On the structure of graded Leibniz triple systems

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Abstract

We study the structure of a Leibniz triple system $\mathcal{E}$ graded by an arbitrary abelian group G which is considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. We show that $\mathcal{E}$ is of the form $\mathcal{E} = U + \sum_{[j]\in\sum^1/\sim} I_{[j]}$ with U a linear subspace of the 1-homogeneous component $\mathcal{E}_1$ and any ideal $I_{[j]}$ of $\mathcal{E}$, satisfying $\{I_{[j]}, \mathcal{E}, I_{[k]}\} = \{I_{[j]}, I_{[k]}, \mathcal{E}\} = \{\mathcal{E}, I_{[j]}, I_{[k]}\} = 0$ if $[j] \neq [k]$, where the relation $\sim$ in $\sum^1 = \{g \in G\backslash\{1\} : L_g \neq 0\}$, defined by $g \sim h$ if and only if g is connected to h.

Key words: graded Leibniz triple system, Lie triple system, Leibniz algebra
MSC(2010): 17A32, 17A60, 17B22, 17B65

1 Introduction

Leibniz triple systems were introduced by Bremner and Sánchez-Ortega [1]. Leibniz triple systems were defined in a functorial manner using the Kolesnikov-Pozhidaev algorithm, which took the defining identities for a variety of algebras and produced the defining identities for the corresponding variety of dialgebras [2]. In [1], Leibniz triple systems were obtained by applying the Kolesnikov-Pozhidaev algorithm to Lie triple systems. The study of gradings on Lie algebras begins in the 1933 seminal Jordan's work, with the purpose of formalizing Quantum Mechanics [3]. Since then, the interest on gradings on different classes of algebras has been remarkable in the recent years, motivated in

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part by their application in physics and geometry [4–7]. Recently, in [7–12], the structure of arbitrary graded Lie algebras, graded Lie superalgebras, graded commutative algebras, graded Leibniz algebras and graded Lie triple systems have been determined by the techniques of connections of roots. Our work is essentially motivated by the work on graded Lie triple systems [12] and the work on split Leibniz triple systems [13].

Throughout this paper, Leibniz triple systems $\mathcal{E}$ are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. This paper proceeds as follows. In section 2, we establish the preliminaries on graded Leibniz triple systems theory. In section 3, we show that such an arbitrary Leibniz triple system is of the form $\mathcal{E} = U + \sum_{[j]\in\sum^1/\sim} I_{[j]}$ with U a subspace of $\mathcal{E}_1$ and any ideal $I_{[j]}$ of $\mathcal{E}$, satisfying $\{I_{[j]}, \mathcal{E}, I_{[k]}\} = \{I_{[j]}, I_{[k]}, \mathcal{E}\} = \{\mathcal{E}, I_{[j]}, I_{[k]}\} = 0$ if $[j] \neq [k]$, where the relation $\sim$ in $\sum^1$, defined by $g \sim h$ if and only if g is connected to h.

2 Preliminaries

Definition 2.1. [14] *A* ***right Leibniz algebra*** *L is a vector space over a base field $\mathbb{K}$ endowed with a bilinear product $[\cdot,\cdot]$ satisfying the Leibniz identity*

$$[[y, z], x] = [[y, x], z] + [y, [z, x]],$$

for all $x, y, z \in L$.

Definition 2.2. [1] *A* ***Leibniz triple system*** *is a vector space $\mathcal{E}$ endowed with a trilinear operation $\{\cdot,\cdot,\cdot\} : \mathcal{E} \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ satisfying*

$$\{a, \{b, c, d\}, e\} = \{\{a, b, c\}, d, e\} - \{\{a, c, b\}, d, e\} - \{\{a, d, b\}, c, e\} + \{\{a, d, c\}, b, e\}, \quad (2.1)$$

$$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{\{a, b, d\}, c, e\} - \{\{a, b, e\}, c, d\} + \{\{a, b, e\}, d, c\}, \quad (2.2)$$

for all $a, b, c, d, e \in \mathcal{E}$.

Example 2.3. *A Lie triple system gives a Leibniz triple system with the same ternary product. If L is a Leibniz algebra with product $[\cdot,\cdot]$, then L becomes a Leibniz triple system by putting $\{x, y, z\} = [[x, y], z]$. More examples refer to [1].*

Definition 2.4. [1] *Let I be a subspace of a Leibniz triple system $\mathcal{E}$. Then I is called a* ***subsystem*** *of $\mathcal{E}$, if $\{I, I, I\} \subseteq I$; I is called an* ***ideal*** *of $\mathcal{E}$, if $\{I, \mathcal{E}, \mathcal{E}\} + \{\mathcal{E}, I, \mathcal{E}\} + \{\mathcal{E}, \mathcal{E}, I\} \subseteq I$.*

