Complex Analysis
Math 147—Winter 2006

Bernard Russo

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Contents

1 Friday January 6—Course information; complex numbers; Assignment 1

1.1 Course information

- Course: Mathematics 147 MWF 1:00–1:50 SSTR 100
- Prerequisite: Math 140AB or consent of the instructor (if you have had 140A and are taking 140B concurrently, that is acceptable)
- Instructor: Bernard Russo MSTB 263 Office Hours MW 2:30-3:30 or by appointment (a good time for short questions is right after class just outside the classroom)
- There is a link to this course on Russo's web page: www.math.uci.edu/∼ brusso
- Discussion section: TuTh 1:00–1:50 SSTR 103
- Teaching Assistant: P. C. Lin
- Homework: There will be approximately 10-12 assignments (10-20 problems from the text for each assignment) with at least one week notice before the due date.
- Grading:

First midterm	February 3 (Friday of week 4)	20 percent
Second midterm	March 3 (Friday of week 8)	20 percent
Final Exam	March 22 (Wednesday 1:30-3:30)	40 percent
Homework	approximately 12 assignments	20 percent

- Holidays: January 16, February 20
- Text: George Cain "Complex Analysis", Freely available on the web (see Russo's web page or go directly to http://www.math.gatech.edu/ cain/winter99/complex.html)
- Material to be Covered: All of the text
- Catalog description: Rigorous treatment of basic complex analysis: complex numbers, analytic functions, Cauchy integral theory and its consequences (Morera's Theorem, The Argument Principle, The Fundamental Theorem of Algebra, The Maximum Modulus Principle, Liouville's Theorem), power series, residue calculus, harmonic functions, conformal mapping. Students are expected to do proofs.
- Math 147 is replacing the old Math 114B, and is intended for mathematics majors. The sequence 114A-147 is acceptable for the specialization in applied mathematics. You cannot take 114A after taking 147.
- Some alternate texts that you may want to look at, in no particular order. There are a great number of such texts at the undergraduate and at the graduate level.

Undergraduate Level

1. S. Fisher: Complex Variables
2. R. Churchill and J. Brown; Complex Variables and Applications
3. J. Marsden and M. Hoffman, Basic Complex Analysis
4. E. Saff and A. Snider: Fundamentals of Complex Analysis

Graduate Level

1. L. Ahlfors; Complex Analysis
2. J. Conway; Functions of one Complex Variable
3. J. Bak and D. Newman; Complex Analysis

1.2 Complex numbers

algebra: sum, product, difference, quotient
geometry: modulus, conjugate, triangle inequality

Assignment 1 (Due January 13)
Problems 1-12 of chapter 1 of Cain.

2 Monday January 9—Polar form; complex derivative, Assignment 2

2.1 Polar coordinates

argument, principal argument, geometric interpretation of sum and product

2.2 Complex functions

Derivatives of functions:

- $f : \mathbf{R} \to \mathbf{R}$, $f'(x)$, graph is a curve
- $f : \mathbf{R}^2 \to \mathbf{R}$, $\partial f/\partial x$, $\partial f/\partial y$, graph is a surface
- $f : \mathbf{R} \to \mathbf{R}^2$, $f'(t) = (x'(t), y'(t))$, parametric equations of a curve in $\mathbf{R}^2$
- $f : \mathbf{C} \to \mathbf{C}$, $f'(z)$ (new idea)

limits of functions, derivative of a complex valued function of a complex variable.

Assignment 2 (Due January 20)
Problems 1-16 of chapter 2 of Cain (except #4). You may skip section 2.1 except for exercises 1–3.

3 Wednesday, January 11—Limits and continuous functions

Continuous function at a point, on a set. Sums, products, differences, and quotients of continuous functions are continuous (whenever defined),

We proved two propositions:

- If $f = u + iv$ is a complex valued function of a complex variable (with real part u and imaginary part v, then f is continuous if and only if both u and v are continuous.
- If $\lim_{z\to z_0} f(z) = L_1$ and $\lim_{z\to z_0} g(z) = L_2$, then $\lim_{z\to z_0} f(z)g(z)$ exists and equals L_1L_2

We showed that the function $f(z) = \overline{z}$ is not differentiable at any point z_0.

4 Friday January 13—The Cauchy-Riemann equations

We proved two propositions:

- If $f = u + iv$ is a complex valued function of a complex variable, and f is differentiable at $z_0 = x_0 + iy_0$, then u and v satisfy the Cauchy Riemann equations at (x_0, y_0).
- If $f = u+iv$ is a complex valued function of a complex variable, and u and v satisfy the Cauchy Riemann equations at (x_0, y_0), and if u_x and u_y are continuous at (x_0, y_0), then f is differentiable at $z_0 = x_0 + iy_0$,

5 Monday January 16—Holiday

6 Wednesday January 18—Exponential and Trigonometric functions

Definition and properties of the functions $\exp z$, $\sin z$, $\cos z$.

7 Friday January 20—The complex logarithm and complex powers; Assignment 3

Definition and properties of the multi-valued functions $\log z$, z^c (c a complex number). Principal logarithm $\operatorname{Log} z$, principal value of z^c.

Assignment 3 (Due January 27)

Problems 1-17 of chapter 3 of Cain (except #7).

8 Monday January 23—More on chapters 2 and 3

Some remarks on chapter 2: differentiable vs. analytic, algebra of differentiable functions, chain rule

Some remarks on chapter 3: Discussion of problems 12 and 13 in chapter 3.

9 Wednesday January 25—-Line integrals, Assignment 4

We decided to ignore section 4.1 and use the formula

$$\int_C f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)\,dt$$

for the definition of the contour integral of $f : D \to \mathbf{C}$ over the curve C given by $\gamma : [\alpha, \beta] \to D$ for $\alpha \le t \le \beta$.

