m} m_{ij} A(R_{ij}) = \sum_{j=1}^{r} m_{i_0 j} A(R_{i_0 j}) + \sum_{j=1}^{r}\sum_{i=1, i\ne i_0}^{m} m_{ij} A(R_{ij}),$$

and

$$\underline{S}(\tilde{N}) = \sum_{j=1}^{r} [m'_{i_0 j} A(R'_{i_0 j}) + m''_{i_0 j} A(R''_{i_0 j})] + \sum_{j=1}^{r}\sum_{i=1, i\ne i_0}^{m} m_{ij} A(R_{ij}).$$

Thus, $\underline{S}(N) \le \underline{S}(\tilde{N})$ if for each j, $m_{i_0 j} A(R_{i_0 j}) \le m'_{i_0 j} A(R'_{i_0 j}) + m''_{i_0 j} A(R''_{i_0 j})$. This last statement is true since $A(R_{i_0 j}) = A(R'_{i_0 j}) + A(R''_{i_0 j})$, and by virtue of the observation above, $m_{i_0 j} \le m'_{i_0 j}$, $m_{i_0 j} \le m''_{i_0 j}$.

This completes the proof of (a) in case the new point u occurs on the "x-axis". You need a similar proof in case the new point occurs on the "y-axis". Then you need to prove (b) in each of these two cases. These proofs can be omitted since no new ideas are needed for them.

Step 2: Assume that the lemma is true in the special case. Write

$$N_0 = \tilde{N} \supset N_1 \supset N_2 \supset \cdots \supset N_s \supset N_{s+1} := N,$$

where N_k is obtained from N_{k+1} by adding a single point $(0 \le k \le s)$.

By assumption, for $0 \le k \le s$,

$$\underline{S}(N_{k+1}) \le \underline{S}(N_k) \le \overline{S}(N_k) \le \overline{S}(N_{k+1}).$$

Therefore,

$$\begin{aligned} \underline{S}(N) &= \underline{S}(N_{s+1}) \le \underline{S}(N_s) \le \underline{S}(N_{s-1}) \le \cdots \le \underline{S}(N_1) \le \underline{S}(N_0) = \underline{S}(\tilde{N}) \\ &\le \overline{S}(N_0) \le \overline{S}(N_1) \le \overline{S}(N_2) \le \cdots \le \overline{S}(N_{s-1}) \le \overline{S}(N_s) \le \overline{S}(N_{s+1}) = \overline{S}(N). \end{aligned}$$

This completes the proof of step 2 and hence of Lemma 2.1.

2.4 Examples (not mentioned in class)

Definition 2.4 A bounded function on a closed and bounded rectangle $R \subset \mathbf{R}^2$ is *integrable on* R if it satisfies the condition of Theorem 1.2, that is, there is a unique real number v (depending on f and R) with the following property:

For every $\epsilon > 0$, there exists $\delta > 0$ such that for all grids N with $d(N) < \delta$ and for every choice C, we have $|S(f, N, C) - v| < \epsilon$.

We can restate Theorem 1.2 as: every continuous function on a compact rectangle is integrable on that rectangle. The question arises: does the converse hold? The answer is no. A discontinuous function can be integrable. There are non-integrable functions, necessarily discontinuous. The next two examples illustrate these two facts.

EXAMPLE 1: Let $f : [0,2] \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 5 & 0 \le x < 1 \\ \alpha & x = 1 \\ 0 & 1 < x \le 2 \end{cases}$$

where $\alpha \in \mathbf{R}$ is arbitrary. Then f is integrable on $[0,2]$ and $\int_{[0,2]} f = 5$.

Proof: Let $P = \{0 = x_0 < x_1 < \cdots < x_m = 2\}$ be any partition of $[0,2]$. The number 1 falls in a unique subinterval $(x_{j-1}, x_j]$, so $x_{j-1} < 1 \le x_j$. Let $C = \{t_1, \ldots, t_m\}$ be any choice of points with $t_k \in I_k = [x_{k-1}, x_k]$ for $1 \le k \le m$. Then

$$\begin{aligned} S(f,P,C) &= 5\ell(I_1) + 5\ell(I_2) + \cdots + \ell(I_{j-1}) + f(t_j)\ell(I_j) + 0 \cdot \ell(I_{j+1}) + \cdots + 0 \cdot \ell(I_m) \\ &= 5(x_{j-1} - x_0) + f(t_j)(x_j - x_{j-1}), \end{aligned}$$

and therefore

$$S(f,P,C) - 5 = 5(x_{j-1} - 1) + f(t_j)(x_j - x_{j-1}). \tag{2}$$

Now let $\epsilon > 0$. Let $M := \max\{5, |\alpha|\}$ and choose $\delta = \epsilon/(5+M)$. By (2), if $d(P) < \delta$ (and C is arbitrary),

$$|S(f,P,C) - 5| \le |5(x_{j-1} - 1)| + |f(t_j)|(x_j - x_{j-1}) < 5\delta + M\delta < \epsilon.$$

EXAMPLE 2: Let $f : [a,b] \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 5 & x \text{ rational} \\ 4 & x \text{ irrational} \end{cases}$$

Then f is not integrable on $[a,b]$.

Proof: To be given in discussion, October 2.

Assignment 2 (Due October 8) Buck, §4.2, page 178 #1,2,3,4,5,6. In problems 3,5 and 6, take $D = R$, a compact rectangle. (Ignore the assumption that D is open in problem 6). Problems 3,5 and 6 will be assigned again later with the set D as stated.

3 Wednesday October 3, 2007

3.1 Proofs of the second and third lemmas (not covered in class)

Proof of Lemma 2.2: Every grid

$$N = \{a = x_0 < \cdots < x_m = b\} \times \{c = y_0 < \cdots < y_r = d\}$$

contains the trivial grid $N_0 = \{a, b\} \times \{c, d\}$. Therefore, with $m := \inf\{f(p) : p \in R\}$ and $M := \sup\{f(p) : p \in R\}$, by Lemma 2.1,

$$mA(R) = \underline{S}(N_0) \leq \underline{S}(N) \leq \overline{S}(N) \leq \overline{S}(N_0) = MA(R).$$

Thus the two sets $\{\underline{S}(N) : N \in \mathcal{N}(R)\}$ and $\{\overline{S}(N) : N \in \mathcal{N}(R)\}$ are bounded, and the numbers s and S exist.

For every N, since $\underline{S}(N) \leq s$ we have $-\underline{S}(N) \geq -s$. Add this inequality to the inequality $\overline{S}(N) \geq S$ and you get $\overline{S}(N) - \underline{S}(N) \geq S - s$, which proves the second statement of the lemma.

To prove the first statement of the lemma, we shall make use of the following:

Claim: for any two grids N_1, N_2, $\underline{S}(N_1) \leq \overline{S}(N_2)$.

Let's assume this claim for the moment. Thinking of N_2 as fixed and N_1 as varying, and taking the supremum over N_1, you get, for every N_2,

$$\sup_{N_1 \in \mathcal{N}(R)} \underline{S}(N_1) \leq \overline{S}(N_2).$$

Thus $s \leq \overline{S}(N)$ for every grid N so taking the infimum over all grids N, you get

$$s \leq \inf_{N \in \mathcal{N}(R)} \overline{S}(N),$$

which proves the first statement.

To prove the claim, we use Lemma 2.1 again. Given any two grids N_1, N_2, let $N = N_1 \cup N_2$. Then $N_1 \subset N$ and $N_2 \subset N$, so that by Lemma 2.1,

$$\underline{S}(N_1) \leq \underline{S}(N) \leq \overline{S}(N) \leq \overline{S}(N_2).$$

This completes the proof of Lemma 2.2.

Proof of Lemma 2.3: Since f is continuous on the compact rectangle R, it is uniformly continuous on R. For any $\epsilon > 0$, let $\delta = \delta(\epsilon/A(R), f, R)$, that is,

$$|f(p) - f(q)| < \epsilon/A(R) \text{ for all } p, q \in R \text{ with } |p - q| < \delta.$$

Let N be any grid with $d(N) < \delta$. Since f is continuous on the each compact subrectangle R_{ij} of R, by the extreme values theorem, there exist points $p_{ij}, q_{ij} \in R_{ij}$ such that $M_{ij} = f(p_{ij})$ and $m_{ij} = f(q_{ij})$. Since $p_{ij}, q_{ij} \in R_{ij}$, $|p_{ij} - q_{ij}| < \delta$, and so $M_{ij} - m_{ij} = f(p_{ij}) - f(q_{ij}) < \epsilon/A(R)$. We now have

$$0 \leq \overline{S}(N) - \underline{S}(N) = \sum_{i,j} (M_{ij} - m_{ij}) A(R_{ij}) \leq \epsilon/A(R) \sum_{ij} A(R_{ij}) = [\epsilon/A(R)] \cdot A(R) = \epsilon.$$

This proves Lemma 2.3.

