

UNIVERSAL CENTRAL EXTENSIONS OF SUPERDIALGEBRAS OF MATRICES

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ABSTRACT. We complete the problem of finding the universal central extension in the category of Leibniz superalgebras of $\mathfrak{sl}(m, n, D)$ when $m + n \geq 3$ and D is a superdialgebra, solving in particular the problem when D is an associative algebra, superalgebra or dialgebra. To accomplish this task we use a different method than the standard studied in the literature. We introduce and use the non-abelian tensor square of Leibniz superalgebras and its relations with the universal central extension.

1. INTRODUCTION

Leibniz algebras, the non-antisymmetric analogue of Lie algebras, were first defined by Bloh [1] and later recovered by Loday in [22] when he handled periodicity phenomena in algebraic K -theory. Many authors have studied this structure and it has some interesting applications in Geometry and Physics ([16], [25], [6]). On the other hand, the theory of superalgebras arises directly from supersymmetry, a part of the theory of elementary particles, in order to have a better understanding of the geometrical structure of spacetime and to complete the substantial meaningful task of the unification of quantum theory and general relativity ([32]). The study of Lie or Leibniz superalgebras has been a very active field in the recent years since the classification of simple complex finite-dimensional Lie superalgebras by Kac in [14].

The study of central extensions is a very important topic in mathematics. There is a direct connection between central extensions and (co)homology, and they also have relations with Physics ([30]). In particular, universal central extensions have been studied in many different structures as groups [27], Lie algebras [11], [31] or Lie superalgebras [28]. A very interesting tool in the study of universal central extensions is the non-abelian tensor product introduced in [2] and extended to Lie algebras in [5] and to Lie superalgebras in [9].

The theory related with the universal central extension of the special linear algebra $\mathfrak{sl}(n, A)$ has been very active due its relation with cyclic homology and its relevance in algebraic K -theory. The first approach was in the category of Lie algebras by Kassel and Loday in [15] where they described it when $n \geq 5$ and A is an associative algebra, and in [8] it was obtained for $n \geq 3$. For the Lie superalgebra

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$\mathfrak{sl}(n, A)$ and A an associative superalgebra was given in [3]. For the special linear superalgebra $\mathfrak{sl}(m, n, A)$ it was worked out for A an associative algebra in [26] and [29]; and for A an associative superalgebra in [4] and [10]. In the category of Leibniz algebras, the universal central extension of $\mathfrak{sl}(n, A)$ (seen as a Leibniz algebra), where A is an associative algebra, was found in [24] when $n \geq 5$ and in [13] when $n \geq 3$. For the Leibniz superalgebra $\mathfrak{sl}(m, n, A)$, when $m + n \geq 5$ and A is an associative algebra it was calculated in [21]. In [18] it was found for the Leibniz algebra $\mathfrak{sl}(m, D)$ and for the Leibniz superalgebra $\mathfrak{sl}(m, n, D)$, where $m \geq 5$ and $m + n \geq 5$, respectively, and D is an associative dialgebra.

The aim of this paper is to complete the task, finding the universal central extension of $\mathfrak{sl}(m, n, D)$ where D is a superdialgebra and $m + n \geq 3$. Since associative algebras, associative superalgebras and dialgebras are all examples of associative superdialgebras, we will solve all cases at once. Moreover, we obtain a result contradicting a specific point of a theorem given in [18]. The most interesting part of this paper is that the method used is not the same as in all the papers cited above. Due its relation with central extensions, we introduce and use the non abelian tensor square of Leibniz superalgebras providing another point of view to this topic.

2. PRELIMINARIES

In what follows we fix a unital commutative ring R .

2.1. Dialgebras. We recall from [23] the definitions and basic examples of (super)dialgebras.

Definition 2.1. An *associative dialgebra* (*dialgebra* for short) is an R -module equipped with two R -linear maps

$$\begin{aligned} \vdash: D \otimes_R D &\rightarrow D, \\ \dashv: D \otimes_R D &\rightarrow D, \end{aligned}$$

where $\vdash$ and $\dashv$ are associative and satisfy the following conditions:

$$\begin{cases} a \dashv (b \dashv c) = a \dashv (b \vdash c), \\ (a \vdash b) \dashv c = a \vdash (b \dashv c), \\ (a \dashv b) \vdash c = (a \vdash b) \vdash c, \end{cases}$$

for all $a, b, c \in D$.

A *bar-unit* in D is an element $e \in D$ such that for all $x \in D$,

$$a \dashv e = a = e \vdash a.$$

Note that a bar-unit may not be unique. A *unital dialgebra* is a dialgebra with a chosen bar-unit, that will be denoted by 1. An *ideal* $I \subset D$ is an R -submodule such that if x or y belong to I then $x \dashv y \in D$ and $x \vdash y \in D$.

An *associative superdialgebra* (*superdialgebra* for short) is a dialgebra equipped with a $\mathbb{Z}_2$ -graded structure compatible with the two operations, i.e. $D_{\bar{\alpha}} \vdash D_{\bar{\beta}} \subseteq D_{\bar{\alpha}+\bar{\beta}}$ and $D_{\bar{\alpha}} \dashv D_{\bar{\beta}} \subseteq D_{\bar{\alpha}+\bar{\beta}}$, for $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}_2$. The concepts of bar-unit, unital and ideal are analogous in superdialgebras. Note that the bar-unit is always even.

Example 2.2. An associative (super)algebra defines a (super)dialgebra structure in a canonical way, where $a \dashv b = ab = a \vdash b$. If it is unital, then the superdialgebra is unital.

Example 2.3. Let (A, d) a differential associative (super)algebra, i.e., $d(ab) = d(a)b + ad(b)$ and $d^2 = 0$. We define the two operations by

$$\begin{aligned} x \dashv y &= xd(y) \\ x \vdash y &= d(x)y. \end{aligned}$$

It is immediate to check that with these operations (A, d) is a (super)dialgebra.