Estimate for a line integral:

Proposition 9.1 *$|\int_C f(z)\,dz| \le ML$, where $M = \sup_{z\in C} |f(z)|$ and $L = \int_\alpha^\beta |\gamma'(t)|\,dt$ is the length of C.*

Assignment 4 (Due February 3 (grace period to February 6)
Problems 1-11 of chapter 4 of Cain (except #3 which does not exist).

Discussion of problems 1 and 5 in chapter 4.

10 Friday January 27—-Line integrals, Independence of the path

Lemma 10.1 *If $G : [\alpha, \beta] \to \mathbf{C}$ is integrable, then so is $|G(t)|$ and*

$$\left|\int_\alpha^\beta G(t)\,dt\right| \le \int_\alpha^\beta |G(t)|\,dt.$$

Lemma 10.2 *If $G : [\alpha, \beta] \to \mathbf{C}$ is differentiable and if $G'(t)$ is integrable, then*

$$\int_\alpha^\beta G'(t)\,dt = G(\beta) - G(\alpha).$$

Lemma 10.3 (Chain Rule) *Let γ be a real valued function defined on an open interval containing $a \in \mathbf{R}$ and suppose that γ is differentiable at a with derivative $\gamma'(a)$. Let f be a real valued function defined on an open interval containing $\gamma(a)$ and suppose that f is differentiable at $\gamma(a)$ with derivative $f'(\gamma(a))$. Then $f \circ \gamma$ is differentiable at a with derivative*

$$(f \circ \gamma)'(a) = f'(\gamma(a))\,\gamma'(a).$$

Proof: Since γ is differentiable at a, $\forall \epsilon' > 0, \exists \delta' > 0$ such that

$$|\gamma(x) - \gamma(a) - \gamma'(a)(x-a)| < \epsilon'|x-a| \quad \text{if } |x-a| < \delta'. \tag{1}$$

Since f is differentiable at $\gamma(a)$, $\forall \epsilon'' > 0, \exists \delta'' > 0$ such that

$$|f(y) - f(\gamma(a)) - f'(\gamma(a))(y - \gamma(a))| < \epsilon''|y - \gamma(a)| \quad \text{if } |y - \gamma(a)| < \delta''. \tag{2}$$

We need to prove: $\forall \epsilon > 0, \exists \delta > 0$ such that

$$|f(\gamma(x)) - f(\gamma(a)) - f'(\gamma(a))\gamma'(a)(x-a)| < \epsilon|x-a| \quad \text{if } |x-a| < \delta. \tag{3}$$

Since γ is continuous at a, $\exists \delta_c > 0$ such that

$$|\gamma(x) - \gamma(a)| < \delta'' \text{ if } |x-a| < \delta_c. \tag{4}$$

Using (4), we may replace y in (2) by $\gamma(x)$ to obtain

$$|f(\gamma(x)) - f(\gamma(a)) - f'(\gamma(a))(\gamma(x) - \gamma(a))| < \epsilon''|\gamma(x) - \gamma(a)| \quad \text{if } |x-a| < \delta_c. \tag{5}$$

Now set $\delta := \min\{\delta_c, \delta'\}$ and $\eta(x) := \gamma(x) - \gamma(a) - \gamma'(a)(x-a)$ so that

$$\gamma(x) - \gamma(a) = \gamma'(a)(x-a) + \eta(x) \tag{6}$$

and by (1),

$$|\eta(x)| < \epsilon'|x-a| \text{ if } |x-a| < \delta. \tag{7}$$

Now substitute (6) into (5) (in two places!) and set

$$A := f(\gamma(x)) - f(\gamma(a)) - f'(\gamma(a))[\gamma'(a)(x-a) + \eta(x)] \tag{8}$$

to obtain from (5)

$$|A| < \epsilon''|\gamma'(a)(x-a) + \eta(x)| \quad \text{if } |x-a| < \delta. \tag{9}$$

Finally, if $|x-a| < \delta$, we have,

$$\begin{aligned}
& |f(\gamma(x)) - f(\gamma(a)) - f'(\gamma(a))\gamma'(a)(x-a)| \\
= \;& |A + f'(\gamma(a))\eta(x)| \qquad \text{(by (8))} \\
\leq \;& |A| + |f'(\gamma(a))\eta(x)| \\
\leq \;& \epsilon''|\gamma'(a)||x-a| + \epsilon''|\eta(x)| + |f'(\gamma(a))||\eta(x)| \qquad \text{(by (9))} \\
\leq \;& [\epsilon''|\gamma'(a)| + \epsilon''\epsilon' + |f'(\gamma(a))|\epsilon']|x-a| \qquad \text{(by (7))} \\
< \;& \epsilon|x-a|,
\end{aligned}$$

the last step provided we simply choose ϵ' and ϵ'' so that $[\epsilon''|\gamma'(a)| + \epsilon''\epsilon' + |f'(\gamma(a))|\epsilon'] < \epsilon$. This proves (3). □

Lemma 10.1 is used in the proof of Proposition 9.1. Lemma 10.2 and the chain rule are used in the proof of the following proposition.

Proposition 10.4 *If $F, f : D \to \mathbf{C}$ are such that $F'(z) = f(z)$ for all $z \in D \subset \mathbf{C}$, then $\int_C f(z)\,dz = F(b) - F(a)$, where C is any curve in D starting at $a \in D$ and ending at $b \in D$. In other words, if the function f has an antiderivative in D, then the contour integral of f over any curve lying in D depends only on the end points of C.*

Begin the discussion of problem 8 in chapter 4.

11 Monday January 30—-More on Independence of the path

Proposition 11.1 *Suppose that f is a continuous complex valued on an open connected set D and that f is path independent, that is, for every curve C lying in D, the value of the contour integral $\int_C f(z)\,dz$ depends only on the endpoints of C. Then f has an antiderivative in D.*

12 Wednesday February 1—Antiderivatives of $1/z$ and z^c

Conclude the discussion of problem 8(a,c) in chapter 4.

