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On non-abelian extensions of Leibniz algebras*

Jiefeng Liu[1], Yunhe Sheng[2] and Qi Wang[2]
[1]Department of Mathematics, Xinyang Normal University,
Xinyang 464000, Henan, China

[2]Department of Mathematics, Jilin University, Changchun 130012, Jilin, China

Email: jfliu13@mails.jlu.edu.cn, shengyh@jlu.edu.cn, wangq13@mails.jlu.edu.cn

Abstract

In this paper, first we classify non-abelian extensions of Leibniz algebras by the second non-abelian cohomology. Then, we construct Leibniz 2-algebras using derivations of Leibniz algebras, and show that under a condition on the center, a non-abelian extension of Leibniz algebras can be described by a Leibniz 2-algebra morphism. At last, we give a description of non-abelian extensions in terms of Maurer-Cartan elements in a differential graded Lie algebra.

1 Introduction

Eilenberg and Maclane developed a theory of non-abelian extensions of abstract groups in the 1940s [12], leading to the low dimensional non-abelian group cohomology. Then there are a lot of analogous results for Lie algebras [1, 18, 19]. Abelian extensions of Lie algebras can be classified by the second cohomology. Non-abelian extensions of Lie algebras can be described by some linear maps regarded as derivations of Lie algebras. This result is generalized to the case of super Lie algebras in [2, 15], and to the case of Lie algebroids in [7]. In [13], the author gave a description of non-abelian extensions of Lie algebras in terms of the Deligne groupoid of a differential graded Lie algebra (DGLA for short). In [25], a non-abelian extension of the Lie algebra $\mathfrak{g}$ by the Lie algebra $\mathfrak{h}$ is explained by a Lie 2-algebra morphism from $\mathfrak{g}$ to the strict Lie 2-algebra $\mathfrak{h} \xrightarrow{\mathrm{ad}} \mathrm{Der}(\mathfrak{h})$, where $\mathrm{Der}(\mathfrak{h})$ is the Lie algebra of derivations on $\mathfrak{h}$. See [5] for more information about a Lie 2-algebra, which is a categorification of a Lie algebra.

The notion of a Leibniz algebra was introduced by Loday [22, 23], which is a noncommutative generalization of a Lie algebra. Central extensions of Leibniz algebras are studied from different aspects [10, 11, 17]. Abelian extensions of Leibniz algebras is studied in [9]. Non-abelian cohomology and extensions by crossed modules of Leibniz algebras are studied in [8, 16, 20]. Associated to a Leibniz algebra, there is a differential graded Lie algebra structure on the graded vector space of the cochain complex, which plays an important role in studying cohomology and deformations of Leibniz algebras, see [6, 14] for details.

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In this paper, we study non-abelian extensions of Leibniz algebras. First we introduce the second non-abelian cohomology $H^2(\mathfrak{g},\mathfrak{h})$ of a Leibniz algebra $\mathfrak{g}$ with the coefficients in a Leibniz algebra $\mathfrak{h}$, by which we classify non-abelian extensions of $\mathfrak{g}$ by $\mathfrak{h}$. Then we construct two Leibniz 2-algebras using left derivations and right derivations of Leibniz algebras. One is a natural generalization of the Lie 2-algebra associated to derivations of a Lie algebra; The other can be used to describe non-abelian extensions of Leibniz algebras that satisfy a condition on centers. Whether there is a Leibniz 2-algebra associated to derivations of Leibniz algebras, that could be used to describe all the non-abelian extensions of Leibniz algebras, like the case of Lie algebras, remains to be an interesting problem. Finally, we generalize the result in [13] to the case of Leibniz algebras. We construct a DGLA associated to Leibniz algebras $\mathfrak{g}$ and $\mathfrak{h}$, and describe non-abelian extensions of $\mathfrak{g}$ by $\mathfrak{h}$ by its Maurer-Cartan elements.

The paper is organized as follows. In Section 2, we study properties of left derivations $\mathrm{Der}^L(\mathfrak{g})$ and right derivations $\mathrm{Der}^R(\mathfrak{g})$ of a Leibniz algebra $\mathfrak{g}$, and construct a semidirect product Leibniz algebra $\mathrm{Der}^L(\mathfrak{g})\oplus\mathrm{Der}^R(\mathfrak{g})$. Then we introduce the second non-abelian cohomology of Leibniz algebras, by which we classify non-abelian extensions of Leibniz algebras (Theorem 2.8). In Section 3, first we recall some basic definitions regarding Leibniz 2-algebras and morphisms between them. Then we construct Leibniz algebras $\Pi(\mathfrak{h})$ and $\Xi(\mathfrak{h})$ associated to a Leibniz algebra $\mathfrak{h}$, which are subalgebras of $\mathrm{Der}^L(\mathfrak{h})\oplus\mathrm{Der}^R(\mathfrak{h})$. We go on constructing strict Leibniz 2-algebras $(\mathfrak{h},\Pi(\mathfrak{h}),(\mathrm{ad}^L,\mathrm{ad}^R),l_2)$ and $(\mathfrak{h},\Xi(\mathfrak{h}),(\mathrm{ad}^L,\mathrm{ad}^R),l_2)$. The former is a natural generalization of the Lie 2-algebra associated to derivations of a Lie algebra, and the latter can be used to describe some non-abelian extensions of $\mathfrak{g}$ by $\mathfrak{h}$. More precisely, we show that there is a one-to-one correspondence between non-abelian extensions $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathfrak{h}$ with $\mathrm{Z}(\mathfrak{h})=\mathrm{Z}(\hat{\mathfrak{g}})\cap\mathfrak{h}$ and morphisms from $\mathfrak{g}$ to $(\mathfrak{h},\Xi(\mathfrak{h}),(\mathrm{ad}^L,\mathrm{ad}^R),l_2)$ (Theorem 3.12). In Section 4, we construct a DGLA $(L,[\cdot,\cdot]_C,\overline{\partial})$, which is a sub-DGLA of $(C(\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_C,\overline{\partial})$, where $\mathfrak{g}\oplus\mathfrak{h}$ is the Leibniz algebra direct sum of $\mathfrak{g}$ and $\mathfrak{h}$. Then we show that the equivalence relation for non-abelian cohomology coincides with the equivalence relation for Maurer-Cartan elements in the DGLA $(L,[\cdot,\cdot]_C,\overline{\partial})$ (Theorem 4.7).

2 Classification of non-abelian extensions of Leibniz algebras in terms of non-abelian cohomology

A Leibniz algebra is a vector space $\mathfrak{g}$, endowed with a linear map $[\cdot,\cdot]_{\mathfrak{g}}:\mathfrak{g}\otimes\mathfrak{g}\longrightarrow\mathfrak{g}$ satisfying

$$[x,[y,z]_{\mathfrak{g}}]_{\mathfrak{g}}=[[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}}+[y,[x,z]_{\mathfrak{g}}]_{\mathfrak{g}},\quad\forall\ x,y,z\in\mathfrak{g}. \tag{1}$$

This is in fact a left Leibniz algebra. In this paper, we only consider left Leibniz algebras. Denote by $\mathrm{Z}(\mathfrak{g})$ the left center of $\mathfrak{g}$, i.e.

$$\mathrm{Z}(\mathfrak{g})=\{x\in\mathfrak{g}\mid[x,y]_{\mathfrak{g}}=0,\ \forall y\in\mathfrak{g}\}. \tag{2}$$

A representation of the Leibniz algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ is a triple (V,l,r), where V is a vector space equipped with two linear maps $l:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ and $r:\mathfrak{g}\longrightarrow\mathfrak{gl}(V)$ such that the following equalities hold:

$$l_{[x,y]_{\mathfrak{g}}}=[l_x,l_y],\quad r_{[x,y]_{\mathfrak{g}}}=[l_x,r_y],\quad r_y\circ l_x=-r_y\circ r_x,\quad\forall x,y\in\mathfrak{g}. \tag{3}$$

The Leibniz cohomology of $\mathfrak{g}$ with coefficients in V is the cohomology of the cochain complex $C^k(\mathfrak{g},V)=\mathrm{Hom}(\otimes^{k+1}\mathfrak{g},V),(k\geq 0)$ with the coboundary operator $\partial:C^{k-1}(\mathfrak{g},V)\longrightarrow C^k(\mathfrak{g},V)$

defined by

$$\begin{aligned}\partial c(x_1,\dots,x_{k+1}) &= \sum_{i=1}^{k}(-1)^{i+1}l_{x_i}(c(x_1,\dots,\widehat{x_i},\dots,x_{k+1}))+(-1)^{k+1}r_{x_{k+1}}(c(x_1,\dots,x_k))\\ &\quad+\sum_{1\le i<j\le k+1}(-1)^i c(x_1,\dots,\widehat{x_i},\dots,x_{j-1},[x_i,x_j]_{\mathfrak{g}},x_{j+1},\dots,x_{k+1}). \qquad (4)\end{aligned}$$