Having proved Lemmas 2.1, 2.2, 2.3, the proof of Theorem 1.2 is now complete[4].

[4]Hallelujah!

3.2 Points of continuity of an integrable function

How discontinuous can an integrable function be?

Assignment 3 (Due October 10) Let f be an integrable function on a closed interval $[a, b] \subset \mathbf{R}$. Then f has at least one point of continuity in $[a, b]$.

The following theorem involves the notion of Lebesgue measure and is deferred to the graduate course in real analysis (Mathematics 210ABC).

Theorem 3.1 *Let f be a bounded function on a closed rectangle R in $\mathbf{R}^2$. Then f is integrable if and only the the set of discontinuities of f in R is a set of Lebesgue measure zero.*

The following theorem, a generalization of Theorem 1.2, will be proved here. First, we need to make precise what a set of area zero is.

Theorem 3.2 (Theorem 2 on page 172 of Buck ($n = 2$)) *If the set of points of discontinuity of a bounded function f on a closed rectangle R has zero area then f is integrable on R.*

3.3 Jordan measurable sets

Let D be a bounded subset of $\mathbf{R}^2$ and choose a closed rectangle R with sides parallel to the axes such that $D \subset R$. A grid N of R gives rise to a decomposition $R = \cup_{i,j} R_{ij}$ as we have seen.

Definition 3.3 The *inner* (or *inscribed*) set for D with respect to the grid N of R is $\cup\{R_{ij} : R_{ij} \subset \operatorname{int} D\}$ and $\underline{S}(N, D, R)$ will denote its area (which is equal to $\sum\{A(R_{ij}) : R_{ij} \subset \operatorname{int} D\}$).

The *outer* (or *circumscribed*) set for D with respect to the grid N of R is $\cup\{R_{ij} : R_{ij} \cap \overline{D} \neq \emptyset\}$ and $\overline{S}(N, D, R)$ will denote its area (which is equal to $\sum\{A(R_{ij}) : R_{ij} \cap \overline{D} \neq \emptyset\}$).

Note that $0 \leq \underline{S}(N, D, R) \leq \overline{S}(N, D, R) \leq A(R)$, so that we can define the outer and inner area of D with respect to R as follows:

- $\overline{A}(D, R) := \inf\{\overline{S}(N, D, R) : N \text{ a grid of } R\}$
- $\underline{A}(D, R) := \sup\{\underline{S}(N, D, R) : N \text{ a grid of } R\}$

Assignment 4 (Due October 10) Show that $\overline{A}(D, R)$ and $\underline{A}(D, R)$ do not depend on the rectangle R containing D. Hereafter, we shall denote these quantities by $\overline{A}(D)$ and $\underline{A}(D)$

Note also that $\underline{S}(\cdot, D, R)$ is "increasing" and $\overline{S}(\cdot, D, R)$ is "decreasing," that is, if $N_1 \subset N_2$, then $\underline{S}(N_1, D, R) \leq \underline{S}(N_2, D, R)$ and $\overline{S}(N_1, D, R) \geq \overline{S}(N_2, D, R)$. It follows from this that for any two grids N_1, N_2 of R, we have

$$\underline{S}(N_1, D, R) \leq \underline{S}(N_1 \cup N_2, D, R) \leq \overline{S}(N_1 \cup N_2, D, R) \leq \overline{S}(N_2, D, R)$$

which proves that $\underline{A}(D) \leq \overline{A}(D)$.

Definition 3.4 If $\underline{A}(D) = \overline{A}(D)$, we say that D is a *Jordan measurable set*, or "has area."

4 Friday October 5, 2007

Proposition 4.1 *For any set $D \subset \mathbf{R}^2$,*

$$\overline{A}(\mathit{bdy}\, D) = \overline{A}(D) - \underline{A}(D).$$

Proof: For any grid N with decomposition $R = \cup R_{ij}$, it is clear from the fact that $\operatorname{bdy} D$ is closed (so that $\operatorname{bdy} D = \overline{\operatorname{bdy} D}$) and $\overline{D} = \operatorname{int} D \cup \operatorname{bdy} D$ (disjoint union) that $R_{ij} \cap \overline{\operatorname{bdy} D} \neq \emptyset$ if and only if $R_{ij} \cap \overline{D} \neq \emptyset$ and $R_{ij} \not\subset \operatorname{int} D$. Thus

$$\overline{S}(N, \operatorname{bdy} D) = \overline{S}(N, D) - \underline{S}(N, D).$$

For $\epsilon > 0$, pick grids N_1 and N_2 such that $\overline{A}(D) + \epsilon \geq \overline{S}(N_1, D)$ and $\underline{A}(D) - \epsilon \leq \underline{S}(N_2, D)$. Then

$$\begin{aligned}
\overline{A}(\operatorname{bdy} D) &\leq \overline{S}(N_1 \cup N_2, \operatorname{bdy}\ D) \\
&= \overline{S}(N_1 \cup N_2, D) - \underline{S}(N_1 \cup N_2, D) \\
&\leq \overline{S}(N_1, D) - \underline{S}(N_2, D) \\
&\leq \overline{A}(D) - \underline{A}(D) + 2\epsilon,
\end{aligned}$$

proving that

$$\overline{A}(\operatorname{bdy} D) \leq \overline{A}(D) - \underline{A}(D).$$

On the other hand, for any grid N

$$\overline{S}(N, \operatorname{bdy} D) = \overline{S}(N, D) - \underline{S}(N, D) \geq \overline{A}(D) - \underline{A}(D)$$

and therefore

$$\overline{A}(\operatorname{bdy} D) \geq \overline{A}(D) - \underline{A}(D),$$

proving the proposition. □

5 Monday October 8, 2007

Proposition 5.1 *A set D has area zero if and only if for every $\epsilon > 0$ there exist rectangles $R_1, \ldots, , R_m$ with sides parallel to the coordinate axes such that $D \subset \cup_{j=1}^m R_j$ and $\sum_{j=1}^m A(R_j) < \epsilon$. This implies that a finite set and a horizontal or vertical finite line segment has area zero.*

Proof:

Suppose first that $\overline{A}(D) = 0$. Given $\epsilon > 0$ take a grid N_ϵ of a rectangle $R_\epsilon \supset D$ with $\overline{S}(N_\epsilon, D) < \epsilon$. Write $R_\epsilon = \cup R_{ij}$ and note that $D \subset \overline{D} \subset \overline{D} \cap [\cup R_{ij}] = \cup [R_{ij} \cap \overline{D}]$. Thus $D \subset \cup \{R_{ij} : R_{ij} \cap \overline{D} \neq \emptyset\}$ and $\sum\{A(R_{ij} : R_{ij} \cap \overline{D} \neq \emptyset\} = \overline{S}(N_\epsilon) < \epsilon$.