8(a) Let $F(z) = \log|z| + i\arg z$ where $0 < \arg z < 2\pi$. Thus F is defined on $D := \mathbf{C} - \{z = x + iy : x \geq 0, y = 0\}$. We need to prove that $F'(z)$ exists and equals $1/z$ for all $z \in D$. We shall use the following three facts to justify the steps following them.

(1) $z = \exp(F(z))$ for $z \in D$
[Proof: $\exp(\log|z| + i\arg z) = \exp(\log|z|)\exp(i\arg z) = |z|\exp(i\arg z) = z$]

(2) $F(z) \neq F(z_0)$ for $z, z_0 \in D$ and $z \neq z_0$
[Proof: If $F(z) = F(z_0)$, then $z = \exp(F(z)) = \exp(F(z_0)) = z_0$]

(3) F is continuous on D.
[Proof: It suffices to prove that $\arg z$ is continuous on D. The argument for this is similar to the solution of problem 12 on the review problems for chapters 1-3. First of all, the function $\arg z$ is not defined for $z = 0$. Let $z_0 = x_0$ be a positive real number. If $y > 0$, then $\arg(x_0 + iy) = \tan^{-1}(y/x_0) \to 0$ as $y \to 0$.
Also, if $y < 0$, then $\arg(x_0 + iy) = 2\pi - \tan^{-1}(-y/x_0) \to 2\pi$ as $y \to 0$. Therefore, $\lim_{z\to x_0} \arg z$ does not exist, and so $\arg z$ is not continuous at $z_0 = x_0$ if $x_0 > 0$.]

Now, using (1)–(3), and the fact that $\exp'(w) = \exp w$, we prove that $F'(z) = 1/z$:

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{\frac{z-z_0}{F(z)-F(z_0)}} = \frac{1}{\frac{\exp(F(z))-\exp(F(z_0))}{F(z)-F(z_0)}}$$

so that

$$\lim_{z\to z_0}\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{\lim_{F(z)\to F(z_0)}\frac{\exp(F(z))-\exp(F(z_0))}{F(z)-F(z_0)}} = \frac{1}{\exp'(F(z_0))} = \frac{1}{\exp(F(z_0))} = \frac{1}{z_0}.$$

8(c) $\int_{C_1} \frac{1}{z}\,dz = \mathrm{Log}\,(i) - \mathrm{Log}\,(-i) = \pi i$

Discussion of problem 10a in chapter 4.

10(a) $H'(z) = \exp(c\,\mathrm{Log}\,z)\,c/z$. Since $H(z) = \exp(c\,\mathrm{Log}\,z)$ and since $z^c z^d = z^{c+d}$ (proof?), this can be written as $H'(z) = cz^{c-1}$.

13 Friday February 3—First Midterm

14 Monday February 6

Discussion of Problem 10 (c) in chapter 4.

The following two theorems were stated but not proved in class (yet!).

Theorem 14.1 (Pre-Cauchy Theorem) *If C_1 and C_2 are homotopic closed curves in a region D and f is analytic in D, then*

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Theorem 14.2 (Cauchy's Theorem) *If D is a simply connected region andf is analytic in D, then for every closed curve C in D,*

$$\int_C f(z)\,dz = 0.$$

Corollary 14.3 (Fundamental Application) *Let D be a region which is contained in the complement of the union of two closed sets (call them holes!) in* **C** *and let f be analytic in D. Let C_1 and C_2 be closed curves in D each containing one of the holes in its "interior" and let C be a curve containing both C_1 and C_2 in its "interior"* (Draw the picture!). *Then*

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

Proof of the Corollary: Let L be a curve starting at a point of C_1 and ending at a point of C_2 and lying entirely in D. Then it is not hard to believe that C is homotopic to $C_1 + L + C_2 + (-L)$ in D, so that by Theorem 14.1,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_L f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{-L} f(z)\,dz.$$

Now recall that $\int_{-L} f(z)\,dz = -\int_L f(z)\,dz$. □

Discussion of Problem 5 in chapter 5.

15 Wednesday February 8

Four examples pertaining to homotopic curves. In each of these examples, the two curves are homotopic to each other.

(a) Two concentric circles in an annulus D, with the same orientation.
(b) A triangle and a circle in an annulus D with the same orientation.
(c) A closed curve and a point outside the curve, in the complex plane $D = \mathbf{C}$.
(d) Two closed curves in the first quadrant D with different orientations.

Discussion of numerical results of first midterm.

16 Friday February 10

Assignment 5 (Due February 17) *This assignment is posted on the webpage. Part 1 consists of 6 problems. Part 2 consists of problems 1–4 of chapter 6 of Cain.*

More discussion of homotopic curves, namely, in **C** every closed curve is homotopic to a point and in $\mathbf{C} - \{0\}$ this is no longer true.

Proof of Theorem 14.1, using 3 leaps of faith (assumptions on the function $H(t, s)$ implementing the continuous deformation of the homotopic curves).

Theorem 16.1 (Cauchy's integral formula) *If f is analytic in a domain D and C is a simple closed curve in D whose inside lies entirely in D, then for any z_0 inside C,*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$

17 Monday February 13

Assignment 6 (Due February 24) *Problems 5–10 and 14–15 of chapter 6 of Cain.* Hint: In two of these problems, use the fact that if f is an entire function, then so is $\exp(f(z))$

Lemma 17.1 *Let C be any curve, and g a continuous complex valued function on C. For any $z \notin C$, let*

$$G(z) := \int_C \frac{g(s)}{s - z}\, ds.$$

Then G is analytic on the complement of C and

$$G'(z) := \int_C \frac{g(s)}{(s - z)^2}\, ds.$$

Theorem 17.2 *If f is analytic on a domain D, then f' is also analytic. It follows that f is infinitely differentiable and its derivatives are given by the formulas*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s - z)^{n+1}}\, ds,$$

where C is any positively oriented simple closed curve, the inside of which lies in D, and z is any point inside C.