Example 2.4. Let A an associative (super)algebra, M an A -(super)bimodule and $f: M \rightarrow A$ an A -(super)bimodule map. Then we can define a (super)dialgebra structure with operations

$$\begin{aligned} m \dashv m' &= mf(m'), \\ m \vdash m' &= f(m)m'. \end{aligned}$$

Example 2.5. Let D and D' be two superdialgebras. Then the tensor product $D \otimes_R D'$ is a superdialgebra where

$$\begin{aligned} (a \otimes a') \dashv (b \otimes b') &= (-1)^{|a'| |b|} (a \dashv b) \otimes (a' \dashv b'), \\ (a \otimes a') \vdash (b \otimes b') &= (-1)^{|a'| |b|} (a \vdash b) \otimes (a' \vdash b'). \end{aligned}$$

Example 2.6. A particular case of the previous example is $\mathcal{M}(n, D) = \mathcal{M}(n, R) \otimes_R D$, the R -supermodule of $(n \times n)$ -matrices. The operations are given by

$$(a \dashv b)_{ij} = \sum_k a_{ik} \dashv b_{kj} \quad \text{and} \quad (a \vdash b)_{ij} = \sum_k a_{ik} \vdash b_{kj}.$$

2.2. Leibniz superalgebras.

Definition 2.7. A *Leibniz superalgebra* L is an R -supermodule with an R -linear even map

$$[-, -]: L \otimes_R L \rightarrow L,$$

satisfying the *Leibniz identity*

$$[x, [y, z]] = [[x, y], z] - (-1)^{|y| |z|} [[x, z], y],$$

for all $x, y, z \in L$.

Note that a Leibniz superalgebra where the identity $[x, y] = -(-1)^{|x| |y|} [y, x]$ also holds, is a Lie superalgebra.

Example 2.8. A Lie superalgebra is in particular a Leibniz superalgebra.

Example 2.9. Let D be a superdialgebra. Then D with the bracket

$$[a, b] = a \dashv b - (-1)^{|a| |b|} b \vdash a,$$

is a Leibniz superalgebra. If the two operations $\dashv$ and $\vdash$ are equal, i.e., D is also an associative superalgebra, this bracket also induces a Lie superalgebra structure.

Definition 2.10. The *centre* of a Leibniz superalgebra L , denoted by $Z(L)$, is the ideal formed by the elements $z \in L$ such that $[z, x] = [x, z] = 0$ for all $x \in L$. The *commutator* of L , denoted by $[L, L]$, is the ideal generated by the elements $[x, y]$ where $x, y \in L$. A Leibniz superalgebra is called *perfect* if $L = [L, L]$.

Definition 2.11. A *central extension* of a Leibniz superalgebra L is a surjective homomorphism $\phi: M \rightarrow L$ such that $\text{Ker } \phi \subseteq Z(M)$. We say that a central extension $\mathbf{u}: U \rightarrow L$ is *universal* if for any central extension $\phi: M \rightarrow L$ there is a unique homomorphism $f: U \rightarrow M$ such that $\mathbf{u} = \phi \circ f$.

The theory of central extensions of Leibniz superalgebras is studied in [20]. We obtain the following straightforward results.

Proposition 2.12. *Let $\phi: E \rightarrow M$ and $\psi: M \rightarrow L$ be two central extensions of Leibniz superalgebras. Then ϕ is universal if and only if $\psi \circ \phi$ is universal.*

Proposition 2.13. *Let M be a Leibniz superalgebra and L an R -supermodule. An R -supermodule homomorphism $\varphi: M \rightarrow L$ such that $\text{Ker } \varphi \subseteq Z(M)$ defines a Leibniz superalgebra structure in L where the bracket is*

$$[x, y] = \varphi([\varphi^{-1}(x), \varphi^{-1}(y)]).$$

for $x, y \in L$.

Now we introduce the homology of Leibniz superalgebras with trivial coefficients adapting it from the non-graded version [24].

Definition 2.14. Let L be a Leibniz superalgebra and $\delta_n: L^{\otimes n} \rightarrow L^{\otimes n-1}$ the R -linear map given by

$$\delta_n(x_1 \otimes \cdots \otimes x_n) = \sum_{i < j} (-1)^{n-j+|x_j|(|x_{i+1}|+\cdots+|x_{j-1}|)} x_1 \otimes \cdots \otimes x_{i-1} \otimes [x_i, x_j] \otimes x_{i+1} \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_n.$$

We define the *homology of Leibniz superalgebras* with trivial coefficients as the homology of the chain complex formed by δ_n , i.e.

$$\text{HL}_n(L) = \frac{\text{Ker } \delta_n}{\text{Im } \delta_{n+1}}$$

Note that $\delta_3(x \otimes y \otimes z) = -[x, y] \otimes z + x \otimes [y, z] + (-1)^{|y||z|} [x, z] \otimes y$.

In [12] it is defined a non-abelian tensor product of Leibniz algebras and in [17] is introduced a variation. They both coincide in the case of perfect Leibniz algebras (i.e. $[L, L] = L$) so for simplicity, we will generalize to Leibniz superalgebras the version of [17].

