Complex Analysis
Math 147—Winter 2008

Bernard Russo

March 14, 2008

Contents

1 Monday January 7—Course information; complex numbers; Assignment 1

1.1 Course information

- Course: Mathematics 147 MWF 11:00–11:50 ET201
- Prerequisite: Math 140AB or consent of the instructor (if you have had 140A and are taking 140B concurrently, that is acceptable)
- Instructor: Bernard Russo MSTB 263 Office Hours MW 10:00-10:40 or by appointment (a good time for short questions is right after class just outside the classroom)
- There is a link to this course on Russo's web page: www.math.uci.edu/∼ brusso
- Discussion section: TuTh 11:00–11:50 HICF 100M
- Teaching Assistant: Kenn Huber
- Homework: There will be approximately 10-12 assignments with at least one week notice before the due date.
- Grading:

First midterm	February 1 (Friday of week 4)	20 percent
Second midterm	February 29 (Friday of week 8)	20 percent
Final Exam	March 21 (Friday 8:00-10:00 am)	40 percent
Homework	approximately 12 assignments	20 percent

- Holidays: January 21, February 18
- Text: George Cain "Complex Analysis", Freely available on the web (see Russo's web page or go directly to http://www.math.gatech.edu/ cain/winter99/complex.html)
- Material to be Covered: All of the text with the possible exception of chapters 8 and 11. However, there will be some material that is not in the text.
- Catalog description: Rigorous treatment of basic complex analysis: complex numbers, analytic functions, Cauchy integral theory and its consequences (Morera's Theorem, The Argument Principle, The Fundamental Theorem of Algebra, The Maximum Modulus Principle, Liouville's Theorem), power series, residue calculus, harmonic functions, conformal mapping. Students are expected to do proofs.
- Math 147 is replacing the old Math 114B, and is intended for mathematics majors. The sequence 114A-147 is acceptable for the specialization in applied mathematics. You cannot take 114A after taking 147.
- Some alternate texts that you may want to look at, in no particular order. There are a great number of such texts at the undergraduate and at the graduate level.

  Undergraduate Level

  1. S. Fisher: Complex Variables
  2. R. Churchill and J. Brown; Complex Variables and Applications
  3. J. Marsden and M. Hoffman, Basic Complex Analysis

4. E. Saff and A. Snider: Fundamentals of Complex Analysis

Graduate Level

1. L. Ahlfors; Complex Analysis
2. J. Conway; Functions of one Complex Variable
3. J. Bak and D. Newman; Complex Analysis

1.2 Complex numbers

algebra: rational numbers, real numbers, complex numbers
sum, product, difference, quotient of complex numbers
geometry: modulus, conjugate, triangle inequality
notation: real and imaginary parts of z
relations: $z\overline{z} = |z|^2$, $|\mathrm{Re}\, z| \le |z|, |\mathrm{Im}\, z| \le |z|$

Proposition 1.1 (Triangle inequality) *For any two complex numbers z and w, we have $|z + w| \le |z| + |w|$.*

Assignment 1 (1A is due January 11; 1B is due January 16)
1A: Problems 1-6 of chapter 1 of Cain.
1B: Problems 7-12 of chapter 1 of Cain.

2 Thursday January 10–Polar form. Assignment 2

Proof of Proposition 1.1

Corollary 2.1 *For any two complex numbers z and w, we have $|z - w| \ge \big|\, |z| - |w| \,\big|$.*

Assignment 2 (2A is due January 18; 2B is due January 23)
2A: Problems 1-3 and 5-9 of chapter 2 of Cain. (You may skip section 2.1 except for exercises 1–3.)
2B: Problems 10-16 of chapter 2 of Cain.

2.1 Polar coordinates

argument, principal argument, geometric interpretation of sum and product

Proposition 2.2 *For any two complex numbers z and w, we have $|zw| = |z| \cdot |w|$ and $\arg zw = \arg z + \arg w$ (modulo $2\pi\mathbf{Z}$).*

3 Friday January 11

3.1 Complex functions

Derivatives of functions:

- $f : \mathbf{R} \to \mathbf{R}$, $f'(x)$, graph is a curve
- $f : \mathbf{R}^2 \to \mathbf{R}$, $\partial f/\partial x$, $\partial f/\partial y$, graph is a surface

- $f : \mathbf{R} \to \mathbf{R}^2$, $f'(t) = (x'(t), y'(t))$, parametric equations of a curve in $\mathbf{R}^2$
- $f : \mathbf{C} \to \mathbf{C}$, $f'(z)$ (new idea)

Definition of derivative of a complex valued function of a complex variable:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

where h represents a complex number.

EXAMPLES:

- $f(z) = z$ is differentiable for every z and $f'(z) = 1$ for all $z \in \mathbf{C}$.
- $g(z) = \overline{z}$ is not differentiable for any z.

4 Monday, January 14—Limits and continuous functions

Continuous function at a point, on a set. Sums, products, differences, and quotients of continuous functions are continuous (whenever defined),

We proved the first of these two propositions (and ignored the second one!):

- If $f = u + iv$ is a complex valued function of a complex variable (with real part u and imaginary part v, then f is continuous if and only if both u and v are continuous.
- If $\lim_{z \to z_0} f(z) = L_1$ and $\lim_{z \to z_0} g(z) = L_2$, then $\lim_{z \to z_0} f(z)g(z)$ exists and equals $L_1 L_2$

We showed that the function $f(z) = \overline{z}$ is not differentiable at any point z_0.