We denote by $\mathrm{Der}^L(\mathfrak{g})$ and $\mathrm{Der}^R(\mathfrak{g})$ the set of left derivations and the set of right derivations of $\mathfrak{g}$ respectively:

$$\begin{aligned}\mathrm{Der}^L(\mathfrak{g}) &= \{D\in\mathfrak{gl}(\mathfrak{g})|D[x,y]_{\mathfrak{g}}=[Dx,y]_{\mathfrak{g}}+[x,Dy]_{\mathfrak{g}},\forall x,y\in\mathfrak{g}\},\\ \mathrm{Der}^R(\mathfrak{g}) &= \{D\in\mathfrak{gl}(\mathfrak{g})|D[x,y]_{\mathfrak{g}}=[x,Dy]_{\mathfrak{g}}-[y,Dx]_{\mathfrak{g}},\forall x,y\in\mathfrak{g}\}.\end{aligned}$$

It is easy to see that for all $x\in\mathfrak{g}$, $\mathrm{ad}^L_x:\mathfrak{g}\longrightarrow\mathfrak{g}$, which is given by $\mathrm{ad}^L_x(y)=[x,y]_{\mathfrak{g}}$, is a left derivation; $\mathrm{ad}^R_x:\mathfrak{g}\longrightarrow\mathfrak{g}$, which is given by $\mathrm{ad}^R_x(y)=[y,x]_{\mathfrak{g}}$, is a right derivation.

Lemma 2.1. *If $x\in\mathrm{Z}(\mathfrak{g})$, then for all $D^L\in\mathrm{Der}^L(\mathfrak{g})$, we have $D^Lx\in\mathrm{Z}(\mathfrak{g})$.*

Proof. It follows from

$$[D^Lx,y]=D^L[x,y]-[x,D^Ly]=0,\quad\forall\ x\in\mathrm{Z}(\mathfrak{g}).\ \blacksquare$$

For all $D_1,D_2\in\mathfrak{gl}(\mathfrak{g})$, $[D_1,D_2]$ denotes their commutator, i.e. $[D_1,D_2]=D_1D_2-D_2D_1$. By straightforward computations, we have

Lemma 2.2. *For all $x\in\mathfrak{g}$, $D^L,D^L_1,D^L_2\in\mathrm{Der}^L(\mathfrak{g})$ and $D^R\in\mathrm{Der}^R(\mathfrak{g})$, we have*

(i) $[D^L_1,D^L_2]\in\mathrm{Der}^L(\mathfrak{g})$;

(ii) $[D^L,D^R]\in\mathrm{Der}^R(\mathfrak{g})$;

(iii) $[D^L,\mathrm{ad}^L_x]=\mathrm{ad}^L_{D^Lx}$, $[D^L,\mathrm{ad}^R_x]=\mathrm{ad}^R_{D^Lx}$, $[\mathrm{ad}^L_x,D^R]=\mathrm{ad}^R_{D^Rx}$.

Corollary 2.3. *With the above notations,* $\mathrm{Der}^L(\mathfrak{g})$ *is a Lie algebra, and* $\mathrm{Der}^R(\mathfrak{g})$ *is a* $\mathrm{Der}^L(\mathfrak{g})$*-module, where the representation ρ is given by*

$$\rho(D^L)(D^R)=[D^L,D^R],\quad\forall D^L\in\mathrm{Der}^L(\mathfrak{g}),D^R\in\mathrm{Der}^R(\mathfrak{g}).$$

Denote by $\mathfrak{Der}(\mathfrak{g})=\mathrm{Der}^L(\mathfrak{g})\oplus\mathrm{Der}^R(\mathfrak{g})$.

View $\mathrm{Der}^L(\mathfrak{g})$ as a Leibniz algebra, then $(\rho,0)$ is a representation of $\mathrm{Der}^L(\mathfrak{g})$ on $\mathrm{Der}^R(\mathfrak{g})$. Thus, we have the semidirect product Leibniz algebra.

Lemma 2.4. $(\mathfrak{Der}(\mathfrak{g}),[\cdot,\cdot]_s)$ *is a Leibniz algebra, where the bracket operation* $[\cdot,\cdot]_s$ *on* $\mathfrak{Der}(\mathfrak{g})$ *is given by*

$$[(D^L_1,D^R_1),(D^L_2,D^R_2)]_s=([D^L_1,D^L_2],[D^L_1,D^R_2]),\quad\forall\ D^L_i\in\mathrm{Der}^L(\mathfrak{g}),\ D^R_i\in\mathrm{Der}^R(\mathfrak{g}). \qquad (5)$$

Definition 2.5. (1) *Let $\mathfrak{g}$, $\mathfrak{h}$, $\hat{\mathfrak{g}}$ be Leibniz algebras. A non-abelian extension of Leibniz algebras is a short exact sequence of Leibniz algebras:*

$$0\longrightarrow\mathfrak{h}\stackrel{\mathbb{i}}{\longrightarrow}\hat{\mathfrak{g}}\stackrel{\mathbb{p}}{\longrightarrow}\mathfrak{g}\longrightarrow 0.$$

We say that $\hat{\mathfrak{g}}$ is a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$.

(2) *A splitting of* $\hat{\mathfrak{g}}$ *is a linear map* $\sigma : \mathfrak{g} \to \widehat{\mathfrak{g}}$ *such that* $\mathbb{p} \circ \sigma = \mathrm{id}$.

(3) *Two extensions of* $\mathfrak{g}$ *by* $\mathfrak{h}$*,* $\hat{\mathfrak{g}}_1$ *and* $\hat{\mathfrak{g}}_2$*, are said to be isomorphic if there exists a Leibniz algebra morphism* $\theta : \hat{\mathfrak{g}}_2 \longrightarrow \hat{\mathfrak{g}}_1$ *such that we have the following commutative diagram:*

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r, "\mathbb{i}_2"] \arrow[d, equal] & \hat{\mathfrak{g}}_2 \arrow[r, "\mathbb{p}_2"] \arrow[d, "\theta"] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathfrak{h} \arrow[r, "\mathbb{i}_1"] & \hat{\mathfrak{g}}_1 \arrow[r, "\mathbb{p}_1"] & \mathfrak{g} \arrow[r] & 0. \end{tikzcd} \tag{6}$$

Let $\hat{\mathfrak{g}}$ be a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$, and $\sigma : \mathfrak{g} \to \widehat{\mathfrak{g}}$ a splitting. Define $\omega : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{h}$, $l : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{h})$ and $r : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{h})$ respectively by

$$\omega(x, y) = [\sigma(x), \sigma(y)]_{\hat{\mathfrak{g}}} - \sigma[x, y]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}, \tag{7}$$

$$l_x(\beta) = [\sigma(x), \beta]_{\hat{\mathfrak{g}}}, \quad \forall x \in \mathfrak{g},\ \beta \in \mathfrak{h}, \tag{8}$$

$$r_y(\alpha) = [\alpha, \sigma(y)]_{\hat{\mathfrak{g}}}, \quad \forall y \in \mathfrak{g},\ \alpha \in \mathfrak{h}. \tag{9}$$

Given a splitting, we have $\hat{\mathfrak{g}} \cong \mathfrak{g} \oplus \mathfrak{h}$, and the Leibniz algebra structure on $\hat{\mathfrak{g}}$ can be transferred to $\mathfrak{g} \oplus \mathfrak{h}$:

$$[x + \alpha, y + \beta]_{(l,r,\omega)} = [x, y]_{\mathfrak{g}} + \omega(x, y) + l_x\beta + r_y\alpha + [\alpha, \beta]_{\mathfrak{h}}. \tag{10}$$

Proposition 2.6. $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(l,r,\omega)})$ *is a Leibniz algebra if and only if* l, r, ω *satisfy the following equalities:*

$$l_x[\alpha, \beta]_{\mathfrak{h}} = [l_x\alpha, \beta]_{\mathfrak{h}} + [\alpha, l_x\beta]_{\mathfrak{h}}, \tag{11}$$

$$r_x[\alpha, \beta]_{\mathfrak{h}} = [\alpha, r_x\beta]_{\mathfrak{h}} - [\beta, r_x\alpha]_{\mathfrak{h}}, \tag{12}$$

$$[l_x\alpha + r_x\alpha, \beta]_{\mathfrak{h}} = 0, \tag{13}$$

$$[l_x, l_y] - l_{[x,y]_{\mathfrak{g}}} = \mathrm{ad}^L_{\omega(x,y)}, \tag{14}$$

$$[l_x, r_y] - r_{[x,y]_{\mathfrak{g}}} = \mathrm{ad}^R_{\omega(x,y)}, \tag{15}$$

$$r_y(r_x(\alpha) + l_x(\alpha)) = 0, \tag{16}$$

$$l_x\omega(y, z) - l_y\omega(x, z) - r_z\omega(x, y) = \omega([x, y]_{\mathfrak{g}}, z) - \omega(x, [y, z]_{\mathfrak{g}}) + \omega(y, [x, z]_{\mathfrak{g}}). \tag{17}$$