Theorem 17.3 (Morera) *Suppose that f is a continuous function on a domain D. If $\int_C f(z)\, dz = 0$ for every closed curve C lying in D, then f is analytic in D.*

Theorem 17.4 (Liouville) *Every bounded entire function is constant.*

18 Wednesday February 15—Maximum Modulus Theorem

Definition 18.1 *A set $G \subset \mathbf{C}$ is* connected *if for each pair of points $a, b \in G$, there is a polygonal path in G starting at a and ending at b.*

Theorem 18.2 *If G is open and connected, then it cannot be the disjoint union of two non-empty open sets.*

Theorem 18.3 (Maximum Modulus) *If D is open and connected, and f is analytic and bounded on D, then either f is a constant or it has no maximum modulus in D. Stated precisely, if there exists a point $z_0 \in D$ with $|f(z_0)| = \sup\{|f(z)| : z \in D\}$, then $f(z) = f(z_0)$ for every $z \in D$.*

19 Friday February 17—Schwarz's Lemma

Proof of Theorem 18.2: Suppose that $G = A \cup B$ where A, B are open, non-empty, and disjoint. We seek a contradiction. Let $z_0, z_1, \ldots, z_n$ be a finite sequence of points in D with $z_0 = a$, $z_n = b$ and such that the line segments $[z_{k-1}, z_k] := \{sz_k + (1-s)z_{k-1} : s \in [0,1]\}$ all lie in D. Choose one of these segments which has one endpoint in A and the other in B, and denote it by $[p, q]$. Then $[0,1] = S \cup T$ where $S = \{s \in [0,1] : sq + (1-s)p \in A\}$ and $T = \{t \in [0,1] : tq + (1-t)p \in B\}$. S and T are each non-empty, since $0 \in S$ and $1 \in T$. The rest of the proof is contained in Assignment 7 □

Assignment 7 (Due March 3) *Two parts*

(a) *Prove that S is either an open set, or is equal to a set of the form $\{0\} \cup S'$ where S' is open. Prove that T is either an open set, or is equal to a set of the form $\{1\} \cup T'$ where T' is open.*

(b) *Complete the proof of Theorem 18.2 by deriving a contradiction* (Hint: Consider $\alpha = \sup S$ and $\beta = \sup T$).

We shall denote the open unit disc $\{z \in \mathbf{C} : |z| < 1\}$ simply by $\{|z| < 1\}$.

Theorem 19.1 (Schwarz's Lemma) *Suppose that $f : \{|z| < 1\} \to \mathbf{C}$ is analytic and satisfies $|f(z)| \leq 1$ for all $z \in \{|z| < 1\}$, and $f(0) = 0$. Then*

(a) *$|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in \{|z| < 1\}$.*

(b) *In* (a)*, if $|f'(0)| = 1$, then there exists a constant c, $|c| = 1$ such that $f(z) = cz$ for all $z \in \{|z| < 1\}$.*

(c) *In* (a)*, if there exists z_0 with $|f(z_0)| = |z_0| \neq 0$, then there exists a constant c, $|c| = 1$ such that $f(z) = cz$ for all $z \in \{|z| < 1\}$.*

Proof: Define $g : \{|z| < 1\} \to \mathbf{C}$ by $g(z) = f(z)/z$ if $z \neq 0$ and $g(0) = f'(0)$. As defined, g is analytic on $\{0 < |z| < 1\}$ and continuous on $\{|z| < 1\}$. By the result of Assignment 10, g is in fact analytic on $\{|z| < 1\}$. Now for any $0 < r < 1$, and $|z| \leq r$, by the maximum modulus theorem, $|g(z)| \leq \max_{|w|=r} |g(w)| \leq 1/r$. Since this is true for any $r < 1$, we obtain $|g(z)| \leq 1$ for all $|z| < 1$. Thus $|f(z)| \leq |z|$ and $|f'(0)| = |g(0)| \leq 1$. This proves (a).

If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $|g(z_0)| = 1$ and g is constant by the maximum modulus theorem, so that $f(z) = cz$ with $|c| = 1$. If $|f'(0)| = 1$, then $|g(0)| = 1$ and again by the maximum modulus theorem, g is a constant. This proves (b) and (c).□

Assignment 8 (due March 3) *For a fixed complex number a with $|a| < 1$, define a function φ_a by*

$$\varphi_a(z) = \frac{z + a}{1 + \overline{a}z}.$$

Although $\varphi_a(z)$ is defined for all $z \neq -1/\overline{a}$, we shall consider it as a function on the closed unit disk $|z| \leq 1$. Prove the following statements.

(a) *If $|z| < 1$ then $|\varphi_a(z)| < 1$.*

(b) *If $|z| = 1$ then $|\varphi_a(z)| = 1$.*

(c) *φ_a is a one to one function, that is, if $|z_1| < 1, |z_2| < 1$ and if $f(z_1) = f(z_2)$, then $z_1 = z_2$.*

(d) *φ_a is an onto function, that is, if $|w_0| < 1$, then there is a z_0 with $|z_0| < 1$ and $f(z_0) = w_0$.*

(e) *What is the inverse of φ_a?*

Assignment 9 (due March 3) *Let f be an arbitrary analytic function on the unit disk $|z| < 1$ which is one to one and onto, that is, if $|z_1| < 1, |z_2| < 1$ and if $f(z_1) = f(z_2)$, then $z_1 = z_2$; and if $|w_0| < 1$, then there is a z_0 with $|z_0| < 1$ and $f(z_0) = w_0$. Prove the following statements.*

(a) *If $f(0) = 0$, then $f(z) = e^{i\theta} z$ for some real θ.*

(b) *If $f(0) = a \neq 0$, let $g(z)$ be defined by $g(z) = \varphi_{-a}(f(z))$. Then $g(z) = e^{i\theta} z$ for some real θ.*

(c) *The function f has the form*

$$f(z) = e^{i\theta} \varphi_a(z),$$

for some θ real and $|a| < 1$.