5 Wednesday January 16—The Cauchy-Riemann equations; Assignment 3

5.1 Cauchy-Riemann equations

We proved the first of these two propositions and started the proof of the second one:

- If $f = u + iv$ is a complex valued function of a complex variable, and f is differentiable at $z_0 = x_0 + iy_0$, then u and v satisfy the Cauchy Riemann equations at (x_0, y_0).
- If $f = u + iv$ is a complex valued function of a complex variable, and u and v satisfy the Cauchy Riemann equations at (x_0, y_0), and if u_x and u_y are continuous at (x_0, y_0), then f is differentiable at $z_0 = x_0 + iy_0$,

5.2 The exponential function

Definition: $\exp z = e^x \cos y + ie^x \sin y$ for $z = x + iy$.

We proved the following two propositions:

- $\frac{d}{dz} \exp z = \exp z$

- $\exp(z+w) = \exp z \exp w$

Assignment 3 (3A is due January 23; 3B is due January 28; 3C is due January 30)
3A: Problems 1-6 of chapter 3 of Cain.
3B: Problems 8-11 of chapter 3 of Cain.
3C: Problems 12-17 of chapter 3 of Cain.

6 Friday January 18—Cauchy-Riemann equations-revisited

Finished the proof of the sufficiency of the Cauchy-Riemann equations.

7 Monday January 21—Holiday

8 Wednesday January 23—Trigonometric functions-(Prof. R. Reilly)

Definition and properties of the functions $\sin z,\ \cos z$.

9 Friday January 25—The complex logarithm and complex powers;-(Prof. A. Figotin)

Definition and properties of the multi-valued functions $\log z,\ z^c$ (c a complex number). Principal logarithm $\operatorname{Log} z$, principal value of z^c.

10 Monday January 28—-Line integrals, Assignment 4

We decided to ignore section 4.1 and use the formula

$$\int_C f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)\,dt$$

for the definition of the contour integral of $f : D \to \mathbf{C}$ over the curve C given by $\gamma : [\alpha, \beta] \to D$ for $\alpha \le t \le \beta$.

Estimate for a line integral:

Proposition 10.1 *$|\int_C f(z)\,dz| \le ML$, where $M = \sup_{z\in C} |f(z)|$ and $L = \int_\alpha^\beta |\gamma'(t)|\,dt$ is the length of C.*

Assignment 4 (4A is due February 6; 4B is due February 8)
4A: Problems 1-6 of chapter 4 of Cain.
4B: Problems 7-11 of chapter 4 of Cain.

11 Wednesday January 30—-Line integrals, Independence of the path

Lemma 11.1 *If $G : [\alpha, \beta] \to \mathbf{C}$ is integrable, then so is $|G(t)|$ and*

$$\left| \int_\alpha^\beta G(t)\, dt \right| \le \int_\alpha^\beta |G(t)|\, dt.$$

Lemma 11.2 *If $G : [\alpha, \beta] \to \mathbf{C}$ is differentiable and if $G'(t)$ is integrable, then*

$$\int_\alpha^\beta G'(t)\, dt = G(\beta) - G(\alpha).$$

Lemma 11.3 (Chain Rule) *Let γ be a real valued function defined on an open interval containing $a \in \mathbf{R}$ and suppose that γ is differentiable at a with derivative $\gamma'(a)$. Let f be a real valued function defined on an open interval containing $\gamma(a)$ and suppose that f is differentiable at $\gamma(a)$ with derivative $f'(\gamma(a))$. Then $f \circ \gamma$ is differentiable at a with derivative*

$$(f \circ \gamma)'(a) = f'(\gamma(a))\, \gamma'(a).$$

Proof: Since γ is differentiable at a, $\forall \epsilon' > 0, \exists \delta' > 0$ such that

$$|\gamma(x) - \gamma(a) - \gamma'(a)(x - a)| < \epsilon' |x - a| \quad \text{if } |x - a| < \delta'. \tag{1}$$

Since f is differentiable at $\gamma(a)$, $\forall \epsilon'' > 0, \exists \delta'' > 0$ such that

$$|f(y) - f(\gamma(a)) - f'(\gamma(a))(y - \gamma(a))| < \epsilon'' |y - \gamma(a)| \quad \text{if } |y - \gamma(a)| < \delta''. \tag{2}$$

We need to prove: $\forall \epsilon > 0, \exists \delta > 0$ such that

$$|f(\gamma(x)) - f(\gamma(a)) - f'(\gamma(a))\gamma'(a)(x - a)| < \epsilon |x - a| \quad \text{if } |x - a| < \delta. \tag{3}$$

Since γ is continuous at a, $\exists \delta_c > 0$ such that

$$|\gamma(x) - \gamma(a)| < \delta'' \text{ if } |x - a| < \delta_c. \tag{4}$$

Using (4), we may replace y in (2) by $\gamma(x)$ to obtain

$$|f(\gamma(x)) - f(\gamma(a)) - f'(\gamma(a))(\gamma(x) - \gamma(a))| < \epsilon'' |\gamma(x) - \gamma(a)| \quad \text{if } |x - a| < \delta_c. \tag{5}$$