Conversely, let $\epsilon > 0$ and take rectangles $R_1, \ldots, R_m$ (not necessarily non-overlapping) such that $D \subset \cup_{j=1}^m R_j$ and $\sum_{j=1}^m A(R_j) < \epsilon$. Replace the R_j by non-overlapping rectangles R'_j, $j = 1, \ldots, m'$ with $\cup_{j=1}^m R_j = \cup_{j=1}^{m'} R'_j$ and $\sum A(R'_j) \leq \sum A(R_j)$. Next choose a rectangle $R \supset R'_j$ for all $j = 1, \ldots, m'$ and a grid N of R containing all of the R'_j as subrectangles of the grid. Then $\overline{S}(N, D) = \sum\{A(R_{ij} : R_{ij} \cap \overline{D} \neq \emptyset\} = \sum_{j=1}^{m'} A(R'_j) < \epsilon$. This proves that $\overline{A}(D) = 0$. □

Theorem 5.2 (Theorem 2 on page 172 of Buck ($n = 2$)) *If f is a bounded function on a closed bounded rectangle $R \subset \mathbf{R}^2$ and f is continuous on $R - E$ where E is a subset of R of area zero, then f is integrable on R.*

Proof: [5] Let $\epsilon > 0$ and pick a grid N_ϵ of R such that $\overline{S}(N_\epsilon, E) < \epsilon$. Write $R = \cup R_{ij}^\epsilon$ and define sets $S = \cup\{R_{ij}^\epsilon : R_{ij}^\epsilon \cap \overline{E} \neq \emptyset\}$ and $T = \cup\{R_{ij}^\epsilon : R_{ij}^\epsilon \cap \overline{E} = \emptyset\}$. Note that $E \subset S$ and f is continuous on T. By the uniform continuity of f on T, there exists $\delta_1 > 0$ such that $|f(p) - f(q)| < \epsilon$ if $p, q \in T$ and $|p - q| < \delta_1$.

Now take an arbitrary grid N of R and write $R = \cup R_{ij}$. Then

$$\overline{S}(N, f) - \underline{S}(N, f) = \sum_{R_{ij} \not\subset T} (M_{ij} - m_{ij}) A(R_{ij}) + \sum_{R_{ij} \subset T} (M_{ij} - m_{ij}) A(R_{ij}). \quad (3)$$

If $d(N) < \delta_1$, then the second term on the right side of (3) is at most $\epsilon A(T) \leq \epsilon A(R)$. The first term on the right side of (3) is at most $2M \sum\{A(R_{ij}) : R_{ij} \not\subset T\}$ where M is the bound of f on R. However, this latter sum is the area of the circumscribing set for S corresponding to the grid N so it can be made less than $A(S) + \epsilon < 2\epsilon$ provided $d(N) < \delta_2$. Thus

$$\overline{S}(N, f) - \underline{S}(N, f) \leq 4M\epsilon + \epsilon A(R)$$

provided $d(N) < \min(\delta_1, \delta_2)$. It now follows, as in the proof of Theorem 1.2 that f is integrable on R. □

[5] This proof follows the book closely and differs slightly from the proof I gave in class

6 Wednesday October 10, 2007

6.1 Integration over arbitrary subsets

Let f be a bounded function on a bounded set $D \subset \mathbf{R}^2$. Choose a rectangle $R \supset D$ and define F on R as follows: $F(p) = f(p)$ for $p \in D$ and $F(p) = 0$ for $p \in R - D$. We shall say that f is integrable over D if F is integrable over R. In this case, we define $\int\int_D f = \int\int_R F$.

It is easy to see that this definition does not depend on the choice of the rectangle R; indeed, if $D \subset R$ and $D \subset R'$ and F and F' are the corresponding extensions of f to R and R' respectively, then $D \subset R'' := R \cap R'$ and $\int\int_R F = \int\int_{R''} F'' = \int\int_{R'} F'$.

Theorem 6.1 (Theorem 3 on page 175 of Buck) *If f is a bounded function on a bounded Jordan measurable set $D \subset \mathbf{R}^2$ and f is continuous on $D - E$ where E is a subset of D of area zero, then f is integrable on D.*

Proof: Choose a rectangle R containing D and define F on R by $F(p) = f(p)$ for $p \in D$ and $F(p) = 0$ for $p \in R - D$. Then F is continuous at least on $R - (E \cup \operatorname{bdy} D)$. Since the union of two sets of zero area has zero area, and $\operatorname{bdy} D$ has zero area by Proposition 5.1, it follows that F is integrable over R by Theorem 5.2. Hence f is integrable over D. □

6.2 More on integration (not covered in lecture)

6.2.1 About Assignment 3

To deal with Assignment 3, you can use the following lemma:

Lemma 6.2 *Let f be integrable on $[a, b]$ and let $\epsilon > 0$. Then there exists a closed subinterval $J \subset [a, b]$ such that*

$$M(f, J) - m(f, J) < \epsilon.$$

Here, we are using the notation $M(f, S) = \sup\{f(x) : x \in S\}$ and $m(f, S) = \inf\{f(x) : x \in S\}$.

Proof: (Sketch) For our given ϵ, choose a partition P_ϵ such that $\overline{S}(P_\epsilon) - \underline{S}(P_\epsilon) < \epsilon$. Choose J to be the subinterval I_i determined by this partition for which the value $M_i - m_i$ is the smallest, that is,

$$M_i - m_i \leq M_j - m_j \text{ for all } 1 \leq j \leq m.$$

Since for all j, $(M_i - m_i)(x_j - x_{j-1}) \leq (M_j - m_j)(x_j - x_{j-1})$, we get

$$(b - a)(M_i - m_i) \leq \overline{S}(P_\epsilon) - \underline{S}(P_\epsilon) < \epsilon,$$

so $M_i - m_i < \epsilon/(b - a)$, which is just as good as $M_i - m_i < \epsilon$.

Here is the solution to Assignment 3(A): Apply the lemma successively with $\epsilon = 1/n$, $n = 1, 2, \ldots$. You get a nested sequence of closed intervals

$$[a, b] \supset J_1 \supset J_2 \supset \cdots \supset J_n \supset \cdots$$

such that $M(f, J_n) - m(f, J_n) < 1/n$ for $n = 1, 2, \ldots$. By a well known property of the real number system, the intersection $\cap_{k=1}^{\infty} J_k$ is not empty. You now need to show that f is continuous at any point x_0 of this intersection.

There are several remarks to be made in connection with the foregoing. First of all, if you examine the last part of the solution, you find that you may only get one-sided continuity of f at x_0. Not to worry; this is enough to solve part (B) of Assignment 3. On the other hand, the above proof can be modified to show that f is indeed continuous at some point x_0. You should verify for yourself these last two statements.

The next remark is that the above proof actually shows that f has infinitely many points of continuity. Do you see why?

Assignment 5 Let f be an integrable function on $[a, b]$. Prove that if $f(x) > 0$ for every $x \in [a, b]$, then $\int_a^b f > 0$. (Note that by Theorem 6.3, $\int_a^b f \geq 0$; the point is to prove that $\int_a^b f \neq 0$.)

6.2.2 Properties of integrals

The following theorem differs from Theorem 4 on page 176 of Buck in the following respects. In Theorem 4,page 176 of Buck, f and g are assumed continuous, and the integration is over arbitrary sets, not necessarily compact rectangles. Moreover, there is a fifth statement, which I shall state and prove separately, but only (for convenience) for $n = 1$ (see Theorem 6.4 below).

Theorem 6.3 (Theorem 4 on page 176 of Buck) *Let f and g be integrable functions on the compact rectangle $R \subset \mathbf{R}^2$. Let c be any real number. Then:*

1. *$f + g$ is integrable on R and $\int_R (f + g) = \int_R f + \int_R g$*
2. *cf is integrable on R and $\int_R cf = c \int_R f$*
3. *If $f(x) \geq 0$ for all $x \in R$, then $\int_R f \geq 0$*
4. *$|f|$ is integrable on R and $|\int_R f| \leq \int_R |f|$.*

Proof: It is trivial to verify that for any grid N and choice C,

$$S(f + g, N, C) = S(f, N, C) + S(g, N, C).$$

For $\epsilon > 0$, choose $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|S(f, N, C) - \int_R f| < \frac{\epsilon}{2} \text{ for all } C \text{ and all } N \text{ with } d(N) < \delta_1,$$

and

$$|S(g,N,C) - \int_R g| < \frac{\epsilon}{2} \text{ for all } C \text{ and all } N \text{ with } d(N) < \delta_2.$$

Then, with $\delta = \min\{\delta_1, \delta_2\}$, we have, for all choices C and all grids N with $d(N) < \delta$,

$$\begin{aligned} |S(f+g,N,C) - \int_R f - \int_R g| &= |S(f,N,C) + S(g,N,C) - \int_R f - \int_R g| \\ &\leq |S(f,N,C) - \int_R f| + |S(g,N,C) - \int_R g| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This proves the first statement and the proof of the second statement is similar. We refer [Buck, page 177] for the proofs of the third and fourth statements.

The following theorem contains two important properties of the integral. The first one will be part of a homework assignment (see Assignment 6 below). The second one will be proved later in this subsection.