Proposition 2.5. [15] *Let $\mathcal{E}$ be a Leibniz triple system. Then the following assertions hold.*

(1) *J is generated by $\{\{a, b, c\} - \{a, c, b\} + \{b, c, a\} : a, b, c \in \mathcal{E}\}$, then J is an ideal of $\mathcal{E}$ satisfying $\{\mathcal{E}, \mathcal{E}, J\} = \{\mathcal{E}, J, \mathcal{E}\} = 0$.*

(2) *J is generated by $\{\{a, b, c\} - \{a, c, b\} + \{b, c, a\} : a, b, c \in \mathcal{E}\}$, then $\mathcal{E}$ is a Lie triple system if and only if $J = 0$.*

(3) *$\{\{c, d, e\}, b, a\} - \{\{c, d, e\}, a, b\} - \{\{c, b, a\}, d, e\} + \{\{c, a, b\}, d, e\} - \{c, \{a, b, d\}, e\} - \{c, d, \{a, b, e\}\} = 0$, for all $a, b, c, d, e \in \mathcal{E}$.*

Definition 2.6. [1] *The **standard embedding** of a Leibniz triple system $\mathcal{E}$ is the two-graded right Leibniz algebra $L = L^0 \oplus L^1$, L^0 being the* $\mathbb{K}$-span *of $\{x \otimes y,\ x, y \in \mathcal{E}\}$, $L^1 := \mathcal{E}$ and where the product is given by*

$$[(x\otimes y, z), (u\otimes v, w)] := (\{x, y, u\}\otimes v - \{x, y, v\}\otimes u + z\otimes w, \{x, y, w\} + \{z, u, v\} - \{z, v, u\}).$$

Let us observe that L^0 with the product induced by the one in $L = L^0 \oplus L^1$ becomes a right Leibniz algebra.

Definition 2.7. *Let $\mathcal{E}$ be a Leibniz triple system. It is said that $\mathcal{E}$ is graded by means of an abelian group G if it decomposes as the direct sum of linear subspaces*

$$\mathcal{E} = \bigoplus_{g\in G} \mathcal{E}_g$$

where the homogeneous components satisfy $\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\} \subset \mathcal{E}_{ghk}$ for any $g, h, k \in G$ (denoting by juxtaposition the product in G). We call the support of the grading the set $\sum^1 := \{g \in G \setminus \{1\} : \mathcal{E}_g \neq 0\}$.

The usual regularity conditions will be understood in the graded sense. That is, a subtriple of $\mathcal{E}$ is a linear subspace S satisfying $\{S, S, S\} \subset S$ and such that splits as $S = \oplus_{g\in G} S_g$ with any $S_g = S \cap \mathcal{E}_g$.

Let L be an arbitrary Leibniz algebra over $\mathbb{K}$. As usual, the term grading will always mean abelian group grading, that is, a decomposition in linear subspaces $L = \oplus_{g\in G} L_g$ where G is an abelian group and the homogeneous spaces satisfy $[L_g, L_h] \subset L_{gh}$. We also call the support of the grading the set $\{g \in G \setminus \{1\} : L_g \neq 0\}$.

Proposition 2.8. *Let $\mathcal{E}$ be a G-graded Leibniz triple system and let $L = L^0 \oplus L^1$ be its standard embedding algebra, then L^0 is a G-graded Leibniz algebra.*

Proof. Define $L^0_1 := \sum_{g\in G} [\mathcal{E}_g, \mathcal{E}_{g^{-1}}]$ and $L^0_g := \sum_{h\in G} [\mathcal{E}_h, \mathcal{E}_{h^{-1}g}]$ for any $g \in G \setminus \{1\}$. Clearly $L^0_1 + \sum_{g\in G\setminus\{1\}} L^0_g \subseteq L^0$. Conversely, since $L^0 = [\mathcal{E}, \mathcal{E}] = [\bigoplus_{g\in G} \mathcal{E}_g, \bigoplus_{h\in G} \mathcal{E}_h] \subseteq$

$$L^0_1 + \sum_{g\in G\setminus\{1\}} L^0_g,$$

we get $L^0 = L^0_1 + \sum_{g\in G\setminus\{1\}} L^0_g$.