Now set $\delta := \min\{\delta_c, \delta'\}$ and $\eta(x) := \gamma(x) - \gamma(a) - \gamma'(a)(x - a)$ so that

$$\gamma(x) - \gamma(a) = \gamma'(a)(x - a) + \eta(x) \tag{6}$$

and by (1),

$$|\eta(x)| < \epsilon' |x - a| \text{ if } |x - a| < \delta. \tag{7}$$

Now substitute (6) into (5) (in two places!) and set

$$A := f(\gamma(x)) - f(\gamma(a)) - f'(\gamma(a))[\gamma'(a)(x - a) + \eta(x)] \tag{8}$$

to obtain from (5)

$$|A| < \epsilon'' |\gamma'(a)(x - a) + \eta(x)| \quad \text{if } |x - a| < \delta. \tag{9}$$

Finally, if $|x-a|<\delta$, we have,

$$|f(\gamma(x)) - f(\gamma(a)) - f'(\gamma(a))\gamma'(a)(x-a)|$$

$$\begin{aligned}
&= |A + f'(\gamma(a))\eta(x)| \qquad \text{(by (8))} \\
&\leq |A| + |f'(\gamma(a))\eta(x)| \\
&\leq \epsilon''|\gamma'(a)||x-a| + \epsilon''|\eta(x)| + |f'(\gamma(a))||\eta(x)| \qquad \text{(by (9))} \\
&\leq [\epsilon''|\gamma'(a)| + \epsilon''\epsilon' + |f'(\gamma(a))|\epsilon']|x-a| \qquad \text{(by (7))} \\
&< \epsilon|x-a|,
\end{aligned}$$

the last step provided we simply choose ϵ' and ϵ'' so that $[\epsilon''|\gamma'(a)| + \epsilon''\epsilon' + |f'(\gamma(a))|\epsilon'] < \epsilon$. This proves (3). □

Lemma 11.1 is used in the proof of Proposition 10.1. Lemma 11.2 and the chain rule are used in the proof of the following proposition.

Proposition 11.4 *If $F, f : D \to \mathbf{C}$ are such that $F'(z) = f(z)$ for all $z \in D \subset \mathbf{C}$, then $\int_C f(z)\,dz = F(b) - F(a)$, where C is any curve in D starting at $a \in D$ and ending at $b \in D$. In other words, if the function f has an antiderivative in D, then the contour integral of f over any curve lying in D depends only on the end points of C.*

12 Friday February 1—Review for Midterm

12.1 Review Problems

12.2 Highlights of Assignments 1,2,3

13 Monday February 4—First Midterm

14 Wednesday February 6—-More on Independence of the path

Proposition 14.1 *Suppose that f is a continuous complex valued on an open connected set D and that f is path independent, that is, for every curve C lying in D, the value of the contour integral $\int_C f(z)\,dz$ depends only on the endpoints of C. Then f has an antiderivative in D.*

15 Friday February 8—Homotopic curves

Some examples pertaining to homotopic curves. In each of these examples, the two curves are homotopic to each other.

(a) Two concentric circles in an annulus D, with the same orientation.
(b) A triangle and a circle in an annulus D with the same orientation.
(c) A closed curve and a point outside the curve, in the complex plane $D = \mathbf{C}$.
(d) Two closed curves in the first quadrant D with different orientations.

16 Monday February 11—Cauchy's theorem; Assignment 5

Theorem 16.1 (Pre-Cauchy Theorem) *If C_1 and C_2 are homotopic closed curves in a region D and f is analytic in D, then*

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Theorem 16.2 (Cauchy's Theorem) *If D is a simply connected region andf is analytic in D, then for every closed curve C in D,*

$$\int_C f(z)\,dz = 0.$$

Corollary 16.3 (Fundamental Application) *Let D be a region which is contained in the complement of the union of two closed sets (call them holes!) in* **C** *and let f be analytic in D. Let C_1 and C_2 be closed curves in D each containing one of the holes in its "interior" and let C be a curve containing both C_1 and C_2 in its "interior"* (Draw the picture!). *Then*

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

Proof of the Corollary: Let L be a curve starting at a point of C_1 and ending at a point of C_2 and lying entirely in D. Then it is not hard to believe that C is homotopic to $C_1 + L + C_2 + (-L)$ in D, so that by Theorem 16.1,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_L f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{-L} f(z)\,dz.$$

Now recall that $\int_{-L} f(z)\,dz = -\int_L f(z)\,dz$. □

Assignment 5 (5A is due February 15; 5B is due February 20)

5A: Problems 6,10 of chapter 5 of Cain. (Exercises 4 and 12 were done in class today.)

5B: This is on the web page at https://math.uci.edu/ brusso/Assign5.pdf

17 Wednesday February 13

Some proofs of statements of the previous lecture.

18 Friday February 15

Proof of Theorem 16.1, using 3 leaps of faith (assumptions on the function $H(t,s)$ implementing the continuous deformation of the homotopic curves).

Exercises 5,8,9,11 from chapter 5 were done in class today.

19 Monday February 18—holiday

20 Wednesday February 20

Theorem 20.1 *Let C be any curve, and g a continuous complex valued function on C. For any $z \notin C$, let*

$$G(z) := \int_C \frac{g(s)}{s-z}\,ds.$$

Then G is analytic on the complement of C and

$$G'(z) := \int_C \frac{g(s)}{(s-z)^2}\, ds.$$

Theorem 20.2 (Cauchy's integral formula) *If f is analytic in a domain D and C is a simple closed curve in D whose inside lies entirely in D, then for any z_0 inside C,*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$

Theorem 20.3 *If f is analytic on a domain D, then f' is also analytic. It follows that f is infinitely differentiable and its derivatives are given by the formulas*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s-z)^{n+1}}\, ds,$$

where C is any positively oriented simple closed curve, the inside of which lies in D, and z is any point inside C.