Proof. Assume that $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(l,r,\omega)})$ is a Leibniz algebra. By

$$[x, [\alpha, \beta]_{(l,r,\omega)}]_{(l,r,\omega)} = [[x, \alpha]_{(l,r,\omega)}, \beta]_{(l,r,\omega)} + [\alpha, [x, \beta]_{(l,r,\omega)}]_{(l,r,\omega)},$$

we deduce that (11) holds. By

$$[\alpha, [\beta, x]_{(l,r,\omega)}]_{(l,r,\omega)} = [[\alpha, \beta]_{(l,r,\omega)}, x]_{(l,r,\omega)} + [\beta, [\alpha, x]_{(l,r,\omega)}]_{(l,r,\omega)},$$

we deduce that (12) holds. By

$$[x, [y, \alpha]_{(l,r,\omega)}]_{(l,r,\omega)} = [[x, y]_{(l,r,\omega)}, \alpha]_{(l,r,\omega)} + [y, [x, \alpha]_{(l,r,\omega)}]_{(l,r,\omega)},$$

we deduce that (14) holds. By

$$[x, [\alpha, y]_{(l,r,\omega)}]_{(l,r,\omega)} = [[x, \alpha]_{(l,r,\omega)}, y]_{(l,r,\omega)} + [\alpha, [x, y]_{(l,r,\omega)}]_{(l,r,\omega)},$$

we deduce that (15) holds. By

$$[x,[y,z]_{(l,r,\omega)}]_{(l,r,\omega)}=[[x,y]_{(l,r,\omega)},z]_{(l,r,\omega)}+[y,[x,z]_{(l,r,\omega)}]_{(l,r,\omega)},$$

we deduce that (17) holds. By

$$[\alpha,[x,\beta]_{(l,r,\omega)}]_{(l,r,\omega)}=[[\alpha,x]_{(l,r,\omega)},\beta]_{(l,r,\omega)}+[x,[\alpha,\beta]_{(l,r,\omega)}]_{(l,r,\omega)},$$

we obtain that $l_x[\alpha,\beta]_{\mathfrak{h}}=[\alpha,l_x\beta]_{\mathfrak{h}}-[r_x\alpha,\beta]_{\mathfrak{h}}$. Combining with (11), we deduce that (13) holds. By

$$[\alpha,[x,y]_{(l,r,\omega)}]_{(l,r,\omega)}=[[\alpha,x]_{(l,r,\omega)},y]_{(l,r,\omega)}+[x,[\alpha,y]_{(l,r,\omega)}]_{(l,r,\omega)},$$

we obtain that $l_xr_y-r_{[x,y]_{\mathfrak{g}}}+r_yr_x=\mathrm{ad}^R_{\omega(x,y)}$. Combining with (15), we deduce that (16) holds.

Conversely, if (11)-(17) hold, it is straightforward to see that $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot]_{(l,r,\omega)})$ is a Leibniz algebra. The proof is finished. ■

Note that (11) means that $l_x\in\mathrm{Der}^L(\mathfrak{h})$, and (12) means that $r_x\in\mathrm{Der}^R(\mathfrak{h})$.

Definition 2.7. *A non-abelian 2-cocycle on $\mathfrak{g}$ with values in $\mathfrak{h}$ is a triple (l,r,ω) of linear maps $l:\mathfrak{g}\to\mathrm{Der}^L(\mathfrak{h})$, $r:\mathfrak{g}\to\mathrm{Der}^R(\mathfrak{h})$ and $\omega:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{h}$ satisfying Eqs.* (13)-(17). *We denote by $Z^2(\mathfrak{g},\mathfrak{h})$ the set of non-abelian 2-cocycles.*

Two 2-cocycles (l^1,r^1,ω^1) and (l^2,r^2,ω^2) are equivalent, if there exists a linear map $\varphi:\mathfrak{g}\to\mathfrak{h}$ such that for all $x,y\in\mathfrak{g}$, the following equalities hold:

$$\begin{aligned} l^1_x-l^2_x &= \mathrm{ad}^L_{\varphi(x)}, &(18)\\ r^1_x-r^2_x &= \mathrm{ad}^R_{\varphi(x)}, &(19)\\ \omega^1(x,y)-\omega^2(x,y) &= l^2_x\varphi(y)+r^2_y\varphi(x)+[\varphi(x),\varphi(y)]_{\mathfrak{h}}-\varphi([x,y]_{\mathfrak{g}}). &(20)\end{aligned}$$

The second non-abelian cohomology $H^2(\mathfrak{g},\mathfrak{h})$ is the quotient of $Z^2(\mathfrak{g},\mathfrak{h})$ by the above equivalence relation.

Theorem 2.8. *Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ and $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}})$ be Leibniz algebras. Then non-abelian extensions of $\mathfrak{g}$ by $\mathfrak{h}$ are classified by the second non-abelian cohomology $H^2(\mathfrak{g},\mathfrak{h})$.*

Proof. Let $\hat{\mathfrak{g}}$ be a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$. By choosing a splitting $\sigma_1:\mathfrak{g}\longrightarrow\hat{\mathfrak{g}}$, we obtain a 2-cocycle (l^1,r^1,ω^1). First we show that the cohomological class of (l^1,r^1,ω^1) does not depend on the choice of splittings. In fact, let σ_1 and σ_2 be two different splittings. Define $\varphi:\mathfrak{g}\longrightarrow\mathfrak{h}$ by $\varphi(x):=\sigma_1(x)-\sigma_2(x)$. By $l^j_x(\beta)=[\sigma_j(x),\beta]_{\hat{\mathfrak{g}}}$ and $r^j_y(\alpha)=[\alpha,\sigma_j(y)]_{\hat{\mathfrak{g}}}$, $j=1,2$, it is obvious that (18) and (19) hold. Furthermore, we have

$$\begin{aligned}\omega^1(x,y) &= [\sigma_1(x),\sigma_1(y)]_{\hat{\mathfrak{g}}}-\sigma_1[x,y]_{\mathfrak{g}}\\ &= [\sigma_2(x)+\varphi(x),\sigma_2(y)+\varphi(y)]_{\hat{\mathfrak{g}}}-\sigma_2[x,y]_{\mathfrak{g}}-\varphi([x,y]_{\mathfrak{g}})\\ &= [\sigma_2(x),\sigma_2(y)]_{\hat{\mathfrak{g}}}+[\sigma_2(x),\varphi(y)]_{\hat{\mathfrak{g}}}+[\varphi(x),\sigma_2(y)]_{\hat{\mathfrak{g}}}+[\varphi(x),\varphi(y)]_{\mathfrak{h}}-\sigma_2[x,y]_{\mathfrak{g}}-\varphi[x,y]_{\mathfrak{g}}\\ &= \sigma_2[x,y]_{\mathfrak{g}}+\omega^2(x,y)+l^2_x\varphi(y)+r^2_y\varphi(x)+[\varphi(x),\varphi(y)]_{\mathfrak{h}}-\sigma_2[x,y]_{\mathfrak{g}}-\varphi([x,y]_{\mathfrak{g}})\\ &= \omega^2(x,y)+l^2_x\varphi(y)+r^2_y\varphi(x)+[\varphi(x),\varphi(y)]_{\mathfrak{h}}-\varphi([x,y]_{\mathfrak{g}}),\end{aligned}$$

which implies that (20) holds. Thus, (l^1,r^1,ω^1) and (l^2,r^2,ω^2) are in the same cohomological class.