Definition 2.15. Let L be a perfect Leibniz superalgebra. The *non-abelian tensor product* of L is

$$L \otimes L = \frac{L \otimes_R L}{\text{Im } \delta_3},$$

where δ_3 is the map defined on the chain complex of Leibniz homology and the bracket is $[x \otimes y, x' \otimes y'] = [x, y] \otimes [x', y']$. Therefore, we have a short exact sequence.

$$0 \longrightarrow \text{HL}_2(L) \longrightarrow L \otimes L \xrightarrow{\delta_2} L \longrightarrow 0.$$

Theorem 2.16. *Let L be a perfect Leibniz superalgebra. Then $\delta_2: L \otimes L \rightarrow L$ is the universal central extension of L and its kernel is $\text{HL}_2(L)$.*

Proof. Let $\sum_i x_i \otimes y_i$ be in the kernel of δ_2 . Then $\sum_i [x_i, y_i] = 0$, so $[\sum_i x_i \otimes y_i, x' \otimes y'] = \sum_i [x_i, y_i] \otimes [x', y'] = 0$. Therefore, δ_2 is a central extension. Let

$0 \longrightarrow K \xrightarrow{\iota} M \xrightarrow{\phi} L \longrightarrow 0$ be a central extension. We define a homomorphism $u: L \otimes L \rightarrow M$, $x \otimes y \mapsto [\bar{x}, \bar{y}]$, where $\bar{x}$ and $\bar{y}$ are preimages by ϕ of x and y respectively. This homomorphism is well defined since $\text{Ker } \phi \subseteq Z(M)$. If u, u' are two homomorphisms such that $\phi \circ u = \phi \circ u'$, then $u - u' = \iota \circ \eta$ where $\eta: L \otimes L \rightarrow K$ and $\eta([L, L]) = 0$. Since L is perfect, $L \otimes L$ is also perfect and u is unique. $\square$

2.3. Matrix Leibniz superalgebras. Let $\{1, \dots, m\} \cup \{m+1, \dots, m+n\}$ be a graded set and $D = D_{\bar{0}} \oplus D_{\bar{1}}$ unital superdialgebra. We consider the set $\mathcal{M}(m, n, D)$ of $(m+n) \times (m+n)$ -matrices. Let $E_{ij}(a)$ be the matrix with $a \in D$ in the position (i, j) and zeros elsewhere. We define a grading in $\mathcal{M}(m, n, D)$ where the homogeneous elements are $E_{ij}(a)$ with a homogeneous and the grading is given by $E_{ij}(a) = |i| + |j| + |a|$. Now we define the *general Leibniz superalgebra* $\mathfrak{gl}(m, n, D)$ which has $\mathcal{M}(m, n, D)$ with the previous grading as underlying set and the Leibniz bracket is given by $[x, y] = x \dashv y - (-1)^{|x||y|} y \vdash x$. If $m+n \geq 2$, we define the *special linear Leibniz superalgebra* $\mathfrak{sl}(m, n, D) = [\mathfrak{gl}(m, n, D), \mathfrak{gl}(m, n, D)]$. It is easy to see that $\mathfrak{sl}(m, n, D)$ is generated by $E_{ij}(a)$ with $a \in D_{\bar{0}} \cup D_{\bar{1}}$ and $1 \leq i \neq j \leq m+n$ and the bracket is given by

$$[E_{ij}(a), E_{kl}(b)] = \delta_{jk} E_{il}(a \dashv b) - (-1)^{|E_{ij}(a)||E_{kl}(b)|} \delta_{il} E_{kj}(b \vdash a).$$

Following [18], if $m+n \geq 3$ then $\mathfrak{sl}(m, n, D)$ is perfect. We define the *supertrace* as the R -bilinear homomorphism $\text{Str}_1 : \mathfrak{gl}(m, n, D) \rightarrow D$ with

$$\text{Str}_1(x) = \sum_{i=1}^{m+n} (-1)^{|i|(|i|+|x_{ii}|)} x_{ii}.$$

Note that $\mathfrak{sl}(m, n, D) = \{x \in \mathfrak{gl}(m, n, D) : \text{Str}_1(x) \in [D, D]\}$.

Definition 2.17. Let D be a superdialgebra and m and n non-negative integers such that $m+n \geq 3$. We define the *Steinberg Leibniz superalgebra* denoted by $\mathfrak{stl}(m, n, D)$ as the Leibniz superalgebra generated by the elements $F_{ij}(a)$ with $a \in D_{\bar{0}} \cup D_{\bar{1}}$, $1 \leq i \neq j \leq m+n$, where the grading is given by $|F_{ij}(a)| = |i| + |j| + |a|$, subject to the relations

$$\begin{aligned} a &\mapsto F_{ij}(a) \text{ is } R\text{-linear,} \\ [F_{ij}(a), F_{kl}(b)] &= F_{ik}(a \dashv b), & \text{if } i \neq l \text{ and } j = k, \\ [F_{ij}(a), F_{kl}(b)] &= -(-1)^{|F_{ij}(a)||F_{kl}(b)|} F_{kj}(b \vdash a), & \text{if } i = l \text{ and } j \neq k, \\ [F_{ij}(a), F_{kl}(b)] &= 0, & \text{if } i \neq l \text{ and } j \neq k. \end{aligned}$$

We recall from [19] that $\mathfrak{stl}(m, n, D)$ is perfect and the canonical Leibniz superalgebra homomorphism $\phi : \mathfrak{stl}(m, n, D) \rightarrow \mathfrak{sl}(m, n, D)$, $F_{ij}(a) \mapsto E_{ij}(a)$ is a central extension.

3. UNIVERSAL CENTRAL EXTENSION OF $\mathfrak{sl}(m, n, D)$

In this section we are going to show that $\mathfrak{stl}(m, n, D)$ is the universal central extension of $\mathfrak{sl}(m, n, D)$ when $m+n \geq 5$. We are going to use a slightly different method than usual found in the literature. The strategy is to prove that the non-abelian tensor product $\mathfrak{stl}(m, n, D) \otimes \mathfrak{stl}(m, n, D)$ is isomorphic to $\mathfrak{stl}(m, n, D)$ itself. Then Proposition 2.12 implies that $\mathfrak{stl}(m, n, D)$ is the universal central extension of $\mathfrak{sl}(m, n, D)$.