Theorem 20.4 (Morera) *Suppose that f is a continuous function on a domain D. If $\int_C f(z)\, dz = 0$ for every closed curve C lying in D, then f is analytic in D.*

Assignment 6 (6A is due February 27; 6B is due February 29, 6C is due March 3)

6A: Problems 1-4 of chapter 6 of Cain.

6B: Problems 5-8 of chapter 6 of Cain

6C: Problems 9,10,14,15 of chapter 6 of Cain (Hint: In two of these problems, use the fact that if f is an entire function, then so is $\exp f(z)$)

21 Friday February 22

Theorem 21.1 (Liouville) *Every bounded entire function is constant.*

Corollary 21.2 *Every non constant polynomial has at least one root.*

22 Monday February 25—Maximum Modulus Theorem

22.1 Announcements

The second midterm will be on Friday March 7 during class 11:00-11:50 am. It will emphasize chapters 4,5,6. One of the two midterms will be dropped and the one with the higher score will be counted double.

The final exam will be on Friday March 21, 8:00-10:00 am. Chapter 7 will be skipped. There will be review problems with solutions for the second midterm and for the final examination. (No take home exams will be given)

We will skip chapter 7.

22.2 Maximum modulus theorem

Definition 22.1 *A set $G \subset \mathbf{C}$ is* polygonally connected *if for each pair of points $a, b \in G$, there is a polygonal path in G starting at a and ending at b.*

Proposition 22.2 Let D be an open polygonally connected set and let f be analytic on D. Then

- If $f'(z) = 0$ for all $z \in D$, then f is a constant (see problem 13 of chapter 2)
- If $|f(z)| = c$, a non-negative constant for all $z \in D$, then f is a constant (see problem 16 of chapter 2)
- If $\operatorname{Re} f$ (or $\operatorname{Im} f$) is a constant, then f is a constant (see problem 15 of chapter 2)

Lemma 22.3 *If G is an open set then it is polygonally connected if and only if it cannot be the disjoint union of two non-empty open sets.*

Theorem 22.4 (Maximum Modulus) *If D is open and connected, and f is analytic and bounded on D, then either f is a constant or it has no maximum modulus in D. Stated precisely, if there exists a point $z_0 \in D$ with $|f(z_0)| = \sup\{|f(z)| : z \in D\}$, then $f(z) = f(z_0)$ for every $z \in D$.*

23 Wednesday February 27

Proof of one direction in Lemma 22.3. The other direction is in Assignment 7.

Proof of the Maximum Modulus Theorem.

24 Friday February 29

Assignments 7,8,9,10 are posted on the webpage. They are due on March 10.

24.1 Schwarz's lemma

We shall denote the open unit disc $\{z \in \mathbf{C} : |z| < 1\}$ simply by $\{|z| < 1\}$.

Theorem 24.1 (Schwarz's Lemma) *Suppose that $f : \{|z| < 1\} \to \mathbf{C}$ is analytic and satisfies $|f(z)| \leq 1$ for all $z \in \{|z| < 1\}$, and $f(0) = 0$. Then*

(a) $|f'(0)| \leq 1$ *and* $|f(z)| \leq |z|$ *for all* $z \in \{|z| < 1\}$.

(b) *In* (a), *if $|f'(0)| = 1$, then there exists a constant c, $|c| = 1$ such that $f(z) = cz$ for all $z \in \{|z| < 1\}$.*

(c) *In* (a), *if there exists z_0 with $|f(z_0)| = |z_0| \neq 0$, then there exists a constant c, $|c| = 1$ such that $f(z) = cz$ for all $z \in \{|z| < 1\}$.*

Proof: Define $g : \{|z| < 1\} \to \mathbf{C}$ by $g(z) = f(z)/z$ if $z \neq 0$ and $g(0) = f'(0)$. As defined, g is analytic on $\{0 < |z| < 1\}$ and continuous on $\{|z| < 1\}$. By the result of Assignment 8, g is in fact analytic on $\{|z| < 1\}$. Now for any $0 < r < 1$, and $|z| \leq r$, by the maximum modulus theorem, $|g(z)| \leq \max_{|w|=r} |g(w)| \leq 1/r$. Since this is true for any $r < 1$, we obtain $|g(z)| \leq 1$ for all $|z| < 1$. Thus $|f(z)| \leq |z|$ and $|f'(0)| = |g(0)| \leq 1$. This proves (a).

If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $|g(z_0)| = 1$ and g is constant by the maximum modulus theorem, so that $f(z) = cz$ with $|c| = 1$. If $|f'(0)| = 1$, then $|g(0)| = 1$ and again by the maximum modulus theorem, g is a constant. This proves (b) and (c). □

24.2 Taylor's Theorem

Proposition 24.2 *If f_n is a sequence of analytic function on a domain D, and f_n converges uniformly on compact subsets of D, then the limit function f is analytic.*

The following theorem, as well as Corollary 25.3 below, are needed to solve Assignment 8 (and many other problems).