Theorem 6.4 *Let $[a,b]$ be a compact interval in $\mathbf{R}$ and let $f : [a,b] \to \mathbf{R}$ be a bounded function.*

(a) *If f is integrable on $[a,b]$ then f is integrable on any compact subinterval $[c,d] \subset [a,b]$.*

(b) *If $a < c < b$ and if f is integrable on $[a,c]$ and on $[c,b]$, then f is integrable on $[a,b]$ and*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Assignment 6 Prove the following statements, which will result in a proof of (a) of Theorem 6.4. (These will be stated for $n = 1$ but both the statements and proofs are valid for $n = 2$ and in fact for any n)

(A) Prove that f is integrable on $[a,b]$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\overline{S}(P) - \underline{S}(P) < \epsilon$ for all partitions P with $d(P) < \delta$. (Hint: The proof is contained in the proof of Theorem 1.2 given above.)

(B) (converse of (A)) Suppose that f is integrable on $[a,b]$. Prove that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\overline{S}(P) - \underline{S}(P) < \epsilon$ for all partitions P with $d(P) < \delta$.

> Hint: I shall give an incorrect proof of this statement, which however, will give the correct idea. For any partition P, with $[a,b] = \cup_{i=1}^m I_i$, pick t_i' and t_i'' in I_i such that $m_i := \inf\{f(x) : x \in I_i\} = f(t_i')$ and $M_i := \sup\{f(x) : x \in I_i\} = f(t_i'')$. Then $\overline{S}(P) = S(f,P,C'')$

where $C' = \{t'_1, \dots, t'_m\}$ and $\underline{S}(P) = S(f, P, C'')$ where $C'' = \{t''_1, \dots, t''_m\}$. We now have, for $d(P) < \delta$,

$$\begin{aligned} |\overline{S}(P) - \underline{S}(P)| &= |S(f,P,C') - S(f,P,C'')| \\ &= |S(f,P,C') - \int_I f| + |\int_I f - S(f,P,C'')| \\ &\leq |S(f,P,C') - \int_I f + \int_I f - S(f,P,C'')| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

The problem with this proof is that f is not necessarily continuous, so there is no guarantee that the points t'_i, t''_i exist. This can be corrected by just using the definition of the numbers m_i, M_i—that is what you have to do.

(C) Let $[c, d] \subset [a, b]$ and let P be a partition of $[a, b]$ which includes the two points c, d. Let $P_2 := P \cap [c, d]$ (which is a partition of $[c, d]$). Prove that

$$\overline{S}(P_2) - \underline{S}(P_2) \leq \overline{S}(P) - \underline{S}(P).$$

Note that, for example, $\overline{S}(P_2)$ is an upper Riemann sum for f on the interval $[c, d]$, and $\overline{S}(P)$ is an upper Riemann sum for f on the interval $[a, b]$. You can use the notation $\overline{S}(P_2) = \overline{S}(P_2, [c, d])$ and $\overline{S}(P) = \overline{S}(P, [a, b])$ to remind yourself of these facts. I will use a similar notation in the proof of Theorem 6.4(b) below.

(D) Use (A) and (C) to prove (a) of Theorem 6.4.

We now turn to the proof of (b) of Theorem 6.4.

Proof of (b) of Theorem 6.4: Let $v_1 := \int_a^c f$ and $v_2 := \int_c^b f$, which are assumed to exist. This means that for every $\epsilon > 0$ there exist $\delta_1 > 0, \delta_2 > 0$ such that

$$|S(f, P_1, C_1, [a, c]) - v_1| < \epsilon \text{ if } d(P_1) < \delta_1, \forall C_1, \tag{4}$$

and

$$|S(f, P_2, C_2, [c, b]) - v_2| < \epsilon \text{ if } d(P_2) < \delta_2, \forall C_2. \tag{5}$$

Here, P_1 is a partition of $[a, c]$ and C_1 is a choice of points corresponding to the subintervals of P_1. Similarly for P_2, C_2 on $[c, b]$.

We have to prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|S(f, P, C, [a, b]) - v_1 - v_2| < \epsilon \text{ if } d(P) < \delta, \forall C. \tag{6}$$

Here, P is a partition of $[a, b]$ and C is a choice of points corresponding to the subintervals of P.

Let's get down to business. Let P be any partition of $[a, b]$. Write $P = \{a = x_0 < x_1 < \cdots < x_m = b\}$ and let $C = \{t_1, \dots, t_m\}$ be any choice of points corresponding to P. There is a unique k such that $x_{k-1} \le c < x_k$. Define partitions P_1 of $[a,c]$ and P_2 of $[c,b]$ by

$$P_1 = \{a = x_0 < x_1 < \cdots < x_{k-1}\} \cup \{c\} \text{ and } P_2 = \{c = x_k < x_{k+1} < \cdots < x_m = b\}.$$

Define choices C_1 and C_2 corresponding to P_1 and P_2 respectively as follows:

$$C_1 = \{t_1, \ldots, t_{k-1}, t'_k\} \text{ and } C_2 = \{t''_k, t_{k+1}, \ldots, t_m\},$$

where t'_k and t''_k are chosen so that $\{t'_k, t''_k\} = \{c, t_k\}$, that is, if $t_k \le c$ define $t'_k = t_k$ and $t''_k = c$, whereas, if $c < t_k$, define $t'_k = c$ and $t''_k = t_k$.

Now let's calculate:

- $S(f, P, C, [a,b]) = \sum_{j=1}^{k-1} f(t_j)\ell(I_j) + f(t_k)\ell(I_k) + \sum_{j=k+1}^{m} f(t_j)\ell(I_j)$
- $S(f, P_1, C_1, [a,c]) = \sum_{j=1}^{k-1} f(t_j)\ell(I_j) + f(t'_k)\ell([x_{k-1}, c])$
- $S(f, P_2, C_2, [c,b]) = f(t''_k)\ell([c, x_k]) + \sum_{j=k+1}^{m} f(t_j)\ell(I_j)$.

We have some cancellation here:

$$S(f,P,C,[a,b]) - S(f,P_1,C_1,[a,c]) - S(f,P_2,C_2,[c,b]) = f(t_k)\ell(I_k) - f(t'_k)(c - x_{k-1}) - f(t''_k)(x_k - c). \quad (7)$$

We are now almost done: let $M := \sup\{|f(x)| : x \in [a,b]\}$ and set $\delta = \min\{\delta_1, \delta_2, \epsilon\}$. Then for any P with $d(P) < \delta$ and for any choice C, we have (by (4), (5), and (7)),

$$\begin{aligned}
|S(f,P,C,[a,b]) - v_1 - v_2| &\le |S(f,P,C,[a,b]) - S(f,P_1,C_1,[a,c]) - S(f,P_2,C_2,[c,b])| \\
&+ |S(f,P_1,C_1,[a,c]) - v_1| + |S(f,P_2,C_2,[c,b]) - v_2| \\
&< |f(t_k)\ell(I_k) - f(t'_k)(c - x_{k-1}) - f(t''_k)(x_k - c)| + \epsilon + \epsilon \\
&< 2M\ell(I_k) + 2\epsilon < 2(M+1)\epsilon.
\end{aligned}$$

It is now clear that by a better choice of "ϵ", we will have (6), and the proof of (b) of Theorem 6.4 is complete.

Assignment 7 *Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x. If f integrable on $[0,1]$?*

Assignment 8 *Let f be integrable on $[a,b]$ and suppose that g is a function on $[a,b]$ such that $g(x) = f(x)$ except for finitely many x in $[a,b]$. Show that g is integrable on $[a,b]$ and that $\int_a^b f = \int_a^b g$.*

Assignment 9 **(a)** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $[a,b]$, and suppose that $f_n \to f$ uniformly on $[a,b]$. Why is f integrable on $[a,b]$? Prove that*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

(b) *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of integrable functions on $[a,b]$, and suppose that $f_n \to f$ uniformly on $[a,b]$. Prove that f is integrable on $[a,b]$ and that*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Assignment 10 *Show that f is integrable on $[-1,1]$ if*

(a) $f(x) = \sin(1/x)$ *for* $x \neq 0$ *and* $f(0) = 0$

(b) $f(x) = x\,sgn(\sin(1/x))$ *for* $x \neq 0$ *and* $f(0) = 0$.