Theorem 3.1. *There is an isomorphism $\mathfrak{stl}(m, n, D) \otimes \mathfrak{stl}(m, n, D) \cong \mathfrak{stl}(m, n, D)$ for $m+n \geq 5$.*

Proof. Let be the homomorphisms defined on generators:

$$\begin{aligned} \varphi : \mathfrak{stl}(m, n, D) \otimes \mathfrak{stl}(m, n, D) &\rightarrow \mathfrak{stl}(m, n, D), & F_{ij}(a) \otimes F_{kl}(b) &\mapsto [F_{ij}(a), F_{kl}(b)], \\ \psi : \mathfrak{stl}(m, n, D) &\rightarrow \mathfrak{stl}(m, n, D) \otimes \mathfrak{stl}(m, n, D), & F_{ij}(a) &\mapsto F_{ik}(a) \otimes F_{kj}(1). \end{aligned}$$

It is straightforward that φ is a well defined Leibniz superalgebra homomorphism. For different i, j, k we have

$$F_{ik}(a) \otimes F_{kj}(1) = [F_{is}(a), F_{sk}(1)] \otimes F_{kj}(1) = F_{is}(a) \otimes F_{sj}(1),$$

so ψ does not depend of the choice of k . To check if ϕ preserves the relations it is enough to see that:

(a) If $i \neq l$ and $j = k$,

$$F_{ij}(a) \otimes F_{kl}(b) = F_{ij}(a) \otimes [F_{ks}(b), F_{sl}(1)] = F_{is}(a \dashv b) \otimes F_{sl}(1).$$

(b) If $i = l$ and $j \neq k$,

$$\begin{aligned} F_{ij}(a) \otimes F_{kl}(b) &= [F_{is}(a), F_{sj}(1)] \otimes F_{ki}(1) \\ &= -(-1)^{(|i|+|j|+|a|)(|k|+|s|+|b|)} F_{ks}(b \vdash a) \otimes F_{sj}(1). \end{aligned}$$

(c) If $i \neq l$ and $j \neq k$,

$$F_{ij}(a) \otimes F_{kl}(b) = [F_{is}(a), F_{sj}(1)] \otimes F_{kl}(b) = 0.$$

Moreover, these relations show that $\psi \circ \phi$ is the identity map and it is obvious that $\phi \circ \psi$ is the identity map too. $\square$

4. UNIVERSAL CENTRAL EXTENSION OF $\mathfrak{sl}(m, n, D)$ WHEN $m + n < 5$

In this section we will find the universal central extension of $\mathfrak{sl}(m, n, D)$ when $3 \leq m + n < 5$. We need some preliminary results first. Recall that $[D, D]$ is the subalgebra generated by the elements $a \dashv b - (-1)^{|a||b|} b \vdash a$. It happens that in superdialgebras, this is not necessarily an ideal.

Lemma 4.1. *Let D be a unital superdialgebra. We have that $D \dashv [D, D] \subseteq [D, D] \dashv D$, $[D, D] \vdash D \subseteq D \vdash [D, D]$ and $[D, D] \dashv D = D \vdash [D, D]$. Then the ideal generated by the elements $a \dashv b - (-1)^{|a||b|} b \vdash a$ is just $[D, D] \dashv D$.*

Proof. The results follow, respectively, from the identities

$$\begin{aligned} a \dashv [b, c] &= [a, b] \dashv c - (-1)^{|b||c|} [a \dashv c, b] \dashv 1, \\ [a, b] \vdash c &= -(-1)^{|b||c|} a \vdash [c, b] + (-1)^{|b||c|} 1 \vdash [a \vdash c, b], \\ [a, b] \dashv c &= -(-1)^{|a||b|} b \vdash [a, c] + [a, b \vdash c]. \end{aligned}$$

$\square$

Definition 4.2. Let D be a superdialgebra and m a positive integer. Let $\mathcal{I}_m$ be the ideal of D generated by the elements ma and $a \dashv b - (-1)^{|a||b|} b \vdash a$. We denote the quotient

$$D_m = \frac{D}{\mathcal{I}_m}.$$

We claim that $\mathfrak{stl}(m, n, D) \otimes \mathfrak{stl}(m, n, D) \cong \mathfrak{stl}(m, n, D) \oplus \mathcal{W}(m, n, D)$ where $\mathcal{W}(m, n, D)$ is an R -supermodule which depends on m and n and the Leibniz superalgebra structure is given by an R -supermodule homomorphism $\varphi: \mathfrak{stl}(m, n, D) \otimes \mathfrak{stl}(m, n, D) \rightarrow \mathfrak{stl}(m, n, D) \oplus \mathcal{W}(m, n, D)$ in the conditions of Proposition 2.13. Then we will define an inverse.

4.1. Case of $\mathfrak{sl}(4, 0, D)$. Let $\mathcal{W}(4, 0, D)$ be the direct sum of six copies of D_2 . The elements will be represented by $v_{ijkl}(a)$ where $1 \leq i, j, k, l \leq 4$ are distinct, $a \in D$ and $|v_{ijkl}(a)| = |a|$. They will be related by R -linearity, the equivalence relations of D_2 and by $v_{ijkl}(a) = -v_{ilkj}(a) = -v_{kjil}(a) = v_{klij}(a)$.

Theorem 4.3. *The universal central extension of $\mathfrak{sl}(4, 0, D)$ is $\mathfrak{stl}(4, 0, D) \oplus D_2^6$.*

Proof. Let $\varphi: \mathfrak{stl}(4, 0, D) \otimes \mathfrak{stl}(4, 0, D) \rightarrow \mathfrak{stl}(4, 0, D) \oplus D_2^6$ be the homomorphism defined on generators by $F_{ij}(a) \otimes F_{kl}(b) \mapsto v_{ijkl}(a \dashv b)$, if i, j, k, l are distinct and $F_{ij}(a) \otimes F_{kl}(b) \mapsto [F_{ij}(a), F_{kl}(b)]$, otherwise. It is obvious that it conserves the grading and that the kernel is inside the centre, so we have to check if φ sends the relation of the non-abelian tensor product to zero.