Theorem 24.3 (Taylor's Theorem) *If f is analytic on $B(z_0, R) := \{z \in \mathbf{C} : |z - z_0| < R\}$, then with $a_n := f^{(n)}(z_0)/n!$, the series $\sum_0^\infty a_n(z - z_0)^n$ converges to $f(z)$ on $B(z_0, R)$, and the convergence is uniform on $B(z_0, r)$ for any $0 < r < R$.*

25 Monday March 3

25.1 Announcements

Assignment 11 will be posted on the web page. It is due on March 10.

There will be no second midterm. The 20% of your grade that would have come from the second midterm will come from either your first midterm or the final exam, whichever is higher.

25.2 Power series

The following two results were stated without proof. The proofs are in Chapter 8 of Cain.

Proposition 25.1 *Consider a series $\sum_0^\infty f_j(z)$ of functions on a domain D. For a given subset C of D, if there is a sequence of constants $M_j \geq 0$ with $\sum_j M_j < \infty$, and if $|f_j(z)| \leq M_j$ for all $z \in C$ and all j, then $\sum_0^\infty f_j(z)$ converges uniformly on C.*

Theorem 25.2 *A power series of the form $\sum_0^\infty c_j(z - z_0)^j$ has a radius of convergence $0 \leq R \leq \infty$, that is, the series converges for $|z - z_0| < R$ and diverges for $|z - z_0| > R$. The convergence is uniform on the set $\{|z - z_0| < r\}$ where $0 < r < R$.*

Corollary 25.3 *A power series converges to an analytic function inside the circle of convergence.*

25.3 The Identity Theorem

Theorem 25.4 *Let D be a polygonally connected open set and let f be analytic on D. The following are equivalent:*

(a) *$f \equiv 0$, that is, $f(z) = 0$ for every z in D.*

(b) *There exists a point $z_0 \in D$ such that $f^{(n)}(z_0) = 0$ for every $n \geq 0$.*

(c) *The set $\{z \in D : f(z) = 0\}$ has a limit point in D, that is, there is a sequence of distinct points z_k in D such that $f(z_k) = 0$ and $\lim_{k\to\infty} z_k$ exists and belongs to D.*

Proof: (a) implies (c) is trivial.

(c) implies (b): Let z_0 be a limit point of $\{z \in D : f(z) = 0\}$ and suppose $z_0 \in D$. Since D is open, $\exists R > 0$ such that $B(z_0, R) \subset D$. Let us assume that (b) does not hold for any point of D. Then $\exists n \geq 1$ such that $0 = f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0)$ and $f^{(n)}(z_0) \neq 0$. Expanding f is a Taylor series about the point z_0, we have $f(z) = a_n(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \cdots = (z-z_0)^n(a_n + a_{n+1}(z-z_0) + \cdots) = (z-z_0)^n g(z)$, where g is analytic and $g(z_0) = a_n = f^{(n)}(z_0)/n! \neq 0$. We have now reached a contradiction as follows. Since g is continuous and $g(z_0) \neq 0$, $\exists r,\ 0 < r \leq R$ with $g(z) \neq 0$ for $|z - z_0| < r$. Hence $\{z \in D : f(z) = 0\} \cap B(z_0, r) = \{z_0\}$. This contradicts the fact that z_0 is a limit point of $\{z \in D : f(z) = 0\}$, and thus completes the proof of (c) implies (b).

26 Wednesday March 5—Identity theorem; proof and examples

26.1 List of Named Theorems

(The last three have not been done yet)

1. Cauchy's Theorem(s)
2. Cauchy's Integral Formula(s)
3. Morera's Theorem
4. Liouville's Theorem
5. Maximum Modulus Theorem
6. Taylor's Theorem
7. Schwarz's Lemma
8. Identity Theorem
9. Riemann's Removable Singularity Theorem
10. Laurent's Theorem
11. Casorati-Weierstrass Theorem

26.2 Completion of the proof of the identity theorem

(b) implies (a): Let $A = \{z \in D : \forall n \geq 0,\ f^{(n)}(z) = 0\}$. By assumption $A \neq \emptyset$. We shall prove that both $D - A$ and A are open sets. It will follow from Theorem 22.3 that $D = A$ and therefore f is identically zero in D.

A is open: Let $a \in A$. Since D is open, $\exists R > 0$ with $B(a, R) \subset D$. Write f in a Taylor series $f(z) = \sum_0^\infty a_n(z-a)^n$ for $|z - a| < R$ with $a_n = f^{(n)}(a)/n!$. Since $a \in A$, each $a_n = 0$ and so f is identically zero on $B(a, R)$. This means that $B(a, R) \subset A$ and so A is an open set.

$D - A$ is open: If $z \in D - A$, then there exists n_0 with $f^{(n_0)}(z) \neq 0$. Since $f^{(n_0)}$ is a continuous function, by "persistence of sign", there exists $r > 0$ such that $f^{(n_0)}$ never vanishes on $B(z, r)$. This says that $B(z, r) \subset D - A$ showing that $D - A$ is an open set. □

26.3 Examples

Let f be analytic on the open disk $\{|z| < 2\}$

1. If $f(1/n) = 0$ for all $n = 1, 2, \ldots$, then $f(z) = 0$ for all $|z| < 2$.
2. If $f(1/n) = 1/n^2$ for all $n = 1, 2, \ldots$, then $f(z) = z^2$ for all $|z| < 2$.
3. If $f(1/n) = 1/n^3$ for all $n = 1, 2, \ldots$, then $f(z) = z^3$ for all $|z| < 2$.
4. If $f(1/n) = f(-1/n) = 1/n^2$ for all $n = 1, 2, \ldots$, then $f(z) = z^2$ for all $|z| < 2$.
5. If $f(1/n) = f(-1/n) = 1/n^3$ for all $n = 1, 2, \ldots$, then $f(z) = z^3$ for all $|z| < 2$ and $f(z) = -z^3$ for all $|z| < 2$—-hence no such (analytic) f exists.