Now we go on to prove that isomorphic extensions give rise to the same element in $H^2(\mathfrak{g},\mathfrak{h})$. Assume that $\hat{\mathfrak{g}_1}$ and $\hat{\mathfrak{g}_2}$ are isomorphic extensions of $\mathfrak{g}$ by $\mathfrak{h}$, and $\theta:\hat{\mathfrak{g}_2}\longrightarrow\hat{\mathfrak{g}_1}$ is a Leibniz algebra morphism such that we have the commutative diagram (6). Assume that $\sigma_2:\mathfrak{g}\longrightarrow\hat{\mathfrak{g}_2}$ and $\sigma_1:\mathfrak{g}\longrightarrow\hat{\mathfrak{g}_1}$ are two splittings. Define $\sigma_2':\mathfrak{g}\longrightarrow\hat{\mathfrak{g}_2}$ by $\sigma_2'=\theta^{-1}\circ\sigma_1$. Since $\mathbb{p}_1\circ\theta=\mathbb{p}_2$, we have

$$\mathbb{p}_2\circ\sigma_2'(x)=(\mathbb{p}_1\circ\theta)\circ\theta^{-1}\circ\sigma_1(x)=x.$$

Thus, σ_2' is a splitting of $\hat{\mathfrak{g}_2}$. Define $\varphi_\theta:\mathfrak{g}\longrightarrow\mathfrak{h}$ by $\varphi_\theta(x)=\sigma_2'(x)-\sigma_2(x)$. Since $\theta^{-1}:\mathfrak{g}_1\to\mathfrak{g}_2$ is a morphism of Leibniz algebras, we have

$$\theta^{-1}[\sigma_1(x),\alpha]_{\hat{\mathfrak{g}}_1}=[\theta^{-1}\sigma_1(x),\alpha]_{\hat{\mathfrak{g}}_2}=[\varphi_\theta(x)+\sigma_2(x),\alpha]_{\hat{\mathfrak{g}}_2}=l^2_x\alpha+\mathrm{ad}^L_{\varphi_\theta(x)}\alpha.$$

On the other hand, since $\theta|_{\mathfrak{h}}=\mathrm{id}$, we have

$$\theta^{-1}[\sigma_1(x),\alpha]_{\hat{\mathfrak{g}}_1}=[\sigma_1(x),\alpha]_{\hat{\mathfrak{g}}_1}=l^1_x\alpha.$$

Therefore, we have

$$l^1_x-l^2_x=\mathrm{ad}^L_{\varphi_\theta(x)}. \tag{21}$$

Similarly, we have

$$r^1_x-r^2_x=\mathrm{ad}^R_{\varphi_\theta(x)}. \tag{22}$$

For all $x,y\in\mathfrak{g}$, on one hand, we have

$$\begin{aligned}
\theta^{-1}[\sigma_1(x),\sigma_1(y)]_{\hat{\mathfrak{g}}_1} &= [\theta^{-1}\sigma_1(x),\theta^{-1}\sigma_1(y)]_{\hat{\mathfrak{g}}_2}\\
&= [\varphi_\theta(x)+\sigma_2(x),\varphi_\theta(y)+\sigma_2(y)]_{\hat{\mathfrak{g}}_2}\\
&= [\varphi_\theta(x),\varphi_\theta(y)]_{\mathfrak{h}}+l^2_x\varphi_\theta(y)+r^2_y\varphi_\theta(x)+[\sigma_2(x),\sigma_2(y)]_{\hat{\mathfrak{g}}_2}\\
&= [\varphi_\theta(x),\varphi_\theta(y)]_{\mathfrak{h}}+l^2_x\varphi_\theta(y)+r^2_y\varphi_\theta(x)+\sigma_2[x,y]_{\mathfrak{g}}+\omega^2(x,y).
\end{aligned}$$

On the other hand, we have

$$\theta^{-1}[\sigma_1(x),\sigma_1(y)]_{\hat{\mathfrak{g}}_1} = \theta^{-1}(\sigma_1[x,y]_{\mathfrak{g}}+\omega^1(x,y))=\varphi_\theta[x,y]_{\mathfrak{g}}+\sigma_2[x,y]_{\mathfrak{g}}+\omega^1(x,y).$$

In conclusion, we have

$$\omega^1(x,y)-\omega^2(x,y)=l^2_x\varphi_\theta(y)+r^2_y\varphi_\theta(x)+[\varphi_\theta(x),\varphi_\theta(y)]_{\mathfrak{h}}-\varphi_\theta[x,y]_{\mathfrak{g}}. \tag{23}$$

By (21)-(23), we deduce that (l^1,r^1,ω^1) and (l^2,r^2,ω^2) are in the same cohomological class.

Conversely, given two 2-cocycles (l^1,r^1,ω^1) and (l^2,r^2,ω^2) representing the same cohomological class, i.e. there exists $\varphi:\mathfrak{g}\longrightarrow\mathfrak{h}$ such that (18)-(20) holds, we show that the corresponding extensions $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot]_{(l^1,r^1,\omega^1)})$ and $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot]_{(l^2,r^2,\omega^2)})$ are isomorphic. Define $\theta:\mathfrak{g}\oplus\mathfrak{h}\longrightarrow\mathfrak{g}\oplus\mathfrak{h}$ by

$$\theta(x+\alpha)=x-\varphi(x)+\alpha.$$

Then, we can deduce that θ is a Leibniz algebra morphism such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{h} & \xrightarrow{\mathbb{i}_2} & \mathfrak{g}\oplus\mathfrak{h}_{(l^2,r^2,\omega^2)} & \xrightarrow{\mathbb{p}_2} & \mathfrak{g} & \longrightarrow & 0\\
 & & \Big\| & & \theta\Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & \mathfrak{h} & \xrightarrow{\mathbb{i}_1} & \mathfrak{g}\oplus\mathfrak{h}_{(l^1,r^1,\omega^1)} & \xrightarrow{\mathbb{p}_1} & \mathfrak{g} & \longrightarrow & 0.
\end{array}$$

We omit details. This finishes the proof. ■

3 Another approach to non-abelian extensions of Leibniz algebras

The notion of strongly homotopy (sh) Leibniz algebras, or Lod_∞-algebras was introduced in [21], see also [4, 26] for more details. In [24], the authors introduced the notion of a Leibniz 2-algebra, which is the categorification of a Leibniz algebra, and prove that the category of Leibniz 2-algebras and the category of 2-term Lod_∞-algebras are equivalent.

Definition 3.1. *A Leibniz* 2*-algebra* $\mathcal{V}$ *consists of the following data:*

- *a complex of vector spaces* $\mathcal{V} : V_1 \xrightarrow{\mathrm{d}} V_0$,
- *bilinear maps* $l_2 : V_i \times V_j \longrightarrow V_{i+j}$, *where* $0 \leq i + j \leq 1$,
- *a trilinear map* $l_3 : V_0 \times V_0 \times V_0 \longrightarrow V_1$,

such that for any $w, x, y, z \in V_0$ *and* $m, n \in V_1$, *the following equalities are satisfied:*

(a) $\mathrm{d}l_2(x, m) = l_2(x, \mathrm{d}m)$,

(b) $\mathrm{d}l_2(m, x) = l_2(\mathrm{d}m, x)$,

(c) $l_2(\mathrm{d}m, n) = l_2(m, \mathrm{d}n)$,

(d) $\mathrm{d}l_3(x, y, z) = l_2(x, l_2(y, z)) - l_2(l_2(x, y), z) - l_2(y, l_2(x, z))$,

(e_1) $l_3(x, y, \mathrm{d}m) = l_2(x, l_2(y, m)) - l_2(l_2(x, y), m) - l_2(y, l_2(x, m))$,

(e_2) $l_3(x, \mathrm{d}m, y) = l_2(x, l_2(m, y)) - l_2(l_2(x, m), y) - l_2(m, l_2(x, y))$,

(e_3) $l_3(\mathrm{d}m, x, y) = l_2(m, l_2(x, y)) - l_2(l_2(m, x), y) - l_2(x, l_2(m, y))$,

(f) *the Jacobiator identity:*

$$\begin{aligned}
&l_2(w, l_3(x, y, z)) - l_2(x, l_3(w, y, z)) + l_2(y, l_3(w, x, z)) + l_2(l_3(w, x, y), z) \\
&-l_3(l_2(w, x), y, z) - l_3(x, l_2(w, y), z) - l_3(x, y, l_2(w, z)) \\
&+l_3(w, l_2(x, y), z) + l_3(w, y, l_2(x, z)) - l_3(w, x, l_2(y, z)) = 0.
\end{aligned}$$

We usually denote a Leibniz 2-algebra by $(V_1, V_0, \mathrm{d}, l_2, l_3)$, or simply by $\mathcal{V}$. A Leibniz 2-algebra is a **strict Leibniz 2-algebra** if $l_3 = 0$. If l_2 is skewsymmetric, we obtain **Lie 2-algebras**, see [5] for more details.