20 Monday February 20—Holiday

21 Wednesday February 22—Power series

Proposition 21.1 *If f_n is a sequence of analytic function on a domain D, and f_n converges uniformly on compact subsets of D, then the limit function f is analytic.*

The following three results were stated without proof. The proofs are in Chapter 8 of Cain.

Proposition 21.2 *Consider a series $\sum_0^\infty f_j(z)$ of functions on a domain D. For a given subset C of D, if there is a sequence of constants $M_j \geq 0$ with $\sum_j M_j < \infty$, and if $|f_j(z)| \leq M_j$ for all $z \in C$ and all j, then $\sum_0^\infty f_j(z)$ converges uniformly on C.*

Theorem 21.3 *A power series of the form $\sum_0^\infty c_j(z-z_0)^j$ has a radius of convergence $0 \leq R \leq \infty$, that is, the series converges for $|z-z_0| < R$ and diverges for $|z-z_0| > R$. The convergence is uniform on the set $\{|z-z_0| < r\}$ where $0 < r < R$.*

Theorem 21.4 *For $|z-z_0| < R$,*

$$\frac{d}{dz}\left(\sum_0^\infty c_j(z-z_0)^j\right) = \sum_1^\infty j c_j (z-z_0)^{j-1}.$$

Theorem 21.5 *If f is analytic on $B(z_0, R) := \{z \in \mathbf{C} : |z-z_0| < R\}$, then with $a_n := f^{(n)}(z_0)/n!$, the series $\sum_0^\infty a_n(z-z_0)^n$ converges to $f(z)$ on $B(z_0, R)$, and the convergence is uniform on $B(z_0, r)$ for any $0 < r < R$.*

(Theorem 21.5 will be proved on Friday)

Assignment 10 (due March 3) *Use Theorem 21.5 to prove the following "leap of faith" in the proof of Schwarz's lemma. If f is analytic on the open unit disk and $f(0) = 0$ and $|f(z)| \leq 1$ for all $|z| < 1$, then the function g defined on the unit disk by $g(z) = f(z)/z$ for $z \neq 0$ and $g(0) = f'(0)$, is analytic at 0.*

22 Friday February 24, 2006—The identity theorem

Theorem 22.1 *Let D be a connected open set and let f be analytic on D. The following are equivalent:*

(a) *$f \equiv 0$, that is, $f(z) = 0$ for every z in D.*

(b) *There exists a point $z_0 \in D$ such that $f^{(n)}(z_0) = 0$ for every $n \geq 0$.*

(c) *The set $\{z \in D : f(z) = 0\}$ has a limit point in D, that is, there is a sequence of distinct points z_k in D such that $f(z_k) = 0$ and $\lim_{k\to\infty} z_k$ exists and belongs to D.*

Proof: (a) implies (c) is trivial.

(c) implies (b): Let z_0 be a limit point of $\{z \in D : f(z) = 0\}$ and suppose $z_0 \in D$. Since D is open, $\exists R > 0$ such that $B(z_0, R) \subset D$. Let us assume that (b) does not hold for any point of D. Then $\exists n \geq 1$ such that $0 = f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0)$ and $f^{(n)}(z_0) \neq 0$. Expanding f is a Taylor series about the point z_0, we have $f(z) = a_n(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \cdots = (z-z_0)^n(a_n + a_{n+1}(z-z_0) + \cdots) = (z-z_0)^n g(z)$, where g is analytic and $g(z_0) = a_n = f^{(n)}(z_0)/n! \neq 0$. We have now reached a contradiction as follows. Since g is continuous and $g(z_0) \neq 0$, $\exists r,\ 0 < r \leq R$ with $g(z) \neq 0$ for $|z - z_0| < r$. Hence $\{z \in D : f(z) = 0\} \cap B(z_0, r) = \{z_0\}$. This contradicts the fact that z_0 is a limit point of $\{z \in D : f(z) = 0\}$, and thus completes the proof of (c) implies (b).

(b) implies (a): Let $A = \{z \in D : \forall n \geq 0,\ f^{(n)}(z) = 0\}$. By assumption $A \neq \emptyset$. We shall prove that both $D - A$ and A are open sets. It will follow from Theorem 18.2 that $D = A$ and therefore f is identically zero in D.

A is open: Let $a \in A$. Since D is open, $\exists R > 0$ with $B(a, R) \subset D$. Write f in a Taylor series $f(z) = \sum_0^\infty a_n(z-a)^n$ for $|z - a| < R$ with $a_n = f^{(n)}(a)/n!$. Since $a \in A$, each $a_n = 0$ and so f is identically zero on $B(a, R)$. This means that $B(a, R) \subset A$ and so A is an open set.

$D - A$ is open: We first show that $A = D \cap \overline{A}$, where $\overline{A}$ is the closure of A, that is, $\overline{A}$ is a closed set and consists of all the limits of convergent sequences of points of A. Obviously, $A \subset D \cap \overline{A}$. On the other hand, if $z \in D \cap \overline{A}$, then there is a sequence z_k in A with $\lim_k z_k = z$. For each $n \geq 0$, $f^{(n)}$ is a continuous function on D. Therefore $f^{(n)}(z) = \lim_k f^{(n)}(z_k) = 0$, proving that $z \in A$ and $A = D \cap \overline{A}$. We now show that $D - A$ is open by proving that $D - A = D \cap (\mathbf{C} - \overline{A})$. Since $A \subset \overline{A}$, we have $\mathbf{C} - \overline{A} \subset \mathbf{C} - A$ so that $D - A = D \cap (\mathbf{C} - A) \supset D \cap (\mathbf{C} - \overline{A})$. On the other hand, if $z \in D - A$, then since $A = D \cap \overline{A}$, $z \in \mathbf{C} - \overline{A}$, that is, $D - A \subset D \cap (\mathbf{C} - \overline{A})$, proving that $D - A = D \cap (\mathbf{C} - \overline{A})$. □