27 Friday March 7

27.1 An application of the identity theorem

Prove the following identity using the Identity Theorem:

$$\exp(z + w) = (\exp z)(\exp w) \text{ for all } z, w \in \mathbf{C}$$

Solution:

Step 1. For a fixed complex number w, define an entire function f_w by

$$f_w(z) = \exp(z + w) \text{ for } z \in \mathbf{C}.$$

If w is a real number, $f_w(x) = 0$ for every real number x. Since the set of real numbers has a limit point in $\mathbf{C}$ (NO DUH: every real number is a limit point of the set of real numbers), by the identity theorem, $f_w(z) = 0$ for all $z \in \mathbf{C}$. This proves the identity in the case that w is a real number (and z is an arbitrary complex number).

Step 2. For a fixed complex number z, define an entire function g_z by

$$g_z(w) = f_w(z) \text{ for } w \in \mathbf{C}.$$

By Step 1, g_z is identically zero whenever w is real. Then by the identity theorem again, $g_z(w) = 0$ for every $w \in \mathbf{C}$. This proves the identity in the case that z and w are both arbitrary complex numbers.

27.2 A corollary to the Identity theorem

Corollary 27.1 *Let f and D be as in Theorem 25.4, and suppose that f is not identically zero, that is (a) fails. Also, let $a \in D$ be a "zero" of f, that is, $f(a) = 0$. Then there exists $n \geq 1$, called the "order" of the zero a of f, and an analytic function g on D such tht $g(a) \neq 0$ and $f(z) = (z - a)^n g(z)$ for all $z \in D$.*

27.3 Riemann's Removable Singularity Theorem

Theorem 27.2 (Riemann's Removable Singularity Theorem) *Let f be analytic on a punctured disk $B(a, R) - \{a\}$. Then f has an analytic extension to $B(a, R)$ if and only if $\lim_{z \to a}(z - a)f(z)$ exists and equals 0.*

Proof: If the analytic extension g exists, then $\lim_{z\to a}(z-a)f(z) = \lim_{z\to a}(z-a)g(z) = 0\cdot g(a) = 0$.

Now suppose that $\lim_{z\to a}(z-a)f(z) = 0$. Define a function g by $g(z) = (z-a)f(z)$ for $z \neq a$ and $g(a) = 0$. The function g is analytic for $z \neq a$, and is continuous at a. We shall show using the Triangulated Morera theorem (see below) that g is analytic at a. Assuming for the moment that this is true, let us complete the proof. Since g is analytic and $g(a) = 0$, then by Corollary 27.1, $g(z) = (z-a)h(z)$ where h is analytic in $B(a,R)$. Thus, for $z \neq a$, $(z-a)f(z) = g(z) = (z-a)h(z)$, and thus $f(z) = h(z)$ for $z \neq a$. Thus h is the analytic extension of f to $B(a,R)$.

It remains to prove that g is analytic at a. This will be done in the next lecture.

28 Monday March 10

28.1 More on the fundamental theorem of algebra. Exercise 12 of chapter 6

Let f be a complex polynomial of degree n; $f(z) = \sum_0^n a_k z^k$. By the fundamental theorem of algebra, f has a zero a. We define the order of a to be k where $0 = f(a) = f'(a) = \cdots = f^{(k-1)}(a)$ and $f^{(k)}(a) \neq 0$.

Since f is an entire function, it has a Taylor series expansion about any point $a \in \mathbf{C}$, which, since $f^{(n+1)}(z) = 0$ for all $z \in \mathbf{C}$ is actually a finite series:

$$f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n.$$

We can therefore write

$$f(z) = (z-a)^k\left[\frac{f^{(k)}(a)}{k!} + \frac{f^{(k+1)}(a)}{(k+1)!}(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^{n-k}\right].$$

or $f(z) = (z-a)^k g(z)$ where g is a polynomial of degree $n-k$ which doesn't vanish at a.

Repeating this procedure starting with the polynomial g and continuing until you get a constant polynomial, leads to a factorization of any complex polynomial with distinct zeros a_i of order k_i $(1 \leq i \leq N)$ as follows:

$$f(z) = c\prod_{i=1}^{N}(z-a_i)^{k_i}$$

where c is a constant.

28.2 Completion of the proof of Riemann's removable singularity theorem

We first state a generalization of the theorem of Morera.

Theorem 28.1 (Triangulated Morera Theorem) *Let f be continuous on a domain D and suppose that $\int_T f(z)\,dz = 0$ for every triangle T which together with its inside lies in D. Then f is analytic in D.*

Proof: Let $a \in D$ and let $B(a,R) \subset D$. For $z \in B(a,R)$, let $F(z) := \int_{[a,z]} f(s)\,ds$ where $[a,z]$ denotes the line segment from a to z. For any other point $z_0 \in B(a,R)$, by our assumption, $F(z) = \int_{[a,z_0]} f(s)\,ds + \int_{[z_0,z]} f(s)\,ds$. Therefore

$$\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{z-z_0}\int_{[z_0,z]} [f(s)-f(z_0)]\,ds$$

and

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| \le \sup_{s\in[z_0,z]} |f(s)-f(z_0)|.$$

Since f is continuous at z_0, $F'(z_0)$ exists and equals $f(z_0)$ so f is analytic. □

To prove Riemann's removable singularity theorem, it remains to show that g is analytic using the Triangulated Morera theorem. We must show that if T is any triangle in $B(a,R)$, then $\int_T f(s)\,ds = 0$. There are four possible cases.