Lemma 3.2. *For a Leibniz* 2*-algebra* $(V_1, V_0, \mathrm{d}, l_2, l_3)$, *we have*

$$l_2(l_2(x, m), y) + l_2(l_2(m, x), y) = 0, \quad \forall\ x, y \in V_0, m \in V_1. \tag{24}$$

Definition 3.3. *Let* $\mathcal{V}$ *and* $\mathcal{V}'$ *be Leibniz* 2*-algebras, a morphism* $\mathfrak{f}$ *from* $\mathcal{V}$ *to* $\mathcal{V}'$ *consists of*

- *linear maps* $f_0 : V_0 \longrightarrow V_0'$ *and* $f_1 : V_1 \longrightarrow V_1'$ *commuting with the differential, i.e.*

$$f_0 \circ \mathrm{d} = \mathrm{d}' \circ f_1;$$

- *a bilinear map* $f_2 : V_0 \times V_0 \longrightarrow V_1'$,

such that for all $x, y, z \in L_0,\ m \in L_1$, *we have*

$$\left\{\begin{array}{rcl} l_2'(f_0(x), f_0(y)) - f_0 l_2(x, y) & = & \mathrm{d}' f_2(x, y), \\ l_2'(f_0(x), f_1(m)) - f_1 l_2(x, m) & = & f_2(x, \mathrm{d}m), \\ l_2'(f_1(m), f_0(x)) - f_1 l_2(m, x) & = & f_2(\mathrm{d}m, x), \end{array}\right. \tag{25}$$

and

$$\begin{array}{l} f_1(l_3(x, y, z)) + l_2'(f_0(x), f_2(y, z)) - l_2'(f_0(y), f_2(x, z)) - l_2'(f_2(x, y), f_0(z)) \\ -f_2(l_2(x, y), z) + f_2(x, l_2(y, z)) - f_2(y, l_2(x, z)) - l_3'(f_0(x), f_0(y), f_0(z)) = 0. \end{array} \tag{26}$$

Let $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ be a Leibniz algebra. Define $\Delta(\mathfrak{h}) \subset \mathrm{Z}(\mathfrak{h})$ by

$$\Delta(\mathfrak{h}) = \mathrm{span}\{[\alpha, \alpha]_{\mathfrak{g}} \mid \alpha \in \mathfrak{h}\}. \tag{27}$$

Define $\Pi(\mathfrak{h}) \subset \mathfrak{Der}(\mathfrak{h})$ by

$$\Pi(\mathfrak{h}) = \{(D^L, D^R) \in \mathfrak{Der}(\mathfrak{h}) \mid D^L\alpha + D^R\alpha \in \Delta(\mathfrak{h}),\ \forall \alpha \in \mathfrak{h}\}. \tag{28}$$

Lemma 3.4. *For any* $\beta \in \Delta(\mathfrak{h})$ *and* $D^R \in \mathrm{Der}^R(\mathfrak{h})$, *we have* $D^R\beta = 0$.

Proof. For all $\alpha \in \mathfrak{h}$ and $D^R \in \mathrm{Der}^R(\mathfrak{h})$, we have $D^R[\alpha, \alpha]_{\mathfrak{h}} = 0$, which implies that for all $\beta \in \Delta(\mathfrak{h})$, $D^R\beta = 0$. ■

Proposition 3.5. $\Pi(\mathfrak{h})$ *is a sub-Leibniz algebra of* $(\mathfrak{Der}(\mathfrak{h}), [\cdot, \cdot]_s)$.

Proof. By Lemma 3.4, for any $\beta \in \Delta(\mathfrak{h})$ and $(D^L, D^R) \in \Pi(\mathfrak{h})$, $D^L\beta \in \Delta(\mathfrak{h})$. Thus, for all $(D_1^L, D_1^R), (D_2^L, D_2^R) \in \Pi(\mathfrak{h})$, we have

$$\begin{array}{rcl} [(D_1^L, D_1^R), (D_2^L, D_2^R)]_s\alpha & = & [D_1^L, D_2^L]\alpha + [D_1^L, D_2^R]\alpha \\ & = & D_1^L(D_2^L\alpha + D_2^R\alpha) - (D_2^L D_1^L\alpha + D_2^R D_1^L\alpha) \in \Delta(\mathfrak{h}), \end{array}$$

which implies that $[(D_1^L, D_1^R), (D_2^L, D_2^R)]_s \in \Pi(\mathfrak{h})$. Thus, $\Pi(\mathfrak{h})$ is a sub-Leibniz algebra. ■

It is obvious that for all $\alpha \in \mathfrak{h}$, $(\mathrm{ad}_\alpha^L, \mathrm{ad}_\alpha^R) \in \Pi(\mathfrak{h})$.

Consider the complex $\mathfrak{h} \stackrel{(\mathrm{ad}^L, \mathrm{ad}^R)}{\longrightarrow} \Pi(\mathfrak{h})$, and define l_2 by

$$\left\{\begin{array}{rcll} l_2((D_1^L, D_1^R), (D_2^L, D_2^R)) & = & [(D_1^L, D_1^R), (D_2^L, D_2^R)]_s, & \text{for } (D_i^L, D_i^R) \in \Pi(\mathfrak{h}), \\ l_2((D^L, D^R), \alpha) & = & D^L\alpha, & \text{for } (D^L, D^R) \in \Pi(\mathfrak{h}),\ \alpha \in \mathfrak{h}, \\ l_2(\alpha, (D^L, D^R)) & = & D^R\alpha, & \text{for } (D^L, D^R) \in \Pi(\mathfrak{h}),\ \alpha \in \mathfrak{h}. \end{array}\right. \tag{29}$$

Proposition 3.6. *With the above notations,* $(\mathfrak{h}, \Pi(\mathfrak{h}), (\mathrm{ad}^L, \mathrm{ad}^R), l_2)$ *is a strict Leibniz* 2*-algebra, where* l_2 *is given by* (29).

Proof. By Lemma 2.2, we have

$$\begin{array}{rcl} (\mathrm{ad}^L, \mathrm{ad}^R) l_2((D^L, D^R), \alpha) & = & (\mathrm{ad}^L, \mathrm{ad}^R)(D^L\alpha) = \mathrm{ad}^L_{D^L\alpha} + \mathrm{ad}^R_{D^L\alpha} \\ & = & [D^L, \mathrm{ad}_\alpha^L] + [D^L, \mathrm{ad}_\alpha^R] \\ & = & [(D^L, D^R), (\mathrm{ad}^L, \mathrm{ad}^R)\alpha]_s = l_2((D^L, D^R), (\mathrm{ad}^L, \mathrm{ad}^R)\alpha), \end{array}$$

which implies that Condition (a) in Definition 3.1 holds.

Similarly, we have

$$\begin{aligned} \mathrm{d}l_2(\alpha, D) - l_2(\mathrm{d}\alpha, D) &= \mathrm{ad}^L_{D^R\alpha} + \mathrm{ad}^R_{D^R\alpha} - ([\mathrm{ad}^L_\alpha, D^L] + [\mathrm{ad}^L_\alpha, D^R]) \\ &= \mathrm{ad}^L_{D^L\alpha + D^R\alpha} = 0, \end{aligned}$$

which implies that Condition (b) in Definition 3.1 holds.

Condition (c) follows from

$$l_2((\mathrm{ad}^L_\alpha, \mathrm{ad}^R_\alpha), \beta) = l_2(\alpha, (\mathrm{ad}^L_\beta, \mathrm{ad}^R_\beta)) = [\alpha, \beta]_{\mathfrak{h}}.$$

By Proposition 3.5, Condition (d) holds naturally.

For all $(D_1^L, D_1^R), (D_2^L, D_2^R) \in \Pi(\mathfrak{h})$, we have

$$\begin{aligned} & l_2(\alpha, l_2((D_1^L, D_1^R), (D_2^L, D_2^R))) - l_2(l_2(\alpha, (D_1^L, D_1^R)), (D_2^L, D_2^R)) - l_2((D_1^L, D_1^R), l_2(\alpha, (D_2^L, D_2^R))) \\ &= [D_1^L, D_2^R]\alpha - D_2^R D_1^R \alpha - D_1^L D_2^R \alpha = -D_2^R(D_1^L\alpha + D_1^R\alpha) = 0, \end{aligned}$$

which implies that Condition (e_1) in Definition 3.1 holds.

Similarly, Conditions (e_2) and (e_3) also hold. Since $l_3 = 0$, Condition (f) holds naturally. The proof is finished. ■

It is well-known that when $\mathfrak{h}$ is a Lie algebra, there is a strict Lie 2-algebra $(\mathfrak{h}, \mathrm{Der}(\mathfrak{h}), \mathrm{ad}, l_2)$, where $\mathrm{Der}(\mathfrak{h})$ is the set of derivations of $\mathfrak{h}$, and l_2 is given by

$$\begin{cases} l_2(D_1, D_2) = [D_1, D_2], & \text{for } D_1, D_2 \in \mathrm{Der}(\mathfrak{h}), \\ l_2(D, \alpha) = -l_2(\alpha, D) = D\alpha, & \text{for } D \in \mathrm{Der}(\mathfrak{h}),\ \alpha \in \mathfrak{h}. \end{cases} \tag{30}$$

When $\mathfrak{h}$ is a Lie algebra, it is obvious that $\Delta(\mathfrak{h}) = 0$. Therefore, $\Pi(\mathfrak{h})$ reduces to

$$\Pi(\mathfrak{h}) = \{(D, -D) | D \in \mathrm{Der}(\mathfrak{h})\}.$$

Furthermore, the strict Leibniz 2-algebra given in Proposition 3.6 reduces to a Lie 2-algebra. Define $f_0 : \mathrm{Der}(\mathfrak{h}) \longrightarrow \Pi(\mathfrak{h})$ by

$$f_0 : D \longmapsto (D, -D), \quad \forall D \in \mathrm{Der}(\mathfrak{h}).$$

About the relation between the two Lie 2-algebras, we have

Proposition 3.7. *Let $\mathfrak{h}$ be a Lie algebra, then $(f_0, f_1 = \mathrm{id}, f_2 = 0)$ is an isomorphism from the Lie 2-algebra $(\mathfrak{h}, \mathrm{Der}(\mathfrak{h}), \mathrm{ad}, l_2)$ to $(\mathfrak{h}, \Pi(\mathfrak{h}), (\mathrm{ad}^L, \mathrm{ad}^R), l_2)$.*