Assignment 11 (Due March 3) *Use Theorem 22.1 to prove the following statements*

1. *Find all entire functions f that satisfy $f(z) = e^x$ for $z = x \in \mathbf{R}$.*

2. *Let f and g be analytic functions defined on a domain D and suppose $f(z)g(z) = 0$ for every $z \in D$. Show that either $f \equiv 0$ or $g \equiv 0$.*

3. *Suppose that f is analytic on $\{|z| < 2\}$. Show that there must exist some positive integer n such that $f(1/n) \neq 1/(n+1)$.*

23 Monday February 27, 2006—Order of a zero

Proof of (b) implies (a) in Theorem 22.1

Corollary 23.1 *Let f and D be as in Theorem 22.1, and suppose that f is not identically zero, that is (a) fails. Also, let $a \in D$ be a "zero" of f, that is, $f(a) = 0$.*

Then there exists $n \geq 1$, called the "order" of the zero a of f, and an analytic function g on D such tht $g(a) \neq 0$ and $f(z) = (z-a)^n g(z)$ for all $z \in D$.

24 Wednesday March 1, 2006—Riemann's Removable Singularity Theorem

Theorem 24.1 (Triangulated Morera Theorem) *Let f be continuous on a domain D and suppose that $\int_T f(z)\,dz = 0$ for every triangle T which together with its inside lies in D. Then f is analytic in D.*

Proof: Let $a \in D$ and let $B(a,R) \subset D$. For $z \in B(a,R)$, let $F(z) := \int_{[a,z]} f(s)\,ds$ where $[a,z]$ denotes the line segment from a to z. For any other point $z_0 \in B(a,R)$, by our assumption, $F(z) = \int_{[a,z_0]} f(s)\,ds + \int_{[z_0,z]} f(s)\,ds$. Therefore

$$\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{z-z_0}\int_{[z_0,z]} [f(s)-f(z_0)]\,ds$$

and

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| \leq \sup_{s\in[z_0,z]} |f(s)-f(z_0)|.$$

Since f is continuous at z_0, $F'(z_0)$ exists and equals $f(z_0)$ so f is analytic. □

Theorem 24.2 (Riemann's Removable Singularity Theorem) *Let f be analytic on a punctured disk $B(a,R)-\{a\}$. Then f has an analytic extension to $B(a,R)$ if and only if $\lim_{z\to a}(z-a)f(z)$ exists and equals 0.*

Proof: If the analytic extension g exists, then $\lim_{z\to a}(z-a)f(z) = \lim_{z\to a}(z-a)g(z) = 0 \cdot g(a) = 0$.

Now suppose that $\lim_{z\to a}(z-a)f(z) = 0$. Define a function g by $g(z) = (z-a)f(z)$ for $z \neq a$ and $g(a) = 0$. The function g is analytic for $z \neq a$, and is continuous at a. We shall show using Triangulated Morera theorem that g is analytic at a. Assuming for the moment that this is true, let us complete the proof. Since g is analytic and $g(a) = 0$, then by Corollary 23.1, $g(z) = (z-a)h(z)$ where h is analytic in $B(a,R)$. Thus, for $z \neq a$, $(z-a)f(z) = g(z) = (z-a)h(z)$, and thus $f(z) = h(z)$ for $z \neq a$. Thus h is the analytic extension of f to $B(a,R)$.

It remains to prove that g is analytic using the Triangulated Morera theorem. We must show that if T is any triangle in $B(a,R)$, then $\int_T f(s)\,ds = 0$. There are four possible cases.

Case 1: **<u>a is a vertex of T</u>** : In this case let x and y denote points on the two edges for which a is an endpoint. Then $\int_T f(s)\,ds = \int_{[a,y,x]} f(s)\,ds + \int_{[y,x,b,c]} f(s)\,ds$ where b and c are the other two vertices of T and $[\alpha,\beta,\cdots]$ denotes a polygon with vertices $\alpha,\beta,\cdots$. By the continuity of g at a, the first integral approaches zero as x and y approach a. The second integral is zero by Cauchy's theorem.

Case 2: <u>a **is inside** T</u> : In this case, draw lines from a to each of the vertices of T. Then $\int_T f(s)\,ds$ is the sum of three integrals of f over triangles having a as a vertex, each of which is zero by case 1.

Case 3: <u>a **lies on an edge of** T</u> : In this case, draw a line from a to the vertex which is opposite to the edge containing a. Then $\int_T f(s)\,ds$ is the sum of two integrals of f over triangles having a as a vertex, each of which is zero by case 1.

Case 4: <u>a **is outside of** T</u> : In this case, $\int_T f(s)\,ds = 0$ by Cauchy's theorem. □

25 Friday March 3, 2006—Order of a pole

Let f be analytic in $B(a,R)-\{a\}$. We say that a is a *pole* of f if $\lim_{z\to a}|f(z)| = +\infty$.

Proposition 25.1 *If f is analytic in $B(a,R)-\{a\}$ and has a pole at a, then there exists $m \ge 1$ and an analytic function g on $B(a,R)$ such that $f(z) = g(z)/(z-a)^m$ for all $z \in B(a,R)-\{a\}$, and $g(a) \neq 0$.*

Proof: Define $h(z) = 1/f(z)$ for $z \neq a$ and $h(a) = 0$. Obviously h is analytic for $z \neq a$, but in fact it is analytic at a by Theorem 24.2, since $\lim_{z\to a}(z-a)h(z) = 0 \cdot 0 = 0$. Since a is a zero of h, then $\exists m \ge 1$ such that $h(z) = (z-a)^m h_1(z)$, where h_1 is analytic and $h_1(a) \neq 0$. By continuity, there exists $r \le R$ with $h_1(z) \neq 0$ for all $|z-a| < r$. Then for $0 < |z-a| < r$, $f(z) = 1/h(z) = (1/h_1(z))/(z-a)^m$, completing the proof. □

26 Monday March 6, 2006—Classification of Singularities—Laurent series

Some definitions. A point $a \in \mathbf{C}$ is said to be an *isolated singularity* of f if f is analytic in $B(a,R)-\{a\}$ for some $R > 0$. Isolated singularities fall into three cases:

(1) a is a *removable singularity* of f if f has an analytic extension to $B(a,R)$.
(2) a is a *pole* of f if $\lim_{z\to a}|f(z)| = +\infty$.
(3) a is an *essential singularity* of f if it is neither a removable singularity or pole.