Assignment 11 *For each rational number x, write $x = p/q$ where p, q are integers with no common factors and $q > 0$. Define $f(x) = 1/q$ for x rational and $f(x) = 0$ if x is irrational. Show that f is integrable on every compact interval $[a,b]$ and that $\int_a^b f = 0$.*

6.2.3 Another definition of integrability

Theorem 6.5 (A) *A bounded function f on a compact interval $I \subset \mathbf{R}$ is integrable on I if and only if*

$$\text{for every } \epsilon > 0, \text{ there exists a partition } P \text{ such that } \overline{S}(P) - \underline{S}(P) < \epsilon. \quad (8)$$

(B) *A bounded function f on a compact rectangle $R \subset \mathbf{R}^2$ is integrable on R if and only if*

$$\text{for every } \epsilon > 0, \text{ there exists a grid } N \text{ such that } \overline{S}(N) - \underline{S}(N) < \epsilon.$$

Proof: We shall write out the proof of the first statement. The second one involves exactly the same ideas, but the notation is more cumbersome. Maybe you should write out the proof of the second statement for practice.

We shall use the criterion established in Assignment 6(A),(B), namely, that f is integrable if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\overline{S}(P) - \underline{S}(P) < \epsilon$ for all partitions P with $d(P) < \delta$.

Step 1: If f is integrable, and $\epsilon > 0$, then choose δ as in the previous paragraph. Choose any partition P_0 with $d(P_0) < \delta$. Then (8) is satisfied. This is the easy part of the proof.

Step 2: Assume that (8) holds. We shall prove that f is integrable by again using the criterion established in Assignment 6(A),(B). So, let $\epsilon > 0$, so that we can get on with finding an appropriate δ. By assumption, there is a partition P_0 such that

$$\overline{S}(P_0) - \underline{S}(P_0) < \epsilon/2. \quad (9)$$

Let m be the number of subintervals determined by the partition P_0 and let B be a bound for f: $|f(x)| \leq B$ for every $x \in I$. Now set $\delta := \epsilon/8mB$. Miraculously, this

δ does the job. To show this we take any partition P with $d(P) < \delta$ and proceed to show that $\overline{S}(P) - \underline{S}(P) < \epsilon$.

We start with the following

CLAIM: With Q defined by $Q = P_0 \cup P$, we have

$$\underline{S}(Q) - \underline{S}(P) \leq 2mB \cdot d(P) \tag{10}$$

and

$$\overline{S}(P) - \overline{S}(Q) \leq 2mB \cdot d(P).$$

Assume for a moment that this claim has been proved. Then since $d(P) < \delta$, and $\delta = \epsilon/8mB$, (10) implies $\underline{S}(Q) - \underline{S}(P) \leq 2mB \cdot d(P) < \epsilon/4$, and $\underline{S}(P_0) \leq \underline{S}(Q)$ implies $\underline{S}(P_0) - \underline{S}(P) \leq \underline{S}(Q) - \underline{S}(P) < \epsilon/4$. Similarly $\overline{S}(P) - \overline{S}(Q) < \epsilon/4$ and $\overline{S}(P_0) \geq \overline{S}(Q)$ implies $\overline{S}(P) - \overline{S}(P_0) \leq \overline{S}(P) - \overline{S}(Q) < \epsilon/4$. So

$$\overline{S}(P) - \underline{S}(P) < [\epsilon/4 + \overline{S}(P_0)] + [\epsilon/4 - \underline{S}(P_0)] = \overline{S}(P_0) - \underline{S}(P_0) + \epsilon/2$$

and by (9), $\overline{S}(P) - \underline{S}(P) < \epsilon/2 + \epsilon/2 = \epsilon$.

It remains to prove the above claim(s). Well, let's first establish some notation: with

$$P_0 = \{a = s_0 < s_1 < \cdots < s_{m-1} < s_m = b\}$$

and

$$P = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\},$$

we have

$$P \subset Q_1 := P \cup \{s_1\} \subset Q_2 := P \cup \{s_1, s_2\} \subset \cdots \subset Q_{m-1} = P \cup \{s_1, \ldots, s_{m-1}\} = Q.$$

Now $\underline{S}(Q_1) - \underline{S}(P)$ is of the form (let u denote s_1)

$$m(f, [t_{k-1}, u])(u - t_{k-1}) + m(f, [u, t_k])(t_k - u) - m(f, [t_{k-1}, t_k])(t_k - t_{k-1})$$

for some k and so

$$|\underline{S}(Q_1) - \underline{S}(P)| \leq B(u - t_{k-1}) + B(t_k - u) + B(t_k - t_{k-1}) \leq 2B \cdot d(P). \tag{11}$$

Similarly,

$$|\underline{S}(Q_2) - \underline{S}(Q_1)| \leq 2Bd(Q_1) \leq 2B \cdot d(P) \tag{12}$$

$$\cdots$$

$$|\underline{S}(Q) - \underline{S}(Q_{m-2})| = |\underline{S}(Q_{m-1}) - \underline{S}(Q_{m-2})| \leq 2Bd(Q_{m-2}) \leq 2B \cdot d(P). \tag{13}$$

Adding up the $m-1$ inequalities (11)-(13), you get $\underline{S}(Q) - \underline{S}(P) \leq 2(m-1)B \cdot d(P)$. This proves the first statement in the claim. Let's believe the companion statement so we can stop.

6.3 Set functions

A set function is any function $F : \mathcal{S} \to \mathbf{R}$, where $\mathcal{S}$ is a given collection of subsets of $\mathbf{R}^2$. Example: $\mathcal{S} =$ the collection of all bounded subsets, $F(D) = \overline{A}(D)$ for $D \in \mathcal{S}$. A set function F is said to be finitely additive if whenever $S_1, S_2 \in \mathcal{S}$ and $S_1 \cap S_2 = \emptyset$, then $F(S_1 \cup S_2) = F(S_1) + F(S_2)$. Exercises 2 and 3 on page 381 give some properties of finitely additive set functions.

The central example of a finitely additive set function is the following: let $\phi : R^2 \to \mathbf{R}$ be a continuous function. Let $\mathcal{S}$ be the collection of compact subsets of $\mathbf{R}^2$ and for $S \in \mathcal{S}$, let $F(S) = \int\int_S \phi$. In the special case when $\phi(p) = 1$ for every $p \in \mathbf{R}^2$, $F(S) = A(S)$ if S is Jordan measurable.

7 Friday October 12, 2007

7.1 Differentiability of set functions

Let $\mathcal{S}$ be any collection of subsets of $\mathbf{R}^2$ which includes all rectangles and let $F : \mathcal{S} \to \mathbf{R}$ be a set function (not necessarily finitely additive). For $p_0 \in \mathbf{R}^2$, we shall say $\lim_{R\downarrow p_0} F(R)$ exists if there is a real number c (depending on p_0) such that, for every $\epsilon > 0$, there exists $\delta > 0$ (depending on ϵ and p_0) with $|F(R) - c| < \epsilon$ whenever $p_0 \in R$ and $\operatorname{diam} R < \delta$. Here, for any set E, it's diameter is $\operatorname{diam} E = \sup\{|p-q| : p, q \in E\}$.

An arbitrary set function F defined on a collection of subsets of $\mathbf{R}^2$ which includes all rectangles is *differentiable* on a set D if for every point $p \in D$, the following limit exists:

$$\lim_{R\downarrow p} \frac{F(R)}{A(R)}.$$

The set function F is said to be *uniformly differentiable* on the set D if it is differentiable at each point p of D, and if for every $\epsilon > 0$, there exists $\delta > 0$ depending only on D but not p, such that, denoting $\lim_{R\downarrow p} \frac{F(R)}{A(R)}$ by $c(p)$ we have

$$\left|\frac{F(R)}{A(R)} - c(p)\right| < \epsilon \text{ for all } R \text{ with } p \in R \cap D \text{ and } \operatorname{diam} R < \delta.$$

Theorem 7.1 (Theorem 1 on page 378 of Buck ($n = 2$)) *Let $\phi : \mathbf{R}^2 \to \mathbf{R}$ be a continuous function, and let F be the "indefinite integral" of f, that is, $F(S) = \int\int_S \phi$ for S a compact subset of $\mathbf{R}^2$. Then F is finitely additive, is uniformly differentiable on any compact subset E of $\mathbf{R}^2$, and it's derivative equals ϕ everywhere on $\mathbf{R}^2$.*

Proof: There is a compact set $E_1 \supset E$ and $\delta_1 > 0$ such that

$$q \in \mathbf{R}^2, p \in E, |p - q| < \delta_1 \Rightarrow q \in E_1.$$

Indeed, if $E \subset B(0, M)$ let $E_1 = \overline{B(0, 2M)}$ and $\delta_1 = M/2$.