Proof. It is obvious that f_0 and f_1 commute with the differential. Furthermore, we have

$$\begin{aligned} f_0 l_2(D_1, D_2) &= ([D_1, D_2], -[D_1, D_2]) = [(D_1, -D_1), (D_2, -D_2)]_s = l_2(f_0(D_1), f_0(D_2)), \\ f_1 l_2(D, \alpha) &= f_1(D\alpha) = D\alpha = l_2(f_0(D), f_1(\alpha)). \end{aligned}$$

Thus, $(f_0, f_1 = \mathrm{id}, f_2 = 0)$ is an isomorphism between Lie 2-algebras. ■

Remark 3.8. *From the above result, we see that the Leibniz 2-algebra associated to a Leibniz algebra given in Proposition 3.6 is a natural generalization of the Lie 2-algebra associated to a Lie algebra. However, we find that we can construct another Leibniz 2-algebra associated to a Leibniz algebra, which plays an important role when we study non-abelian extensions of Leibniz algebras.*

Define a subspace $\Xi(\mathfrak{h}) \subset \mathfrak{Der}(\mathfrak{h})$ by

$$\Xi(\mathfrak{h}) = \{(D^L, D^R) \in \mathfrak{Der}(\mathfrak{h}) \mid D^L\alpha + D^R\alpha \in \mathrm{Z}(\mathfrak{h}), \text{ for all } \alpha \in \mathfrak{h}, \text{ and } D^R\mathrm{Z}(\mathfrak{h}) = 0\}. \tag{31}$$

It is obvious that for all $\alpha \in \mathfrak{h}$, $(\mathrm{ad}^L_\alpha, \mathrm{ad}^R_\alpha) \in \Xi(\mathfrak{h})$.

Proposition 3.9. *$\Xi(\mathfrak{h})$ is a sub-Leibniz algebra of $(\mathfrak{Der}(\mathfrak{h}), [\cdot,\cdot]_s)$.*

Proof. For all (D_1^L, D_1^R), $(D_2^L, D_2^R) \in \Xi(\mathfrak{h})$ and $\alpha, \beta \in \mathfrak{h}$, by Lemma 2.1, we have

$$\begin{aligned} [[(D_1^L, D_1^R), (D_2^L, D_2^R)]_s\alpha, \beta]_\mathfrak{h} &= [[D_1^L, D_2^L]\alpha + [D_1^L, D_2^R]\alpha, \beta]_\mathfrak{h} \\ &= [D_1^L(D_2^L\alpha + D_2^R\alpha), \beta]_\mathfrak{h} - [D_2^L D_1^L\alpha + D_2^R D_1^L\alpha, \beta]_\mathfrak{h} \\ &= 0. \end{aligned}$$

Furthermore, by Lemma 2.1 and the fact that $D_2^R\mathrm{Z}(\mathfrak{h}) = 0$, for all $\alpha \in \mathrm{Z}(\mathfrak{h})$, we have

$$[D_1^L, D_2^R]\alpha = D_1^L D_2^R\alpha - D_2^R D_1^L\alpha = 0.$$

Thus, $[(D_1^L, D_1^R), (D_2^L, D_2^R)]_s\alpha \in \Xi(\mathfrak{h})$, which implies that $\Xi(\mathfrak{h})$ is a sub-Leibniz algebra. ■

Corollary 3.10. *$\Xi(\mathfrak{h})$ is a right ideal of $(\mathfrak{Der}(\mathfrak{h}), [\cdot,\cdot]_s)$.*

Similar as Proposition 3.6, we have

Proposition 3.11. *With the above notations, $(\mathfrak{h}, \Xi(\mathfrak{h}), (\mathrm{ad}^L, \mathrm{ad}^R), l_2)$ is a strict Leibniz 2-algebra, where l_2 is given by (29) for all $(D_i^L, D_i^R) \in \Xi(\mathfrak{h})$, $(D^L, D^R) \in \Xi(\mathfrak{h})$.*

Define $\mathrm{Pr}_L : \mathfrak{Der}(\mathfrak{h}) \longrightarrow \mathrm{Der}^L(\mathfrak{h})$ and $\mathrm{Pr}_R : \mathfrak{Der}(\mathfrak{h}) \longrightarrow \mathrm{Der}^R(\mathfrak{h})$ by

$$\mathrm{Pr}_L(D^L, D^R) = D^L, \ \mathrm{Pr}_R(D^L, D^R) = D^R.$$

Theorem 3.12. *Let $\hat{\mathfrak{g}} = (\mathfrak{g} \oplus \mathfrak{h}, [\cdot,\cdot]_{(l,r,\omega)})$ be a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$ with $\mathrm{Z}(\mathfrak{h}) = \mathrm{Z}(\hat{\mathfrak{g}}) \cap \mathfrak{h}$. Then, (l, r, ω) give rise to a Leibniz 2-algebra morphism (f_0, f_1, f_2) from $\mathfrak{g}$ to the Leibniz 2-algebra $(\mathfrak{h}, \Xi(\mathfrak{h}), (\mathrm{ad}^L, \mathrm{ad}^R), l_2)$, where f_0, f_1 and f_2 are given by*

$$f_0(x) = (l_x, r_x), \ f_1 = 0, \ f_2(x, y) = \omega(x, y).$$

Conversely, for any morphism (f_0, f_1, f_2) from $\mathfrak{g}$ to $(\mathfrak{h}, \Xi(\mathfrak{h}), (\mathrm{ad}^L, \mathrm{ad}^R), l_2)$, there is a non-abelian extension $\hat{\mathfrak{g}} = (\mathfrak{g} \oplus \mathfrak{h}, [\cdot,\cdot]_{(l,r,\omega)})$ of $\mathfrak{g}$ by $\mathfrak{h}$, with $\mathrm{Z}(\mathfrak{h}) = \mathrm{Z}(\hat{\mathfrak{g}}) \cap \mathfrak{h}$, where l, r, ω are given by

$$l_x = \mathrm{Pr}_L f_0(x), \ r_x = \mathrm{Pr}_R f_0(x), \ \omega(x, y) = f_2(x, y).$$

Proof. Let $\hat{\mathfrak{g}} = (\mathfrak{g} \oplus \mathfrak{h}, [\cdot,\cdot]_{(l,r,\omega)})$ be a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$ with $\mathrm{Z}(\mathfrak{h}) = \mathrm{Z}(\hat{\mathfrak{g}}) \cap \mathfrak{h}$. Then, l, r, ω satisfy Eqs. (11)-(17). By (11) and (12), we have $l_x \in \mathrm{Der}^L(\mathfrak{g})$ and $r_x \in \mathrm{Der}^R(\mathfrak{g})$. By (13) and (16), we have $l_x(\alpha) + r_x(\alpha) \in \mathrm{Z}(\hat{\mathfrak{g}}) \cap \mathfrak{h} \subset \mathrm{Z}(\mathfrak{h})$, which implies that $f_0(x)(\alpha) \in \mathrm{Z}(\mathfrak{h})$, for all $\alpha \in \mathfrak{h}$. Furthermore, it is obvious that $r_x(\mathrm{Z}(\hat{\mathfrak{g}}) \cap \mathfrak{h}) = 0$. By the condition $\mathrm{Z}(\mathfrak{h}) = \mathrm{Z}(\hat{\mathfrak{g}}) \cap \mathfrak{h}$, we get $r_x(\mathrm{Z}(\mathfrak{h})) = 0$. Thus, we deduce that $f_0(x) \in \Xi(\mathfrak{h})$. Furthermore, by (14) and (15), we get

$$\begin{aligned} l_2(f_0(x), f_0(y)) - f_0([x, y]_\mathfrak{g}) &= ([l_x, l_y] - l_{[x,y]_\mathfrak{g}}, [l_x, r_y] - r_{[x,y]_\mathfrak{g}}) = (\mathrm{ad}^L_{\omega(x,y)}, \mathrm{ad}^R_{\omega(x,y)}) \\ &= (\mathrm{ad}^L, \mathrm{ad}^R) f_2(x, y). \end{aligned}$$

At last, by (17), we deduce that (26) holds. Thus, (f_0, f_1, f_2) is a morphism from $\mathfrak{g}$ to the Leibniz 2-algebra $(\mathfrak{h}, \Xi(\mathfrak{h}), (\mathrm{ad}^L, \mathrm{ad}^R), l_2)$.