Some facts.
(1) a is a removable singularity if and only if $\lim_{z\to a}(z-a)f(z) = 0$ (this is Riemann's Removable Singularity Theorem). In this case, f has a power series expansion

$$f(z) = \sum_0^\infty a_n (z-a)^n, \qquad 0 < |z-a| < R.$$

(2) If a is a pole of f of order m, then $m \geq 1$ and because of Proposition 25.1, f has a power series expansion

$$f(z) = \frac{b_{-m}}{(z-a)^m} + \frac{b_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{b_{-1}}{z-a} + \sum_0^\infty b_n (z-a)^n, \qquad 0 < |z-a| < R.$$

Theorem 26.1 *Let $0 \leq R_1 < R_2 \leq \infty$, let $z_0 \in \mathbf{C}$, and let f be analytic in the annulus $D = \{z \in \mathbf{C} : R_1 < |z - z_0| < R_2\}$. Let C be a simple closed curve lying in D such that z_0 is inside C. For each $j = 0, \pm 1, \pm 2, \ldots$ set $c_j = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z_0)^{j+1}}\, ds$. Then for each $z \in D$ not lying on C, we have $f(z) = \sum_{j=-\infty}^{\infty} c_j (z-z_0)^j$.*

27 Wednesday March 8, 2006—Proof of Laurent's theorem

Proof of Theorem 26.1: The data we are given consists of z_0, R_1, R_2, C, f, z. The numbers z_0, R_1, R_2 define the domain (the annulus $\{R_1 < |z - z_0| < R_2\}$), C is a simple closed curve in D with z_0 inside, z is a point of D not lying on C. With this data, we construct three circles C_1, C_2, Γ as follows. C_1 is the circle $|z - z_0| = r_1$ where $R_1 < r_1$ and C and z lie outside of C_1. C_2 is the circle $|z - z_0| = r_2$ where $r_2 < R_2$ and C and z lie inside of C_2. Γ is the circle with center z and radius $\delta > 0$ which does not intersect C, C_1 or C_2. Now draw a curve L connecting C_1 to Γ, draw a picture and convince yourself that C_2 is homotopic to $C_1 + L + \Gamma - L$ in the domain $D - \{z\}$. By the pre-Cauchy theorem (see Corollary 14.3) applied to the function $f(s)/(s-z)$, which is analytic (as a function of s) in $D - \{z\}$,

$$\int_{C_2} \frac{f(s)}{s-z}\, ds = \int_{C_1} \frac{f(s)}{s-z}\, ds + \int_\Gamma \frac{f(s)}{s-z}\, ds.$$

By Cauchy's integral formula, $\int_\Gamma \frac{f(s)}{s-z}\, ds = 2\pi i f(z)$. Hence it remains to show that

$$\int_{C_2} \frac{f(s)}{s-z}\, ds = \sum_{j=0}^{\infty} \left(\int_C \frac{f(s)}{(s-z_0)^{j+1}}\, ds \right) (z-z_0)^j \tag{10}$$

and

$$\int_{C_1} \frac{f(s)}{s-z}\, ds = -\sum_{j=1}^{\infty} \left(\int_C \frac{f(s)}{(s-z_0)^{-j+1}}\, ds \right) \frac{1}{(z-z_0)^j} \tag{11}$$

To prove (10), note that for s on C_2, $|z - z_0| < |s - z_0|$ so that

$$\begin{aligned} \frac{1}{s-z} &= \frac{1}{(s-z_0)-(z-z_0)} = \frac{1}{s-z_0} \left(\frac{1}{1 - \left(\frac{z-z_0}{s-z_0}\right)} \right) \\ &= \frac{1}{s-z_0} \sum_{j=0}^{\infty} \left(\frac{z-z_0}{s-z_0} \right)^j = \sum_{j=0}^{\infty} \frac{(z-z_0)^j}{(s-z_0)^{j+1}}, \end{aligned}$$

and therefore

$$\int_{C_2} \frac{f(s)}{s-z}\, ds = \sum_{j=0}^{\infty} \left(\int_{C_2} \frac{f(s)}{(s-z_0)^{j+1}}\, ds \right) (z-z_0)^j.$$

In the integral on the right side, C_2 may be replaced by C since $f(s)/(s-z_0)^{j+1}$ is analytic between C_2 and C. This proves (10).

To prove (11), note that for s on C_1, $|s-z_0| < |z-z_0|$ so that

$$\begin{aligned}
\frac{1}{s-z} &= \frac{-1}{(z-z_0)-(s-z_0)} = \frac{-1}{z-z_0}\left(\frac{1}{1-\left(\frac{s-z_0}{z-z_0}\right)}\right) \\
&= \frac{-1}{z-z_0}\sum_{j=0}^{\infty}\left(\frac{s-z_0}{z-z_0}\right)^j = -\sum_{j=0}^{\infty}\frac{(s-z_0)^j}{(z-z_0)^{j+1}} \\
&= -\sum_{j=1}^{\infty}\frac{(s-z_0)^{j-1}}{(z-z_0)^j} = -\sum_{j=1}^{\infty}\frac{(1}{(s-z_0)^{-j+1}}\frac{1}{(z-z_0)^j}
\end{aligned}$$

and therefore

$$\int_{C_1} \frac{f(s)}{s-z}\, ds = -\sum_{j=1}^{\infty} \left(\int_{C_1} \frac{f(s)}{(s-z_0)^{-j+1}}\, ds \right) (z-z_0)^{-j}.$$

In the integral on the right side, C_1 may be replaced by C since $f(s)/(s-z_0)^{-j+1}$ is analytic between C_1 and C. This proves (11) and completes the proof of the theorem. □

28 Friday March 10, 2006—Examples of Laurent Series

Let $f(z) = \frac{1}{z(z-1}$. This function is analytic on $\mathbf{C}$ except for isolated singularities at $z = 0$ and at $z = 1$. The following are the possible Laurent expansions for this function.