Since ϕ is uniformly continuous on E_1, given $\epsilon > 0$, choose $\delta > 0$ with $\delta < \delta_1$ such that

$$p, q \in E_1, |p - q| < \delta \Rightarrow |\phi(p) - \phi(q)| < \epsilon.$$

In particular,

$$p \in E, q \in \mathbf{R}^2, |p-q| < \delta \Rightarrow |\phi(p) - \phi(q)| < \epsilon.$$

If R is any rectangle with $\operatorname{diam} R < \delta$ containing the point p of E, then $R \subset E_1$ and by the mean value for integrals (see Exercise 5, page 178 of Buck, $g = 1$)

$$F(R) = \int\int_R \phi = \phi(q)A(R) \text{ for some } q \in R.$$

Thus for any $p \in R$

$$\left|\frac{F(R)}{A(R)} - \phi(p)|\right| = |\phi(q) - \phi(p)| < \epsilon.\square$$

7.2 Characterization of indefinite integral (Undergraduate Radon-Nikodym theorem)

A set function F is *area continuous*, a.c. for short, if $F(S) = 0$ whenever $A(S) = 0$. An example is the indefinite integral of a continuous function (see Exercise 6 on page 381).

Theorem 7.2 (Theorem 2 on page 379 of Buck ($n = 2$)) *If F is a finitely additive and area continuous set function which is differentiable everywhere on $\mathbf{R}^2$ and moreover uniformly differentiable on compact set, then letting f denote the derivative of F*

(a) *f is continuous*

(b) *$F(R) = \int\int_R f$ for all rectanges $R \subset \mathbf{R}^2$*

(c) *If $F(S) \geq 0$ for every S, then (b) holds for all compact sets S which are Jordan measurable.*

Proof: Postponed to a later lecture.

8 Monday October 15, 2007

8.1 Additional tools for the proof of Theorem 8.7

Step 1 Theorem 8.1 (Theorem 3 on page 382 of Buck) *If $L : \mathbf{R}^n \to \mathbf{R}^n$ is a linear transformation and D is a bounded Jordan measurable subset of $\mathbf{R}^n$, then $L(D)$ is Jordan measurable and $v(L(D)) = |\det L|v(D)$.*

Proof: Postponed to a later lecture.

Step 2 Theorem 8.2 (Theorem 4 on page 385 of Buck) *Let $T : \Omega \to \mathbf{R}^n$ be a transformation of class C^1 on an open set $\Omega \subset \mathbf{R}^n$ and let E be a compact subset of Ω with $v(E) = 0$. Then $v(T(E)) = 0$.*

Proof: Postponed to a later lecture

Step 3 Corollary 8.3 (Corollary on page 386 of Buck) *Let $T : \Omega \to \mathbf{R}^n$ be a transformation of class C^1 on an open set $\Omega \subset \mathbf{R}^n$ and suppose that $\det T'(p) \neq 0$ for all $p \in \Omega$. For any compact Jordan measurable set D, $T(D)$ is also Jordan measurable.*[6]

Proof: Postponed to a later lecture

Step 4 Lemma 8.4 (Lemma 1 on page 387 of Buck) *Let $T : \Omega \to \mathbf{R}^3$ be a one-to-one transformation*[7] *of class C^1 on an open set $\Omega \subset \mathbf{R}^3$. For each $p \in \Omega$*

$$\lim_{C \downarrow p} \frac{v(T(C))}{v(C)} = |\det T'(p)| \text{ uniformly on compact subsets of } \Omega.$$

(C denotes a cube with center p).

Proof: Postponed to a later lecture

Step 5 Theorem 8.5 (Theorem 5 on page 386 of Buck) *Let $T : \Omega \to \mathbf{R}^3$ be a one-to-one transformation of class C^1 on an open set $\Omega \subset \mathbf{R}^3$ and suppose that $\det T'(p) \neq 0$ for every $p \in \Omega$*[8]. *For every compact set $D \subset \Omega$,*

$$v(T(D)) = \int\int\int_D |\det T'|.$$

Proof: Postponed to a later lecture

Step 6 Theorem 8.6 (Theorem 8 on page 274 of Buck) (Tietze extension theorem) *If f is a bounded continuous function defined on a closed subset E of $\mathbf{R}^n$, then there is a bounded continuous function $\tilde{f}$ on $\mathbf{R}^n$ with the same bound as f, that is,*

$$\sup_{p \in \mathbf{R}^n} |\tilde{f}(p)| = \sup_{p \in E} |f(p)|,$$

and which extends f: $\tilde{f}(p) = f(p)$ for all $p \in E$.

Proof: Postponed to a later lecture

[6] Correction: In class I mistakenly stated this with the assumption that T was one-to-one on Ω; my apologies to Professor Herr Doktor Buck

[7] Is the assumption of one-to-one really used?

[8] Isn't it true that if T is one-to-one, then $\det T'(p) \neq 0$?

8.2 Change of variables in a multiple integral

Theorem 8.7 (Theorem 6 on page 391 of Buck ($n = 3$)) *Let $T : \Omega \to \mathbf{R}^3$ be a one-to-one transformation of class C^1 on an open set $\Omega \subset \mathbf{R}^3$ satisfying $\det T'(p) \neq 0$ for every $p \in \Omega$. For every compact set $D \subset \Omega$ and for every continuous function f on $T(D)$,*

$$\int\int\int_{T(D)} f = \int\int\int_D f \circ T\,|\det T'|.$$

Proof: In the special case where f is a constant function, this theorem reduces to Step 4.

Since $T(D)$ is compact, f is bounded on $T(D)$, say $|f(p)| \leq M$ for $p \in T(D)$. Writing $g := M - f$, it suffices to prove the theorem for g. In other words, we may assume WLOG that $f(p) \geq 0$ for every $p \in T(D)$. Thus we can use part (c) of Theorem 7.2 in what follows.

Define a set function $F : \{\text{Jordan measurable subsets of } \Omega\} \to \mathbf{R}$ by

$$F(S) = \int\int\int_{T(S)} f.$$

Given the compact set $D \subset \Omega$, take a compact set E with $D \subset \operatorname{int} E \subset E \subset \Omega$ and let $C \subset E$ denote a cube. By the mean value theorem for integrals, there is a point $q_C \in T(C)$ such that $F(C) = f(q_C)v(T(C))$. As $C \downarrow p_0 \in E$, $T(C) \downarrow T(p_0)$ so that $q_C \to T(p_0)$. Thus

$$\lim_{C\downarrow p_0} \frac{F(C)}{v(C)} = \lim_{C\downarrow p_0} f(q_C)\frac{v(T(C))}{v(C)} = \lim_{q_C \to T(p_0)} f(q_C) \lim_{C\downarrow p_0} \frac{v(T(C))}{v(C)} = f(T(p_0))|\det T'(p_0|$$

by the continuity of f and Lemma 8.4.

Since f is uniformly continuous on E, this limit is uniform on E and therefore the derivative of the set function F is $f \circ T \cdot |\det T'|$, so that

$$\int\int\int_{T(D)} f = F(D) = \int\int\int_D f \circ T \cdot |\det T'|.\square$$

9 Wednesday October 17, 2007

9.1 Proof of Theorem 8.2 (Step 2)

This proof is taken from the book: Mathematical Analysis, by T. M. Apostol 1957, pages 257–258.

By the Corollary to the mean value theorem (Theorem 12 on page 350 of Buck) on page 351 of Buck , there exist $M > 0, \delta > 0$ such that

$$|T(p) - T(q)| \leq M|p - q| \text{ for all } p, q \in E \text{ with } |p - q| < \delta.$$

For the details of the proof of the mean value theorem, see Theorem 23.2 on page 43 of the minutes for Math 140C Fall 2006 at *https://math.uci.edu/ brusso/140cdec3.pdf*.