Case 1: <u>**a is a vertex of T**</u> : In this case let x and y denote points on the two edges for which a is an endpoint. Then $\int_T f(s)\,ds = \int_{[a,y,x]} f(s)\,ds + \int_{[y,x,b,c]} f(s)\,ds$ where b and c are the other two vertices of T and $[\alpha,\beta,\cdots]$ denotes a polygon with vertices $\alpha,\beta,\cdots$. By the continuity of g at a, the first integral approaches zero as x and y approach a. The second integral is zero by Cauchy's theorem.

Case 2: <u>**a is inside T**</u> : In this case, draw lines from a to each of the vertices of T. Then $\int_T f(s)\,ds$ is the sum of three integrals of f over triangles having a as a vertex, each of which is zero by case 1.

Case 3: <u>**a lies on an edge of T**</u> : In this case, draw a line from a to the vertex which is opposite to the edge containing a. Then $\int_T f(s)\,ds$ is the sum of two integrals of f over triangles having a as a vertex, each of which is zero by case 1.

Case 4: <u>**a is outside of T**</u> : In this case, $\int_T f(s)\,ds = 0$ by Cauchy's theorem. □

29 Tuesday March 11-Completion of the proof of Riemann's removable singularity theorem and of Schwarz's lemma

29.1 completion of the proof of Triangulated Morera theorem

See Theorem 28.1 above.

29.2 completion of the proof of Riemanns' removable singularity theorem

See Theorem 27.2 above and subsection 28.2.

29.3 completion of the proof of Schwarz's lemma

See Theorem 24.1 above.

30 Wednesday March 12—Classification of Singularities

30.1 The order of a pole

Let f be analytic in $B(a,R)-\{a\}$. We say that a is a *pole* of f if $\lim_{z\to a}|f(z)|=+\infty$.

Proposition 30.1 *If f is analytic in $B(a,R)-\{a\}$ and has a pole at a, then there exists $m\geq 1$ and an analytic function g on $B(a,R)$ such that $f(z)=g(z)/(z-a)^m$ for all $z\in B(a,R)-\{a\}$, and $g(a)\neq 0$. (We say that a is a pole of order m.)*

Proof: Define $h(z)=1/f(z)$ for $z\neq a$ and $h(a)=0$. Obviously h is analytic for $z\neq a$, but in fact it is analytic at a by Theorem 27.2, since $\lim_{z\to a}(z-a)h(z)=0\cdot 0=0$. Since a is a zero of h, then $\exists m\geq 1$ such that $h(z)=(z-a)^m h_1(z)$, where h_1 is analytic and $h_1(a)\neq 0$. By continuity, there exists $r\leq R$ with $h_1(z)\neq 0$ for all $|z-a|<r$. Then for $0<|z-a|<r$, $f(z)=1/h(z)=(1/h_1(z))/(z-a)^m$, completing the proof. □

30.2 Classification of Singularities

Some definitions. A point $a\in\mathbf{C}$ is said to be an *isolated singularity* of f if f is analytic in $B(a,R)-\{a\}$ for some $R>0$. Isolated singularities fall into three cases:

(1) a is a *removable singularity* of f if f has an analytic extension to $B(a,R)$. Example: $f(z)=(\sin z)/z$, $a=0$
(2) a is a *pole* of f if $\lim_{z\to a}|f(z)|=+\infty$. Examples: $f(z)=1/z$, $a=0$; $f(z)=e^z/(z-2)^{47}$, $a=2$.
(3) a is an *essential singularity* of f if it is neither a removable singularity or pole. Example: $f(z)=e^{1/z}$, $a=0$.

30.3 The range of an analytic function on $\mathbf{C}-\{z_0\}$

Proposition 30.2 Let f be analytic on $\mathbf{C}-\{z_0\}$ and suppose that f is a not a constant function. In particular, z_0 is an isolated singularity of f.

(a) If z_0 is a removable singularity of f (so that f is entire), then $f(\mathbf{C})$ is dense in $\mathbf{C}$.

(b) If z_0 is a pole of f, then $f(\mathbf{C}-\{z_0\})$ is dense in $\mathbf{C}$.

(c) If z_0 is an essential singularity of f, then for every $\delta>0$ $f(B(z_0,\delta)-\{z_0\})$ is dense in $\mathbf{C}$. (This is the **Casorati-Weierstrass theorem**)

Proof: Suppose (a) is false. Since f has a removable singularity at z_0, it extends to an entire function. For simplicity of notation, we just let f denote the extension. Then there exist a complex number a_0 and $\delta_0>0$ such that $|f(z)-a_0|\geq\delta_0$ for all $z\in\mathbf{C}$. Then the function g defined by $g(z)=1/(f(z)-a_0)$ is entire and bounded (by $1/\delta_0$), so is a constant by Liouville's theorem. Hence f is a constant, a contradiction which proves (a).