The relation on generators is given by

$$F_{ij}(a) \otimes [F_{kl}(b), F_{st}(c)] - [F_{ij}(a), F_{kl}(b)] \otimes F_{st}(c) + (-1)^{|F_{kl}(b)||F_{st}(c)|} [F_{ij}(a), F_{st}(c)] \otimes F_{kl}(b). \quad (\text{Gen})$$

If is not involved any preimage of $\mathcal{W}(4, 0, D)$, then the image is just the Leibniz identity on $\mathfrak{stl}(4, 0, D)$. To have any $v_{ijkl}(a)$ we need that in i, j, k, l, s, t one element appears three times and the others three once. Using the relation $v_{ijkl}(a \dashv [b, c]) = 0 = v_{ijkl}([a, b] \dashv c)$ and that we do not need to worry about signs ($v_{ijkl}(2a) = 0$) it is easy to go through the different possibilities and check that they all vanish. Therefore, the bracket defined on $\mathfrak{stl}(m, n, D) \oplus D_2^6$ is the standard bracket unless if i, j, k, l are distinct, then $[F_{ij}(a), F_{kl}(b)] = v_{ijkl}(a \dashv b)$. Moreover, the elements $v_{ijkl}(a)$ are in the centre.

Now we define $\psi: \mathfrak{stl}(4, 0, D) \oplus D_2^6 \rightarrow \mathfrak{stl}(4, 0, D) \otimes \mathfrak{stl}(4, 0, D)$ by $F_{ij}(a) \mapsto F_{ik}(a) \otimes F_{kj}(1)$ and $v_{ijkl}(a) \mapsto F_{ij}(a) \otimes F_{kl}(1)$. It is well defined for the elements of $\mathfrak{stl}(4, 0, D)$ (as in Theorem 3.1) so we have to check if it is well defined for the elements of D_2^6 .

$$\begin{aligned} F_{ij}(a) \otimes F_{kl}(1) &= [F_{il}(a), F_{lj}(1)] \otimes F_{kl}(1) = -F_{il}(a) \otimes F_{kj}(1), \\ F_{ij}(a) \otimes F_{kl}(1) &= F_{ij}(a) \otimes [F_{ki}(1), F_{il}(1)] = -F_{kj}(a) \otimes F_{il}(1). \end{aligned}$$

So $v_{ijkl}(a) = -v_{ilkj}(a) = -v_{kjil}(a) = v_{klij}(a)$. Now,

$$\begin{aligned} 0 &= [F_{ij}(a) \otimes F_{ji}(b), F_{ij}(c) \otimes F_{kl}(1)] = [F_{ij}(a), F_{ji}(b)] \otimes [F_{ij}(c), F_{kl}(1)] \\ &= [[F_{ij}(a), F_{ji}(b)], F_{ij}(c)] \otimes F_{kl}(1) = [[F_{ij}(a), F_{ji}(b)], [F_{ik}(c), F_{kj}(1)]] \otimes F_{kl}(1) \\ &= F_{ij}(a \dashv b \dashv c + (-1)^{|a||b|+|a||c|+|b||c|} c \dashv b \dashv a) \otimes F_{kl}(1), \end{aligned}$$

Choosing $b = c = 1$ we have $F_{ij}(2a) \otimes F_{kl}(1) = 0$. Choosing $c = 1$, we have

$$F_{ij}(a \dashv b + (-1)^{|a||b|} b \dashv a) \otimes F_{kl}(1) = 0.$$

Therefore, ψ is a well-defined R -supermodule homomorphism. Moreover, the identity

$$\begin{aligned} F_{ij}(a) \otimes F_{kl}(b) &= F_{ij}(a) \otimes [F_{ki}(b), F_{il}(1)] = -(-1)^{|a||b|} F_{kj}(b \dashv a) \otimes F_{il}(1) \\ &= F_{kj}(a \dashv b) \otimes F_{il}(1), \end{aligned}$$

shows that ψ is a Leibniz superalgebra homomorphism and that φ and ψ are inverses to each other. $\square$

4.2. Case of $\mathfrak{sl}(3, 1, D)$. Let $\mathcal{W}(3, 1, D)$ be the direct sum of six copies of $\Pi(D_2)$, where Π denotes the parity change functor. The elements will be represented by $v_{ijkl}(a)$ with the same relations as in the previous case.

Theorem 4.4. *The universal central extension of $\mathfrak{sl}(3, 1, D)$ is $\mathfrak{stl}(3, 1, D) \oplus \Pi(D_2)^6$.*

Proof. Assuming that $|i| = 0$, then we can adapt the proof of the Theorem 4.3. In the case that $|i| = 1$,

$$0 = [F_{kl}(a) \otimes F_{lk}(1), F_{ij}(1) \otimes F_{kl}(1)],$$

gives us that $F_{kl}(2a) \otimes F_{ij}(1) = 0$, and we adapt again the proof of Theorem 4.3. $\square$

4.3. Case of $\mathfrak{sl}(2, 2, D)$. Let $\mathcal{W}(2, 2, D)$ be the direct sum of four copies of D_2 and two copies of D_0 . The elements will be represented by $v_{ijkl}(a)$ related by $v_{ijkl}(a) = -v_{ilkj}(a) = -v_{kjil}(a) = v_{klij}(a)$, where $v_{1324}(a)$ and $v_{3142}(a)$ will represent the copies of D_0 and the rest will be the copies of D_2 . Note that $v_{ijkl}(a)$ represents one copy of D_0 if and only if $|i| + |j| = \bar{1} = |k| + |l|$ and $|i| + |k| = \bar{0} = |j| + |l|$.