Let $\epsilon > 0$. We shall show that $T(E)$ is contained in a finite union of cubes of total volume no more than $\epsilon n^{n/2}(2M)^n$. By Proposition 5.1, $v(T(E)) = 0$.

Begin by enclosing E in an n-dimensional box $B = I^n$ where $I = [a,b] \subset \mathbf{R}$. Corresponding to our given ϵ, there is a grid N_ϵ of B with $\overline{S}(N,E) < \epsilon$ for all grids $N \supset N_\epsilon$, and we may assume that $d(N_\epsilon) < \delta$.

Choose a grid $N_1 \supset N_\epsilon$ which divides B into cubes with side $\Delta < \delta/\sqrt{n}$. Let m be the number of those cubes (call them $B_1, \ldots, B_m$) which intersect with E. Then we have $m\Delta^n = \overline{S}(N_1, E) < \epsilon$.

For each $1 \le k \le m$, pick a point $x^{(k)} \in B_k \cap E$. The for every $y \in B_k \cap E$, $|y - x^{(k)}| \le \Delta\sqrt{n} < \delta$, so that

$$|T(y) - T(x^{(k)})| \le M|y - x^{(k)}| \le M\Delta\sqrt{n}.$$

This says that $T(y)$ belongs to the ball $B(T(x^{(k)}), M\sqrt{n}\Delta)$ and this ball is in turn contained in a cube with side length $2M\Delta\sqrt{n}$ and hence volume $(2M\Delta\sqrt{n})^n$. Thus $T(E)$ is contained in a union of m cubes each of volume $(2M\Delta\sqrt{n})^n$, and hence of total volume $m\Delta^n n^{n/2}(2M)^n \le \epsilon n^{n/2}(2M)^n$, as required. □

9.2 Proof of Corollary 8.3 (Step 3)

By the open mapping theorem (see Theorem 25.2 on page 49—the proof is on page 52—of the minutes for Math 140C Fall 2006 at *https://math.uci.edu/ brusso/140cdec3.pdf*; or Theorem 15 on page 356 of Buck), $T(\operatorname{int} D)$ is an open subset of $\mathbf{R}^n$ and is therefore contained in $\operatorname{int} T(D)$. Since D and $T(D)$ are both closed, we have disjoint unions

$$D = \operatorname{int} D \cup \operatorname{bdy} D \text{ and } T(D) = \operatorname{int} T(D) \cup \operatorname{bdy} T(D).$$

Thus $\operatorname{bdy} T(D) \subset T(\operatorname{bdy} D)$.

If D is Jordan measurable, then $v(\operatorname{bdy} D) = 0$ so that $v(\operatorname{bdy} T(D)) \le v(T(\operatorname{bdy} D)) = 0$, proving that $T(D)$ is Jordan measurable. □

10 Friday October 19, 2007

10.1 Proof of Lemma 8.4 (Step 4)

We begin by quoting Theorem 10 on page 344 of Buck (see Theorem 18.5 on page 34 of the minutes for Math 140C Fall 2006 at *https://math.uci.edu/ brusso/140cdec3.pdf*, namely

$$\lim_{p \to p_0} \frac{|T(p) - T(p_0) - T'(p_0)(p - p_0)|}{|p - p_0|} = 0 \text{ uniformly on } E \tag{14}$$

Now we make the preliminary assumption that $T'(p_0) = I$. The general case will be reduced to this case. From (14), we can write $T(p) = T(p_0) + (p - p_0) + R(p)$ where $|R(p)| < \epsilon|p - p_0|$ whenever p is such that $|p - p_0| < \delta$.

By the triangle inequality

$$|T(p) - T(p_0)| \le |p - p_0| + |R(p)| \le (1 + \epsilon)|p - p_0|,$$

provided $|p - p_0| < \delta$.

By the "backwards" triangle inequality

$$|T(p) - T(p_0)| = |p - p_0 + R(p)| \geq |p - p_0| - |R(p)| \geq (1 - \epsilon)|p - p_0|,$$

provided $|p - p_0| < \delta$.

Combining these last two equations yields

$$(1 - \epsilon) \leq \frac{|T(p) - T(p_0)|}{|p - p_0|} \leq (1 - \epsilon) \text{ provided } |p - p_0| < \delta.$$

Letting C be a cube centered at p_0 and having side length Δ, this last equation implies that $T(C)$ contains a cube of side length $(1 - 2\epsilon)\Delta$ and is contained in a cube of side length $(1 + 2\epsilon)\Delta$ so that

$$(1 - 2\epsilon)^3 \leq \frac{v(T(C))}{v(C)} \leq (1 + 2\epsilon)^3 \text{ provided } |p - p_0| < \delta,$$

which shows that

$$\lim_{C \downarrow p_0} \frac{v(T(C))}{v(C)} = 1 = \det I = \det T'(p_0).$$

Moving now to the general case of the lemma, note first that by the boundedness of linear transformations (see Theorem 8 on page 338 of Buck, or Lemma 21.1 on page 39 of the minutes for Math 140C Fall 2006 at *https://math.uci.edu/ brusso/140cdec3.pdf*) and a compactness argument, there exists $M_0 > 0$ such that

$$|(T'(p_0)^{-1}(s)| \leq M_0|s| \text{ for all } x \in \mathbf{R}^3 \text{ and all } p_0 \in E.$$

From (14), we can write $T(p) - T(p_0) = T'(p_0)(p - p_0) + R(p)$ where $|R(p)| < \epsilon|p - p_0|$ whenever p is such that $|p - p_0| < \delta$. Hence $(T'(p_0)^{-1}(T(p) - T(p_0)) = p - p_0 + (T'(p_0)^{-1}(R(p))$ and therefore, letting $T^* := (T'(p_0))^{-1} \circ T$, we have

$$T^*(p) = T^*(p_0) + p - p_0 + R^*(p)$$

where $R^*(p) := (T'(p_0)^{-1}(R(p))$ and $|R^*(p)| \leq M_0\epsilon|p - p_0|$ provided $|p - p_0| < \delta$.

By the chain rule for transformations (see Theorem 20.3 on page 36 of the minutes for Math 140C Fall 2006 at *https://math.uci.edu/ brusso/140cdec3.pdf*, or Theorem 11 on page 346 of Buck)

$$(T^*)'(p_0) = (T'(p_0)^{-1})'(T(p_0) \circ T'(p_0) = T'(p_0)^{-1}(T(p_0) \circ T'(p_0) = T'(p_0)^{-1} \circ T'(p_0) = I,$$

so that by the special case considered first we have

$$(1 - 2\epsilon M_0)^3 \leq \frac{v(T^*(C))}{v(C)} \leq (1 + 2\epsilon M_0)^3$$

and therefore

$$\lim_{C \downarrow p_0} \frac{v(T^*(C))}{v(C)} = 1.$$

But $v(T^*(C)) = v(T'(p_0)^{-1}(T(C)) = |\det T'(p_0)^{-1}|v(T(C))$ from which the lemma follows.[9] □

[9]Note that the limit is uniform on E by (14)

11 Monday October 22, 2007

11.1 Proof of Theorem 8.5 (Step 5)

Define a set function F on Jordan measurable compact sets S by $F(S) = v(T(S))$. By Lemma 8.4, F is uniformly differentiable on compact sets with derivative $|\det T'(p)|$ at p. Since T is one-to-one, F is finitely additive and by Theorem 8.2, F is area continuous (volume continuous in our case). By part (c) of Theorem 7.2, $F(D) = \int\int\int_D |\det T'(p)|$ for every Jordan measurable compact set D. Thus $v(T(D)) = F(D) = \int\int\int_D |\det T'(p)|$. □

11.2 Proof of Theorem 7.2

We show first that f is continuous. Let E be a closed cube and let $\epsilon > 0$. Choose $\delta > 0$ such that $|F(C)/v(C) - f(p)| < \epsilon$ for all $p \in E$, and for all cubes C containing p with diameter $< \delta$.

If $p_1, p_1 \in E$ and $|p_1 - p_2| < \delta/2$ pick a cube C of diameter $< \delta$ containing both p_1 and p_2. Then

$$|f(p_1) - f(p_2)| \leq |f(p_1) - F(C)/v(C)| + |F(C)/v(C) - f(p_2)| < 2\epsilon,$$

showing that f is uniformly continuous on E.