Suppose (b) is false. Then there exist a complex number a_0 and $\delta_0>0$ such that $|f(z)-a_0|\geq\delta_0$ for all $z\in\mathbf{C}-\{z_0\}$. Then the function g defined by $g(z)=1/(f(z)-a_0)$ for $z\in\mathbf{C}-\{z_0\}$ is analytic on $\mathbf{C}-\{z_0\}$ and bounded (by $1/\delta_0$) there, so by Riemann's removable singularity theorem, z_0 is a removable singularity of g. Thus g extends to an entire function which is bounded by $1/\delta_0$, and therefore constant by Liouville's theorem. Hence f is a constant, a contradiction which proves (b).

(c) will be proved in the next lecture. □

31 Friday March 14—Laurent's theorem

31.1 Proof of the Casorati-Weierstrass theorem (Part (c) of Proposition 30.2)

Theorem 31.1 (Casorati-Weierstrass) *If f has an essential singularity at z_0, then for every $\delta > 0$, the set $f(B(z_0,\delta) - \{z_0\})$ is dense in $\mathbf{C}$. That is, for all $c \in \mathbf{C}$ and $\epsilon > 0$, there exists a $z \in B(z_0,\delta) - \{z_0\}$ such that $|f(z) - c| < \epsilon$.*

Proof: [1] Suppose not. Then there exist $c_0 \in \mathbf{C}$ and $\epsilon_0 > 0$ such that $|f(z) - c_0| \geq \epsilon_0$ for all $z \in B(z_0,\delta) - \{z_0\}$. It follow that

$$\lim_{z \to z_0} \left| \frac{f(z) - c_0}{z - z_0} \right| = +\infty,$$

so that the function $(f(z) - c_0)/(z - z_0)$ has a pole at z_0. Let $m \geq 1$ be the order of this pole so that there is an analytic function g at z_0 such that

$$\frac{f(z) - c_0}{z - z_0} = \frac{g(z)}{(z - z_0)^m}$$

for $z \neq z_0$.

We have $\lim_{z \to z_0} |z - z_0|^{m+1} |f(z) - c_0| = 0$ and therefore $|z - z_0|^{m+1}|f(z)| \leq |z - z_0|^{m+1}|f(z) - c_0| + |z - z_0|^{m+1}|c_0| \to 0$ as $z \to z_0$. Thus, $(z - z_0)^m f(z)$ has a removable singularity at $z = z_0$ and it follows that for $z \neq z_0$, $f(z) = h(z)/(z - z_0)^m$ for some function h which is analytic at z_0. This says that f has a pole at z_0, which is a contradiction. □

31.2 More properties of singularities

(1) a is a removable singularity if and only if $\lim_{z \to a}(z - a)f(z) = 0$ (this is Riemann's Removable Singularity Theorem). In this case, f has a power series expansion

$$f(z) = \sum_0^\infty a_n (z - a)^n, \qquad 0 < |z - a| < R.$$

(2) If a is a pole of f of order m, then $m \geq 1$ and because of Proposition 30.1, f has a power series expansion

$$f(z) = \frac{b_{-m}}{(z - a)^m} + \frac{b_{-m+1}}{(z - a)^{m-1}} + \cdots + \frac{b_{-1}}{z - a} + \sum_0^\infty b_n (z - a)^n, \qquad 0 < |z - a| < R.$$

(3) By Taylor's theorem,

$$e^z = 1 + z + z^2/2! + z^3/3! + \cdots$$

[1] Not done in class—included here for completeness

This series converges for every $z \in \mathbf{C}$, and therefore, the function $\exp(1/z)$, for which 0 is an essential singularity can be written as

$$\exp(1/z) = 1 + 1/z + 1/2!z^2 + 1/3!z^3 + \cdots$$

These three examples are instances of Laurent's theorem[2].

Theorem 31.2 (Laurent's theorem) *Let $0 \le R_1 < R_2 \le \infty$, let $z_0 \in \mathbf{C}$, and let f be analytic in the annulus $D = \{z \in \mathbf{C} : R_1 < |z - z_0| < R_2\}$. Let C be a simple closed curve lying in D such that z_0 is inside C. For each $j = 0, \pm 1, \pm 2, \ldots$ set $c_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z_0)^{j+1}} \, ds$. Then for each $z \in D$ not lying on C, we have $f(z) = \sum_{j=-\infty}^{\infty} c_j (z - z_0)^j$.*

31.3 About final exam week

10 review problems (with hints) have been posted. This is all you need to prepare for the final examination. You may ask questions in person (see the office hours below) or by email.

Here are the main theorems again; you won't need to know the last two.

1. Cauchy's Theorem(s)
2. Cauchy's Integral Formula(s)
3. Morera's Theorem
4. Liouville's Theorem
5. Maximum Modulus Theorem
6. Taylor's Theorem
7. Schwarz's Lemma
8. Identity Theorem
9. Riemann's Removable Singularity Theorem
10. Laurent's Theorem
11. Casorati-Weierstrass Theorem

Office Hours during finals week:

Wednesday at 4 pm (K. Huber)

Tuesday and Thursday 2-4 pm (B. Russo—You may pick up your graded homework papers at these times)

Solutions to Assignments 7-11 will be posted early next week. Solutions to the Review Problems will be posted on Thursday morning.

Final Exam Friday March 21 8:00-10:00 AM ET 201

[2] Stated a little bit differently in class; proof not done