Theorem 4.5. *The universal central extension of $\mathfrak{sl}(2, 2, D)$ is $\mathfrak{stl}(2, 2, D) \oplus D_2^4 \oplus D_0^2$.*

Proof. Let S_4 be the group of permutations of 4 elements and let $\sigma: S_4 \rightarrow \{-1, 1\}$ be the map that sends $(1423), (2314), (3241), (4132)$ to -1 and the rest to 1 . Note that if $\sigma(ijkl) = -1$, then $v_{ijkl}(a)$ represents a copy of D_0 . Let be the homomorphism

$$\varphi: \mathfrak{stl}(2, 2, D) \otimes \mathfrak{stl}(2, 2, D) \rightarrow \mathfrak{stl}(2, 2, D) \oplus D_2^4 \oplus D_0^2,$$

defined on generators by

$$F_{ij}(a) \otimes F_{kl}(b) \mapsto \begin{cases} (-1)^b \sigma(ijkl) v_{ijkl}(a \dashv b) & \text{if } i, j, k, l \text{ are distinct,} \\ [F_{ij}(a), F_{kl}(b)] & \text{otherwise.} \end{cases}$$

Again we need to check that φ sends the relation of the non-abelian tensor product to zero. If $v_{ijkl}(a)$ represents an element of D_2 , then the proof is similar as the proof given in Theorem 4.3. Therefore, we have to check if relation (Gen) vanishes when an element of D_0 appears. Avoiding symmetries the choices that we have to check are $(i, j, k, l, s, t) = (1, 3, 2, 1, 1, 4), (1, 3, 2, 3, 3, 4), (1, 3, 1, 4, 2, 1)$ and $(1, 3, 3, 4, 2, 3)$. It is a straightforward computation and we omit it.

Now we define $\psi: \mathfrak{stl}(2, 2, D) \oplus D_2^4 \oplus D_0^2 \rightarrow \mathfrak{stl}(2, 2, D) \otimes \mathfrak{stl}(2, 2, D)$ by $F_{ij}(a) \mapsto F_{ik}(a) \otimes F_{kj}(1)$ and $v_{ijkl}(a) \mapsto \sigma(ijkl) F_{ij}(a) \otimes F_{kl}(1)$. To check that is a well-defined homomorphism we can follow the proofs of Theorem 3.1 and Theorem 4.3 and we will cover all the cases unless the two copies of D_0 . We have that

$$\begin{aligned} F_{ij}(a) \otimes F_{kl}(1) &= [F_{il}(a), F_{lj}(1)] \otimes F_{kl}(1) = -F_{il}(a) \otimes F_{kj}(1), \\ F_{ij}(a) \otimes F_{kl}(1) &= F_{ij}(a) \otimes [F_{ki}(1), F_{il}(1)] = -F_{kj}(a) \otimes F_{il}(1). \end{aligned}$$

So $v_{ijkl}(a) = -v_{ilkj}(a) = -v_{kjil}(a) = v_{klij}(a)$. Then,

$$\begin{aligned} 0 &= [F_{ij}(a) \otimes F_{ji}(b), F_{ij}(c) \otimes F_{kl}(1)] = [F_{ij}(a), F_{ji}(b)] \otimes [F_{ij}(c), F_{kl}(1)] \\ &= F_{ij}(a \dashv b \dashv c + (-1)^{(|a|+\bar{1})(|b|+\bar{1})+(|c|+\bar{1})(|a|+|b|)} c \vdash b \vdash a) \otimes F_{kl}(1), \end{aligned}$$

choosing $c = 1$ we have that

$$F_{ij}(a \dashv b - (-1)^{|a||b|} b \vdash a) \otimes F_{kl}(1) = 0.$$

To see that ψ is a Leibniz superalgebra homomorphism,

$$\begin{aligned} F_{ij}(a) \otimes F_{kl}(b) &= F_{ij}(a) \otimes [F_{ki}(b), F_{il}(1)] = [F_{ij}(a), F_{ki}(b)] \otimes F_{il}(1) \\ &= -(-1)^{|a||b|+|b|} F_{kj}(b \vdash a) \otimes F_{il}(1) = -(-1)^{|b|} F_{kj}(a \dashv b) \otimes F_{il}(1) \\ &= (-1)^{|b|} F_{ij}(a \dashv b) \otimes F_{kl}(1). \end{aligned}$$

The previous relation also proves that $\psi \circ \varphi$ is the identity map. Moreover, it is straightforward that $\varphi \circ \psi$ is the identity map, completing the proof. $\square$

4.4. Case of $\mathfrak{sl}(3, 0, D)$. Let $\mathcal{W}(3, 0, D)$ be the direct sum of six copies of D_3 . The elements will be represented by $v_{ijpq}(a)$ where $pq = ik$ or kj and $\{i, j, k\} = \{1, 2, 3\}$ and they will be related by R -linearity, the equivalence relations of D_3 and the additional relation $v_{ijpq}(a) = -v_{pqij}(a)$.

Theorem 4.6. *The universal central extension of $\mathfrak{sl}(3, 0, D)$ is $\mathfrak{stl}(3, 0, D) \oplus D_3^6$.*

Proof. Let be the homomorphism

$$\varphi: \mathfrak{stl}(3, 0, D) \otimes \mathfrak{stl}(3, 0, D) \rightarrow \mathfrak{stl}(3, 0, D) \oplus D_3^6,$$

defined on generators by

$$F_{ij}(a) \otimes F_{pq}(b) \mapsto \begin{cases} v_{ijpq}(a \dashv b) & \text{if } pq = ik \text{ or } kj \\ [F_{ij}(a), F_{pq}(b)] & \text{otherwise.} \end{cases}$$