Define a set function F_0 on Jordan measurable compact sets S by $F_0(S) = \int\int\int_S f$. By Theorem 7.1, F_0 is uniformly differentiable on compact sets with derivative f. Thus the set function $H(S) = F(S) - F_0(S)$ is finitely additive, area continuous and uniformly differentiable on compact sets with derivative 0 everywhere.

We claim next that $H(C) = 0$ if C is any closed cube. For $\epsilon > 0$ and a compact set E, choose $\delta > 0$ such that $|H(C)/v(C) - 0| < \epsilon$ whenever $C \subset E$ and $\operatorname{diam} E < \delta$. For such C we therefore have $|H(C)| \leq \epsilon v(C)$.

An arbitrary cube C can be written as a non-overlapping union $\cup_{j=1}^m C_j$ of cubes C_j each of diameter less than δ. We then have

$$H(C) = \sum_{j=1}^m H(C_j) \leq \sum_{j=1}^m \epsilon v(C_j) = \epsilon v(C).$$

This shows that $H(C) = 0$, proving the claim. (proof continues in the next section)

11.3 Discussion of Theorems 8.1 and 8.6

Theorem 8.1 could be considered as a theorem in linear algebra and will be worked into an Assignment below. Theorem 8.6, which played a small but obviously important role in the proof of Theorem 8.7 is a theorem in topology whose straightforward but not transparent proof can be found in Buck (page 274), and we will skip it.

12 Wednesday October 24, 2007

12.1 Completion of the proof of Theorem 7.2

Let S be a compact Jordan measurable set. By the definition of volume, there exists a finite union of cubes $S_n \subset S$ such that $v(S_n) \to v(S)$. Since F is non-negative valued, we have $F(S_n) \leq F(S)$. But we already know that $F(S_n) = F_0(S_n) = \int\int\int_{S_n} f \to \int\int\int_S f = F_0(S)$. Indeed, $v(S) = v(S_n) + v(S - s_n)$ so that $\int\int\int_S f - \int\int\int_{S_n} f = (\int\int\int_{S_n} f + \int\int\int_{S-S_n} f) - \int\int\int_{S_n} = \int\int\int_{S-S_n} f \leq Mv(S - S_n) \to 0$. Thus $F_0(S) \leq F(S)$.

By the definition of volume, there exists a finite union of cubes $S_n \supset S$ such that $v(S_n) \to v(S)$. Since F is non-negative valued, we have $F(S_n) \geq F(S)$. As in the previous paragraph, we have $F(S_n) = F_0(S_n) = \int\int\int_{S_n} f \to \int\int\int_S f = F_0(S)$. Thus $F_0(S) \geq F(S)$. □

12.2 Curves

A *curve* is a continuous transformation $\gamma : I \to \mathbf{R}^n$, where I is a closed interval in $\mathbf{R}$, not necessarily finite. The trace of the curve γ is $\gamma(I)$. If $I = [a, b]$ where $-\infty < a < b < \infty$, the *endpoints* of γ are $\gamma(a)$ and $\gamma(b)$. The curve γ is *closed* if $\gamma(a) = \gamma(b)$ and *simple* if γ is one-to-one on the interior of I. The curve γ is *smooth* if γ is of class C^1 on I and $\gamma'(t)$ has rank 1 for every $t \in I$. This means $\gamma'(t) \neq 0$ for all $t \in I$.

A *line* is a curve of the form $\gamma(t) = p_0 + tv$, $t \in \mathbf{R}$, where $v, p_0 \in \mathbf{R}^n$ and $v \neq 0$. The direction of the line is v. Motivated by the case of a line, we define the *direction* of a smooth curve at p_0 is $v/|v|$ where $v = \gamma'(t_0)$ and $\gamma(t_0) = p_0$ The *tangent* to a curve γ at p_0 is the line $\alpha(t) = p_0 + vt$ where $v = \gamma'(t_0)$ and $\gamma(t_0) = p_0$.

The *arc length* of a smooth curve $\gamma : [a, b] \to \mathbf{R}^n$ is $L(\gamma) = \int_a^b |\gamma'(t)|\, dt$. The *geometric length* of a curve γ is

$$L'(\gamma) = \sup\{\sum_{j=0}^{n-1} |\gamma(t_{j+1} - \gamma(t_j)| : P = \{t_0, \ldots, t_n\} \text{ a partition of } [a, b]\}.$$

Intuitively, since "a straight line is the shortest distance between two points," $L'(\gamma) \leq L(\gamma)$. A curve is said to be *rectifiable* if $L'(\gamma) < \infty$. By the above, a smooth curve is rectifiable.

Theorem 12.1 (Theorem 7 on page 404 of Buck) *If γ is a smooth curve, then $L'(\gamma) = L(\gamma)$, that is, the arc length is the same as the geometric length.*

13 Friday October 26, 2007

13.1 Proof of Theorem 12.1

For simplicity we will assume that $n = 2$ and write $\gamma(t) = (X(t), Y(t))$ for $a \leq t \leq b$. By the mean value theorem in one variable

$$xxx$$

13.2 Equivalence of curves

Theorem 13.1 (Theorem 8 on page 407 of Buck) *yyy*

Theorem 13.2 (Theorem 9 on page 407 of Buck) *yyy*

13.3 Curvature

TO BE DONE LATER

14 Monday October 29, 2007

14.1 Surfaces

14.2 Normal to a surface

Theorem 14.1 (Theorem 11 on page 422 of Buck) *yyy*

15 Wednesday October 31, 2007

15.1 Tangent plane to a smooth surface

Theorem 15.1 (Theorem 12 on page 424 of Buck) *yyy*

15.2 Area of a smooth surface

15.3 Parametric Equivalence of smooth surfaces

Theorem 15.2 (Theorem 14 on page 432 of Buck) *yyy*

16 Friday November 2, 2007

CLASS CANCELLED

17 Monday November 5, 2007

17.1 Functionals

curve functionals, surface functionals, region functionals

17.2 1-forms in $\mathbf{R}^2$ and in $\mathbf{R}^3$

17.3 2-forms in $\mathbf{R}^2$ and 3-forms in $\mathbf{R}^3$

17.4 The algebra of differential forms

18 Wednesday November 7, 2007

18.1 Differentiation of forms

18.2 2-forms in $\mathbf{R}^3$

19 Friday November 9, 2007

19.1 Product of the derivatives of two 0-forms

Theorem 19.1 (Theorem 1 on page 457 of Buck) *yyy*

19.2 Forms acting on equivalent curves and surfaces

Theorem 19.2 (Theorem 2 on page 460 of Buck) *yyy*

20 Monday November 12, 2007—holiday

21 Wednesday November 14, 2007

21.1 Cross product

21.2 Gradient of a scalar field, divergence and curl of a vector field

22 Friday November 16, 2007

22.1 Vector analysis and differential forms—operations corresponding to multiplication and differentiation of forms

Theorem 22.1 (Theorem 3 on page 474 of Buck) *yyy*

23 Monday November 19, 2007

23.1 General statements of Green, Stokes, Divergence theorems, and the generalized Stokes theorem

Theorem 23.1 (Theorem 7 on page 479 of Buck) *yyy*

24 Wednesday November 21, 2007—no class

25 Friday November 23, 2007—holiday

26 Monday November 26, 2007—no class

27 Wednesday November 28,2007—no class

28 Friday November 30, 2007—class cancelled

29 Monday December 3, 2007

29.1 Green's theorem for equivalent regions

Theorem 29.1 (Theorem 8 on page 487 of Buck) *yyy*

Theorem 29.2 (Theorem 9 on page 483 of Buck) *yyy*

30 Wednesday December 5, 2007

30.1 Another proof of the change of variables theorem

Theorem 30.1 (Theorem 10 on page 488 of Buck) *yyy*

30.2 A version of Stokes' theorem

Theorem 30.2 (Theorem 11 on page 489 of Buck) *yyy*

31 Friday December 7, 2007

31.1 Divergence theorem for a cube

Theorem 31.1 (Theorem 12 on page 491 of Buck) *yyy*

31.2 Restatements of Divergence and Stokes'

Theorem 31.2 (Theorem 4 on page 474 of Buck) yyy

Theorem 31.3 (Theorem 5 on page 475 of Buck) yyy

Theorem 31.4 (Divergence theorem on page 493 of Buck) yyy

Theorem 31.5 (Stokes' theorem on page 493 of Buck) yyy