Conversely, by the fact that (f_0, f_1, f_2) is a morphism from $\mathfrak{g}$ to $(\mathfrak{h}, \Xi(\mathfrak{h}), (\mathrm{ad}^L, \mathrm{ad}^R), l_2)$, it is straightforward to deduce that Eqs. (11)-(17) hold. Furthermore, for all $\alpha \in \mathrm{Z}(\mathfrak{h})$, by the fact that $r_y \mathrm{Z}(\mathfrak{h}) = 0$, we have

$$[\alpha, y + \beta]_{(l,r,\omega)} = r_y \alpha + [\alpha, \beta]_{\mathfrak{h}} = 0,$$

i.e. $\alpha \in \mathrm{Z}(\hat{\mathfrak{g}}) \cap \mathfrak{h}$. Thus, we have $\mathrm{Z}(\mathfrak{h}) = \mathrm{Z}(\hat{\mathfrak{g}}) \cap \mathfrak{h}$. The proof is finished. ■

Remark 3.13. *We can also prove that isomorphic extensions correspond to equivalent Leibniz 2-algebra morphisms similar as Theorem 2.8. We omit details.*

By the results in [3], we can give some examples of non-abelian extensions of Leibniz algebras.

Example 3.14. Let $\mathfrak{h} = \{e_1, e_2, e_3\}$ be the Leibniz algebra with the non-zero brackets $[e_1, e_2] = e_3$, and $\mathfrak{g} = \{x\}$ the trivial Leibniz algebra. Let $\hat{\mathfrak{g}} = \{e_1, e_2, e_3, x\}$ be the Leibniz algebra with the non-zero brackets

$$[e_1, e_2] = e_3,\ [x, e_1] = e_1,\ [x, e_2] = \gamma e_2,\ [x, e_3] = (1 + \gamma)e_3,\ [e_1, x] = -e_1,\ \gamma \in \mathbb{R}.$$

Then one has

$$\mathrm{Z}(\hat{\mathfrak{g}}) = \mathrm{Z}(\mathfrak{h}) = \{e_2, e_3\}.$$

Example 3.15. Let $\mathfrak{h} = \{e_1, e_2, e_3\}$ be the Leibniz algebra with the non-zero brackets $[e_1, e_1] = e_3$, and $\mathfrak{g} = \{x\}$ the trivial Leibniz algebra. Let $\hat{\mathfrak{g}} = \{e_1, e_2, e_3, x\}$ be the Leibniz algebra with the non-zero brackets

$$[e_1, e_1] = e_3,\ [x, e_2] = e_2,\ [e_2, x] = -e_2.$$

Then we have

$$\mathrm{Z}(\mathfrak{h}) = \{e_2, e_3\},\ \mathrm{Z}(\hat{\mathfrak{g}}) = \{e_3\}.$$

Note that $\mathrm{Z}(\hat{\mathfrak{g}}) \subsetneqq \mathrm{Z}(\mathfrak{h})$.

Example 3.16. Let $\mathfrak{h} = \{e_1, e_2, e_3\}$ be the Leibniz algebra with the non-zero brackets $[e_1, e_1] = e_3$, and $\mathfrak{g} = \{x\}$ the trivial Leibniz algebra. Let $\hat{\mathfrak{g}}_1 = \{e_1, e_2, e_3, x\}$ be the Leibniz algebra with the non-zero brackets

$$[e_1, e_1] = e_3, [x, e_1] = e_1 + e_2,\ [x, e_2] = e_2,\ [x, e_3] = 2e_3,\ [e_1, x] = -e_1 - e_2,\ [e_2, x] = -e_2.$$

Let $\hat{\mathfrak{g}}_2 = \{e_1, e_2, e_3, x\}$ be the Leibniz algebra with the non-zero brackets

$$[e_1, e_1] = e_3,\ [x, e_2] = e_2.$$

Then one has $\mathrm{Z}(\mathfrak{h}) = \{e_2, e_3\}$. But $\mathrm{Z}(\hat{\mathfrak{g}}_2) = \{e_2, e_3\}$ and $\mathrm{Z}(\hat{\mathfrak{g}}_1) = \{e_3\}$. They are not equal. Therefore, this two extensions are not equivalent.

4 Non-abelian extensions of Leibniz algebras and Maurer-Cartan elements

In this section, we generalize the results in [13] to the case of Leibniz algebras. Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ be a Leibniz algebra. The graded vector space $C(\mathfrak{g},\mathfrak{g})=\oplus_k C^k(\mathfrak{g},\mathfrak{g})$ equipped with the graded commutator bracket

$$[P,Q]_C=P\circ Q+(-1)^{pq+1}Q\circ P,\ \forall P\in C^p(\mathfrak{g},\mathfrak{g}),\ Q\in C^q(\mathfrak{g},\mathfrak{g}) \tag{32}$$

is a graded Lie algebra, where $C^k(\mathfrak{g},\mathfrak{g})=\mathrm{Hom}(\otimes^{k+1}\mathfrak{g},\mathfrak{g})$ and $P\circ Q\in C^{p+q}(\mathfrak{g},\mathfrak{g})$ is defined by

$$\begin{aligned} P\circ Q(x_1,\cdots,x_{p+q+1}) \quad = \quad & \sum_{k=0}^{p}(-1)^{kq}\Big(\sum_{\sigma\in sh(k,q)}\mathrm{sgn}(\sigma)P(x_{\sigma(1)},\cdots,x_{\sigma(k)},\\ & Q(x_{\sigma(k+1)},\cdots,x_{\sigma(k+q)},x_{k+q+1}),x_{k+q+2},\cdots,x_{p+q+1})\Big). \end{aligned}$$

Furthermore, $(C(\mathfrak{g},\mathfrak{g}),[\cdot,\cdot]_C,\overline{\partial})$ is a DGLA, where $\overline{\partial}$ is given by $\overline{\partial}P=(-1)^p\partial P$ for all $P\in C^p(\mathfrak{g},\mathfrak{g})$, and ∂ is the coboundary operator of $\mathfrak{g}$ with the coefficients in the adjoint representation $(\mathrm{ad}^L,\mathrm{ad}^R)$. See [6, 14] for more details.

Remark 4.1. *The coboundary operator ∂ associated to the adjoint representation of the Leibniz algebra $\mathfrak{g}$ can be written as $\partial P=(-1)^p[\mu_{\mathfrak{g}},P]_C$, for all $P\in C^p(\mathfrak{g},\mathfrak{g})$, where $\mu_{\mathfrak{g}}\in C^1(\mathfrak{g},\mathfrak{g})$ is the Leibniz algebra structure on $\mathfrak{g}$, i.e. $\mu_{\mathfrak{g}}(x,y)=[x,y]_{\mathfrak{g}}$. Thus, we have $\overline{\partial}P=[\mu_{\mathfrak{g}},P]_C$.*

The set $MC(L)$ of **Maurer-Cartan elements** of a DGLA $(L,[\cdot,\cdot],\mathrm{d})$ is defined by

$$MC(L)\triangleq\{P\in L_1\mid \mathrm{d}P+\frac{1}{2}[P,P]=0\}.$$

Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}})$ and $(\mathfrak{h},[\cdot,\cdot]_{\mathfrak{h}})$ be two Leibniz algebras. Let $\mathfrak{g}\oplus\mathfrak{h}$ be the Leibniz algebra direct sum of $\mathfrak{g}$ and $\mathfrak{h}$, where the bracket is defined by $[x+\alpha,y+\beta]=[x,y]_{\mathfrak{g}}+[\alpha,\beta]_{\mathfrak{h}}$. Then there is DGLA $(C(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_C,\overline{\partial})$, where ∂ is the coboundary operator for the Leibniz algebra $\mathfrak{g}\oplus\mathfrak{h}$ with the coefficients in the adjoint representation. It is straightforward to see that $(C(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h}),[\cdot,\cdot]_C,\overline{\partial})$ is a sub-DGLA. Define $C^k_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})\subset C^k(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$ by

$$C^k(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})=C^k_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})\oplus C^k(\mathfrak{h},\mathfrak{h}).$$

Denote by $C_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})=\oplus_k C^k_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$, which is a graded vector space.

Lemma 4.2. *With the above notations, we have $\partial(C^k_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h}))\subset C^{k+1}_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$, and $(C_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h}),[\cdot,\cdot]_C,\overline{\partial})$ is a sub-DGLA of $(C(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_C,\overline{\partial})$. Furthermore, its degree 0 part $C^0_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})=\mathrm{Hom}(\mathfrak{g},\mathfrak{h})$ is abelian.*

We denote by $(L=\oplus_k L_k,[\cdot,\cdot]_C,\overline{\partial})$ this differential graded Lie algebra, where $L_k=C^k_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$.