To check that φ is well defined it only needs to check that relation (Gen) is followed when a $v_{ijpq}(a)$ appears. It is immediate that

$$\varphi(x \otimes [y, z]) = -(-1)^{|y||z|} \varphi(x \otimes [z, y]),$$

so the non straightforward cases are

$$\begin{aligned} \varphi(F_{ji}(a) \otimes [F_{ji}(b), F_{ik}(c)]) &= v_{jijk}(a \dashv b \dashv c) \\ &= v_{jijk}(a \dashv (b \vdash c)) = -(-1)^{|b||c|} v_{jkji}(a \dashv c \dashv b) \\ &= -(-1)^{|b||c|} \varphi(F_{jk}(a \dashv c) \otimes F_{ji}(b)) \\ &= \varphi([F_{ji}(a), F_{ji}(b)] \otimes F_{ik}(c) \\ &\quad - (-1)^{|b||c|} [F_{ji}(a), F_{ik}(c)] \otimes F_{ji}(b)), \end{aligned}$$

and

$$\begin{aligned} \varphi(F_{ij}(a) \otimes [F_{ki}(b), F_{ij}(c)]) &= v_{ijkj}(a \dashv b \dashv c) \\ &= -(-1)^{|a||b|} v_{kji}(b \vdash a) \dashv c \\ &= -(-1)^{|a||b|} \varphi(F_{kj}(b \vdash a) \otimes F_{ij}(c)) \\ &= \varphi([F_{ij}(a), F_{ki}(b)] \otimes F_{ij}(c) \\ &\quad - (-1)^{|b||c|} [F_{ij}(a), F_{ij}(c)] \otimes F_{ki}(b)). \end{aligned}$$

Now we define $\psi: \mathfrak{stl}(3, 0, D) \oplus D_3^6 \rightarrow \mathfrak{stl}(3, 0, D) \otimes \mathfrak{stl}(3, 0, D)$ by $F_{ij}(a) \mapsto F_{ik}(a) \otimes F_{kj}(1)$ and $v_{ijpq}(a) \mapsto F_{ij}(a) \otimes F_{pq}(1)$. There is only one choice for k , but we need to check that it is well defined for elements of $\mathfrak{stl}(3, 0, D)$, since the arguments of Theorem 3.1 do not hold.

Then,

$$\begin{aligned}
F_{ik}(a \dashv b) \otimes F_{kj}(1) &= -F_{ij}(a \dashv b) \otimes [F_{kj}(1), F_{jk}(1)] \\
&= -[F_{ik}(a), F_{kj}(b)] \otimes [F_{kj}(1), F_{jk}(1)] \\
&= F_{ik}(a) \otimes (F_{kj}(b) + F_{kj}(b)) - F_{ik}(a) \otimes F_{kj}(b) \\
&= F_{ik}(a) \otimes F_{kj}(b),
\end{aligned}$$

and similarly for $F_{ik}(a \dashv b) \otimes F_{ji}(1) = F_{ik}(a) \otimes F_{ji}(b)$. Moreover,

$$F_{ij}(a) \otimes F_{ij}(b) = F_{ij}(a) \otimes [F_{ik}(b), F_{kj}(1)] = 0.$$

For the elements of D_3^6 ,

$$0 = [F_{ij}(a), F_{ik}(1)] \otimes [F_{ik}(1), F_{ki}(1)] = F_{ij}(a) \otimes F_{ik}(-3) = F_{ij}(3a) \otimes F_{ik}(1),$$

and

$$\begin{aligned}
0 &= [F_{ij}(a), F_{ji}(b)] \otimes [F_{ij}(c), F_{ik}(1)] \\
&= F_{ij}(a \dashv b \dashv c + (-1)^{|a||b|+|a||c|+|b||c|} c \vdash b \vdash a) \otimes F_{ik}(1) - F_{ik}(a \dashv b) \otimes F_{ij}(c),
\end{aligned}$$

choosing $b = c = 1$,

$$F_{ik}(a) \otimes F_{ij}(1) = -F_{ij}(a) \otimes F_{ik}(1),$$

and choosing $c = 1$,

$$F_{ij}(-a \dashv b + (-1)^{|a||b|} b \vdash a) \otimes F_{ik}(1) = 0.$$

To complete the proof,

$$\begin{aligned}
F_{ij}(a) \otimes F_{ik}(b) &= -F_{ij}(a) \otimes [F_{jk}(b), F_{ij}(1)] = -F_{ik}(a \dashv b) \otimes F_{ij}(1) \\
&= F_{ij}(a \dashv b) \otimes F_{ik}(1).
\end{aligned}$$

□

4.5. Case of $\mathfrak{sl}(2, 1, D)$. In this case, $\mathcal{W}(2, 1, D) = 0$.

Theorem 4.7. *The universal central extension of $\mathfrak{sl}(2, 1, D)$ is $\mathfrak{stl}(2, 1, D)$.*

Proof. Defining the homomorphisms as in Theorem 4.6, we can recover the relations and additionally

$$0 = [F_{ij}(a), F_{ik}(b)] \otimes [F_{ik}(1), F_{ki}(1)] = F_{ij}(a) \otimes F_{ik}(2 + (-1)^{|i|+|k|} 1).$$

Therefore, if $|i| = \bar{1}$ or $|k| = \bar{1}$, we have the relation $F_{ij}(a) \otimes F_{ik}(1) = 0$. If $|j| = \bar{1}$, we do the same calculation but for $F_{ik}(a) \otimes F_{ij}(1)$. It is similar for $F_{ij}(a) \otimes F_{kj}(1)$. □

5. HOCHSCHILD HOMOLOGY AND LEIBNIZ HOMOLOGY

In this section we adapt to the superalgebra case the definition of Hochschild homology of dialgebras introduced in [7] and we relate it with the universal central extension of $\mathfrak{sl}(m, n, D)$.