About $z = 0$ and on $0 < |z| < 1$:

$$f(z) = -\frac{1}{z} + \frac{1}{z-1} = -\frac{1}{z} - (1 + z + z^2 + \cdots).$$

About $z = 0$ and on $1 < |z| < \infty$:

$$f(z) = \frac{1}{z}\cdot\frac{1}{z-1} = \frac{1}{z^2}\left(\frac{1}{1-\frac{1}{z}}\right) = \frac{1}{z^2}(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

About $z = 1$ and on $0 < |z-1| < 1$:

$$f(z) = \frac{1}{z}\cdot\frac{1}{z-1} = \left(\frac{1}{1-(1-z)}\right)\cdot\frac{1}{z-1} = \Big(1 + (1-z) + (1-z)^2 + \cdots\Big)\cdot\frac{1}{z-1}$$

$$= \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + (z-1)^3 - \cdots.$$

About $z = 1$ and on $1 < |z-1| < \infty$:

$$f(z) = \frac{1}{z} \cdot \frac{1}{z-1} = \frac{1}{z-1}\left(\frac{1}{1+\frac{1}{z-1}}\right) \cdot \frac{1}{z-1}$$

$$= \frac{1}{z-1}\left(1 - \frac{1}{z-1} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \cdots\right) \cdot \frac{1}{z-1} = \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \cdots$$

29 Monday March 13, 2006—Casorati-Weierstrass theorem

Theorem 29.1 *If f has an essential singularity at z_0, then for every $\delta > 0$, the set $f(B(z_0, \delta) - \{z_0\})$ is dense in $\mathbf{C}$. That is, for all $c \in \mathbf{C}$ and $\epsilon > 0$, there exists a $z \in B(z_0, \delta) - \{z_0\}$ such that $|f(z) - c| < \epsilon$.*

Proof: Suppose not. Then there exist $c_0 \in \mathbf{C}$ and $\epsilon_0 > 0$ such that $|f(z) - c_0| \geq \epsilon_0$ for all $z \in B(z_0, \delta) - \{z_0\}$. It follow that

$$\lim_{z \to z_0} \left| \frac{f(z) - c_0}{z - z_0} \right| = +\infty,$$

so that the function $(f(z) - c_0)/(z - z_0)$ has a pole at z_0. Let $m \geq 1$ be the order of this pole so that there is an analytic function g at z_0 such that

$$\frac{f(z) - c_0}{z - z_0} = \frac{g(z)}{(z - z_0)^m}$$

for $z \neq z_0$.

We have $\lim_{z \to z_0} |z - z_0|^{m+1} |f(z) - c_0| = 0$ and therefore $|z - z_0|^{m+1}|f(z)| \leq |z - z_0|^{m+1}|f(z) - c_0| + |z - z_0|^{m+1}|c_0| \to 0$ as $z \to z_0$. Thus, $(z - z_0)^m f(z)$ has a removable singularity at $z = z_0$ and it follows that for $z \neq z_0$, $f(z) = h(z)/(z - z_0)^m$ for some function h which is analytic at z_0. This says that f has a pole at z_0, which is a contradiction. □

30 Wednesday March 15, 2006—About the final exam(s); Zeros of entire functions

30.1 About the final exam(s)

Part I of the final exam has been posted. You may download a copy if you wish, but I will bring hard copies for everyone to class on Friday March 17. All 10 questions on Part I are equally weighted. You are to hand in FIVE AND ONLY FIVE QUESTIONS on March 24 at 3 pm. You may choose any five questions. I recommend studying all 10 questions in preparation for Part II of the final exam. Part I is open book and notes, but independent work is required. You are not to share answers or discuss the questions with other students in the class. You may ask questions in person or by email to me or Chris.

Part II of the final exam will take place on March 22 1:30-3:30 in the classroom. It will include problems based on some of the following 11 theorems, on which the problems on part I are also based.

1. Cauchy's Theorem(s)
2. Cauchy's Integral Formula(s)
3. Morera's Theorem
4. Liouville's Theorem
5. Maximum Modulus Theorem
6. Taylor's Theorem
7. Schwarz's Lemma
8. Identity Theorem
9. Riemann's Removable Singularity Theorem
10. Laurent's Theorem
11. Casorati-Weierstrass Theorem

Parts I and II constitute "the final exam" of the course, and carry equal weight.

Special Office Hours; Monday March 20 1:30-3:30 MSTB 263

Tuesday March 21 1:30-3:30 MSTB 263

Final Exam-Part II Wednesday March 22 1:30-3:30 SSTR 100

30.2 Zeros of entire functions

We discussed the following facts:

1. The zeros of an analytic function are isolated

2. An entire function f with no zeros is of the form $f(z) = \exp(g(z))$ for some other entire function g.

3. An entire function f with only finitely many zeros $a_1, \ldots, a_n$ of orders $k_1, \ldots, k_n$ is of the form $f(z) = \prod_{j=1}^{n}(z - a_j)^{k_j} \exp(g(z))$ for some entire function g.

4. The set of zeros of an entire function is at most countable.

31 Friday March 17, 2006

Miscellaneous end of course business (Return and short discussion of second midterm; return of HW 7-11; an example of calculating an integral involving sines and cosines by converting it to a contour integral on the unit circle (see the appendix to the online text by Cain).