Proof. By the fact that $\overline{\partial}=[\mu_{\mathfrak{g}}+\mu_{\mathfrak{h}},\cdot]_C$, we can verify that $\overline{\partial}(C^k_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h}))\subset C^{k+1}_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$. It is straightforward to see that $C_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$ is closed under the bracket operation $[\cdot,\cdot]_C$. Obviously, $C^0_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})=\mathrm{Hom}(\mathfrak{g},\mathfrak{h})$ is abelian. ■

We have $L_1=\mathrm{Hom}(\otimes^2\mathfrak{g},\mathfrak{h})\oplus\mathrm{Hom}(\mathfrak{g}\otimes\mathfrak{h},\mathfrak{h})\oplus\mathrm{Hom}(\mathfrak{h}\otimes\mathfrak{g},\mathfrak{h})$. For all $\varphi_1,\varphi_2\in L_0,\ P\in L_1$, we have

$$[[P,\varphi_1]_C,\varphi_2]_C=0. \tag{33}$$

Definition 4.3. *Let $(L, [\cdot,\cdot]_C, \overline{\partial})$ be the DGLA given in Lemma 4.2. Two elements $c, c' \in L_1$ are said to be equivalent if there is an element $\varphi \in L_0$ such that*

$$c' = c + [c,\varphi]_C + \overline{\partial}\varphi + \frac{1}{2}[\overline{\partial}\varphi, \varphi]_C. \tag{34}$$

Remark 4.4. *By* (33) *and the fact that L_0 is abelian, we can deduce that* (34) *is indeed an equivalence relation. See [13] for more details about this equivalence relation for Lie algebras.*

Now consider $(\mathfrak{g}\oplus\mathfrak{h}, [\cdot,\cdot]_{(l,r,\omega)})$, where $[\cdot,\cdot]_{(l,r,\omega)}$ is given by (10) for any $l, r : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{h})$ and $\omega \in \mathrm{Hom}(\otimes^2\mathfrak{g}, \mathfrak{h})$. View $l, r : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{h})$ as elements in $\mathrm{Hom}(\mathfrak{g}\otimes\mathfrak{h}, \mathfrak{h})$ and $\mathrm{Hom}(\mathfrak{h}\otimes\mathfrak{g}, \mathfrak{h})$ respectively via

$$l(x,\alpha) = l_x(\alpha), \quad r(\alpha, x) = r_x(\alpha). \tag{35}$$

Proposition 4.5. *The following two statements are equivalent:*

(a) $(\mathfrak{g}\oplus\mathfrak{h}, [\cdot,\cdot]_{(l,r,\omega)})$ *is a Leibniz algebra, which is a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$;*

(b) $l + r + \omega$ *is a Maurer-Cartan element, i.e. $\omega + l + r \in MC(L)$.*

Proof. By Proposition 2.6, $(\mathfrak{g}\oplus\mathfrak{h}, [\cdot,\cdot]_{(l,r,\omega)})$ is a Leibniz algebra if and only if Eqs. (11)-(17) hold.

If $c = \omega + l + r$ is a Maurer-Cartan element, we have

$$(\overline{\partial}c + \frac{1}{2}[c,c]_C)(e_1, e_2, e_3) = 0, \quad \forall e_i = x_i + \alpha_i \in \mathfrak{g}\oplus\mathfrak{h}.$$

By straightforward computations, we have

$$\begin{aligned} \partial c(e_1,e_2,e_3) &= [e_1, c(e_2,e_3)] - [e_2, c(e_1,e_3)] - [c(e_1,e_2), e_3] \\ &\quad - c([e_1,e_2],e_3) - c(e_2,[e_1,e_3]) + c(e_1,[e_2,e_3]), \end{aligned}$$

and

$$\begin{aligned} [e_1, c(e_2,e_3)] &= [\alpha_1, \omega(x_2,x_3)]_\mathfrak{h} + [\alpha_1, l_{x_2}\alpha_3]_\mathfrak{h} + [\alpha_1, r_{x_3}\alpha_2]_\mathfrak{h}, \\ [e_2, c(e_1,e_3)] &= [\alpha_2, \omega(x_1,x_3)]_\mathfrak{h} + [\alpha_2, l_{x_1}\alpha_3]_\mathfrak{h} + [\alpha_2, r_{x_3}\alpha_1]_\mathfrak{h}, \\ [c(e_1,e_2), e_3] &= [\omega(x_1,x_2), \alpha_3]_\mathfrak{h} + [l_{x_1}\alpha_2, \alpha_3]_\mathfrak{h} + [r_{x_2}\alpha_1, \alpha_3]_\mathfrak{h}, \\ c([e_1,e_2],e_3) &= \omega([x_1,x_2]_\mathfrak{g}, x_3) + l_{[x_1,x_2]_\mathfrak{g}}\alpha_3 + r_{x_3}[\alpha_1,\alpha_2]_\mathfrak{h}, \\ c(e_2,[e_1,e_3]) &= \omega(x_2,[x_1,x_3]_\mathfrak{g}) + l_{x_2}[\alpha_1,\alpha_3]_\mathfrak{h} + r_{[x_1,x_3]_\mathfrak{g}}\alpha_2, \\ c(e_1,[e_2,e_3]) &= \omega(x_1,[x_2,x_3]_\mathfrak{g}) + l_{x_1}[\alpha_2,\alpha_3]_\mathfrak{h} + r_{[x_2,x_3]_\mathfrak{g}}\alpha_1. \end{aligned}$$

Furthermore, by (32), we have

$$\begin{aligned} \frac{1}{2}[c,c]_C = (c\circ c)(e_1,e_2,e_3) &= c(c(e_1,e_2),e_3) - c(e_1, c(e_2,e_3)) + c(e_2, c(e_1,e_3)) \\ &= r_{x_3}\omega(x_1,x_2) + r_{x_3}l_{x_1}\alpha_2 + r_{x_3}r_{x_2}\alpha_1 - l_{x_1}\omega(x_2,x_3) \\ &\quad - l_{x_1}l_{x_2}\alpha_3 - l_{x_1}r_{x_3}\alpha_2 + l_{x_2}\omega(x_1,x_3) + l_{x_2}l_{x_1}\alpha_3 + l_{x_2}r_{x_3}\alpha_1. \end{aligned}$$

Thus, $c = \omega + l + r$ is a Maurer-Cartan element if and only if (11)-(17) hold. The proof is finished. ■

Corollary 4.6. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Leibniz algebras, then there is a one-to-one correspondence between $Z^2(\mathfrak{g},\mathfrak{h})$ and Maurer-Cartan elements in the DGLA $(L, [\cdot,\cdot]_C, \overline{\partial})$.*

We can further prove that

Theorem 4.7. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Leibniz algebras, the equivalence in $Z^2(\mathfrak{g},\mathfrak{h})$ coincides with the equivalence in $MC(L)$.*

Proof. Two elements $c=\omega+r+l$ and $c'=\omega'+r'+l'$ in $MC(L)$ are equivalent if there exists $\varphi\in\mathrm{Hom}(\mathfrak{g},\mathfrak{h})$ such that

$$c'=c+[c,\varphi]_C+\partial\varphi+[\partial\varphi,\varphi]_C.$$

For all $e_i=x_i+\alpha_i\in\mathfrak{g}\oplus\mathfrak{h}$, we have

$$\begin{aligned}
[c,\varphi]_C(e_1,e_2) &= [\omega+r+l,\varphi]_C(e_1,e_2)=l_{x_1}\varphi(x_2)+r_{x_2}\varphi(x_1),\\
\partial\varphi(e_1,e_2) &= [\alpha_1,\varphi(x_2)]_{\mathfrak{h}}+[\varphi(x_1),\alpha_2]_{\mathfrak{h}}-\varphi([x_1,x_2]_{\mathfrak{g}}),\\
[\partial\varphi,\varphi]_C(e_1,e_2) &= 2[\varphi(x_1),\varphi(x_2)]_{\mathfrak{h}}.
\end{aligned}$$

Therefore, we have

$$c'(e_1,e_2)=c(e_1,e_2)+l_{x_1}\varphi(x_2)+r_{x_2}\varphi(x_1)+\mathrm{ad}^R_{\varphi(x_2)}\alpha_1+\mathrm{ad}^L_{\varphi(x_1)}\alpha_2-\varphi([x_1,x_2]_{\mathfrak{g}})+[\varphi(x_1),\varphi(x_2)]_{\mathfrak{h}},$$

which implies that

$$\begin{aligned}
l'(x_1,\alpha_2)-l(x_1,\alpha_2) &= \mathrm{ad}^L_{\varphi(x_1)}\alpha_2,\\
r'(\alpha_1,x_2)-l(\alpha_1,x_2) &= \mathrm{ad}^R_{\varphi(x_2)}\alpha_1,\\
\omega'(x_1,x_2)-\omega(x_1,x_2) &= l_{x_1}\varphi(x_2)+r_{x_2}\varphi(x_1)-\varphi([x_1,x_2]_{\mathfrak{g}})+[\varphi(x_1),\varphi(x_2)]_{\mathfrak{h}}.
\end{aligned}$$

By (35), they are equivalent to Eqs. (18)-(20), which are the equivalence relation in $Z^2(\mathfrak{g},\mathfrak{h})$. This finishes the proof. ■

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