Let D a superdialgebra with a R -basis containing the bar-unit. Note that now we have to assume that D admits an R -basis. The boundary map $d_n: D^{\otimes n+1} \rightarrow D^{\otimes n}$ is defined on generators by

$$\begin{aligned}
d_n(a_0 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} (a_0 \otimes \cdots \otimes a_i \dashv a_{i+1} \otimes \cdots \otimes a_n) \\
&\quad + (-1)^{n+|a_n| \sum_{i=0}^{n-1} |a_i|} (a_n \vdash a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}),
\end{aligned}$$

where $a_i \in D$. The *Hochschild homology of superdialgebras*, denoted by $\mathrm{HH}_*(D)$, is the homology of the chain complex formed by the boundary maps d_* . Let I be the ideal of D generated by the elements of the form $a \otimes b \dashv c - a \otimes b \vdash c$. We define

$$\mathrm{HHS}_1(D) = \frac{\mathrm{Ker} d_1}{\mathrm{Im} d_2 + I}.$$

Theorem 5.1. *There is an isomorphism of R -supermodules $\mathrm{HL}_2(\mathfrak{sl}(m, n, D)) \cong \mathrm{HHS}_1(D) \oplus \mathcal{W}(m, n, D)$.*

Proof. We have the following diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{HHS}_1(D) \oplus \mathcal{W}(m, n, D) & \longrightarrow & \frac{D \otimes D}{\mathrm{Im} d_2 + I} \oplus \mathcal{W}(m, n, D) & \xrightarrow{d_1} & [D, D] & \longrightarrow 0 \\ & & & \uparrow \downarrow \mu & & \uparrow \downarrow E_{11}(-) & \\ 0 \longrightarrow & \mathrm{HL}_2(\mathfrak{sl}(m, n, D)) & \longrightarrow & \mathfrak{stl}(m, n, D) \otimes \mathfrak{stl}(m, n, D) & \xrightarrow{\omega} & \mathfrak{stl}(m, n, D) & \longrightarrow 0, \end{array}$$

where $\mu(a \otimes b) = F_{1j}(a) \otimes F_{j1}(b) - (-1)^{|a||b|} F_{1j}(b \vdash a) \otimes F_{j1}(1)$, $\mu(v_{ijkl}(a)) = F_{ij}(a) \otimes F_{kl}(b)$ and

$$\mathrm{Str}_2(F_{ij}(a) \otimes F_{kl}(b)) \begin{cases} a \otimes b & \text{if } i = j \text{ and } k = l \\ v_{ijkl}(a \dashv b) & \text{if } i, j, k, l \text{ if it makes sense depending of } m, n \\ 0 & \text{otherwise.} \end{cases}$$

It is a straightforward computation that $\mu \circ \mathrm{Str}_2$ and $E_{11}(-) \circ \mathrm{Str}_1$ are the identity maps and that the diagram is commutative. Then the restrictions of Str_2 to the kernel of ω is also a split epimorphism, with μ restricted to the kernel of d_1 as section. Let us see that these restrictions are indeed isomorphisms. An element in the kernel of ω , is a sum of elements of the form $F_{ij}(a) \otimes F_{ji}(b)$ plus the elements of $\mathcal{W}(m, n, D)$. Any element of $\mathrm{Ker} \omega$ can be written as an element of $\mathrm{Im} \mu$ plus $\sum_{i=2}^{m+n} F_{1i}(a_i) \otimes F_{i1}(1)$, since

$$F_{ij}(a) \otimes F_{ji}(b) = F_{i1}(a) \otimes F_{1i}(b) - (-1)^{(|a|+|i|+|j|)(|b|+|i|+|j|)} F_{j1}(b \vdash a) \otimes F_{1j}(1),$$

and

$$\begin{aligned} F_{1j}(a) \otimes F_{j1}(b) &= F_{1j}(a) \otimes F_{j1}(b) - (-1)^{|a||b|} F_{j1}(b \vdash a) \otimes F_{1j}(1) + \\ &\quad (-1)^{|a||b|} F_{j1}(b \vdash a) \otimes F_{1j}(1). \end{aligned}$$

Furthermore, if it is in the kernel of ω , all the a_i must be zero. Then the restriction of μ to the kernel of d_1 is surjective. $\square$

Remark 5.2. The proof given in [21] can also be adapted since the assumptions on the characteristic of the ring are not used, but we rather give our version of the proof to show its relation with non-abelian tensor product.

6. CONCLUDING REMARKS

Combining the results obtained above we present the following summarizing theorems

Theorem 6.1. *Let R a unital commutative ring and D an associative unital R -superdialgebra with an R -basis containing the identity. Then,*

$$\mathrm{HL}_2(\mathfrak{sl}(m, n, D)) = \begin{cases} \mathrm{HHS}_1(D) & \text{for } m+n \geq 5 \text{ or } m=2, n=1, \\ \mathrm{HHS}_1(D) \oplus D_3^6 & \text{for } m=3, n=0, \\ \mathrm{HHS}_1(D) \oplus D_2^6 & \text{for } m=4, n=0, \\ \mathrm{HHS}_1(D) \oplus \Pi(D_2)^6 & \text{for } m=3, n=1, \\ \mathrm{HHS}_1(D) \oplus D_2^4 \oplus D_0^2 & \text{for } m=2, n=2, \end{cases}$$

where D_m is the quotient of D by the ideal $mD + ([D, D] \dashv D)$ (Definition 4.2) and Π is the parity change functor.

Theorem 6.2. *Let R a unital commutative ring and D an associative unital R -superdialgebra with an R -basis containing the identity. Then,*

$$\mathrm{H}_2(\mathfrak{stl}(m, n, D)) = \begin{cases} 0 & \text{for } m+n \geq 5 \text{ or } m=2, n=1, \\ D_3^6 & \text{for } m=3, n=0, \\ D_2^6 & \text{for } m=4, n=0, \\ \Pi(D_2)^6 & \text{for } m=3, n=1, \\ D_2^4 \oplus D_0^2 & \text{for } m=2, n=2, \end{cases}$$

where D_m is the quotient of D by the ideal $mD + ([D, D] \dashv D)$ (Definition 4.2) and Π is the parity change functor.

Remark 6.3. We recall that in the case that $m = n = 2$, $\mathcal{W}(2, 2, D)$ might not be zero even if $\mathrm{char}(R) \neq 2$ contradicting [18, Theorem 6.2].

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