The direct character of the sum can be checked as follows. If $x \in L^0_g \cap (\sum_{h\in G\setminus\{g\}} L^0_h)$, then for any $q \in G$ and $y \in \mathcal{E}_q$ we have $[x, y] \in \mathcal{E}_{gq} \cap (\sum_{h\in G\setminus\{g\}} \mathcal{E}_{hq})$ and so $[x, y] = 0$. From here $[x, \mathcal{E}] = 0$ and so $x = 0$. Hence we can write

$$L^0 = L^0_1 \oplus (\bigoplus_{g\in G\setminus\{1\}} L^0_g).$$

Finally, we have

$$[L^0_g, L^0_h] \subseteq L^0_{gh}$$

for any $g, h \in G$. Indeed,

$$[L^0_g, L^0_h] = \sum_{k,l\in G} [[\mathcal{E}_k, \mathcal{E}_{k^{-1}g}], [\mathcal{E}_l, \mathcal{E}_{l^{-1}h}]] \subset$$

$$[\mathcal{E}_k, [\mathcal{E}_{k^{-1}g}, [\mathcal{E}_l, \mathcal{E}_{l^{-1}h}]]] + [[\mathcal{E}_k, [\mathcal{E}_l, \mathcal{E}_{l^{-1}h}]], \mathcal{E}_{k^{-1}g}] \subset$$
$$[\mathcal{E}_k, [\mathcal{E}_{k^{-1}g}, L^0_h]] + [[\mathcal{E}_k, L^0_h], \mathcal{E}_{k^{-1}g}] \subset$$
$$[\mathcal{E}_k, \mathcal{E}_{k^{-1}gh}] + [\mathcal{E}_{kh}, \mathcal{E}_{k^{-1}g}] \subset L^0_{gh}.$$

Observe that for any $g, h \in G$ we have $[\mathcal{E}_g, \mathcal{E}_h] \subset L^0_{gh}$. □

In the following, we shall denote by $\sum^0$ the support of the graded Leibniz algebra L^0.

3 Connections and gradings

From now on, $\mathcal{E}$ denotes a graded Leibniz triple system with support $\sum^1$, and

$$\mathcal{E} = \bigoplus_{g\in G} \mathcal{E}_g = \mathcal{E}_1 \oplus (\bigoplus_{g\in\sum^1} \mathcal{E}_g)$$

the corresponding grading. Denote by $-\sum^i = \{-g : g \in \sum^i\}, i = 0, 1$.

Definition 3.1. *Let g and h be two elements in $\sum^1$. We say that g is connected to h if there exist $g_1, g_2, \cdots, g_{2n+1} \in \pm\sum^1 \cup \{1\}$ such that*

(1) $\{g_1, g_1g_2g_3, \cdots, g_1g_2g_3 \cdots g_{2n}g_{2n+1}\} \subset \pm\sum^1$,

(2) $\{g_1g_2, g_1g_2g_3g_4, \cdots, g_1g_2g_3 \cdots g_{2n}\} \subset \pm\sum^0$,

(3) $g_1 = g$ *and* $g_1g_2g_3 \cdots g_{2n}g_{2n+1} \in \{h, h^{-1}\}$.

We also say that $\{g_1, \cdots, g_{2n+1}\}$ is a connection from g to h.

Proposition 3.2. *The relation $\sim$ in $\sum^1$, defined by $g \sim h$ if and only if g is connected to h is an equivalence relation.*

Proof. The proof is completely analogously to [16, Proposition 3.1]. □

By Proposition 3.2 the connection relation is an equivalence relation in $\sum^1$ and so we can consider the quotient set $\sum^1 / \sim = \{[g] : g \in \sum^1\}$, becoming $[g]$ the set of elements in the support of the grading which are connected to g. By the definition of $\sim$, it is clear that if $h \in [g]$ and $h^{-1} \in \sum^1$ then $h^{-1} \in [g]$.

Our goal in this section is to associate an adequate subtriple $I_{[g]}$ to any $[g]$. Fix $g \in \sum^1$, we start by defining

$$\mathcal{E}_{1,[g]} := span_{\mathbb{K}}\{\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_{(hk)^{-1}}\} : h \in [g], k \in [g] \cup \{1\}\} \subset \mathcal{E}_1$$

and $V_{[g]} := \oplus_{h\in[g]}\mathcal{E}_h$. Finally, we denote by $\mathcal{E}_{[g]}$ the direct sum of the two subspaces above, that is,

$$\mathcal{E}_{[g]} := \mathcal{E}_{1,[g]} \oplus V_{[g]}.$$

Proposition 3.3. *For any $g \in \sum^1$, the graded linear subspace $\mathcal{E}_{[g]}$ is a subtriple of $\mathcal{E}$.*

Proof. We have to check that $\mathcal{E}_{[g]}$ satisfies

$$\{\mathcal{E}_{[g]}, \mathcal{E}_{[g]}, \mathcal{E}_{[g]}\} = \{\mathcal{E}_{1,[g]} \oplus V_{[g]}, \mathcal{E}_{1,[g]} \oplus V_{[g]}, \mathcal{E}_{1,[g]} \oplus V_{[g]}\} \subset \mathcal{E}_{[g]}.$$

Since $\mathcal{E}_{1,[g]} \subset \mathcal{E}_1$ we clearly have

$$\{\mathcal{E}_{1,[g]}, \mathcal{E}_{1,[g]}, V_{[g]}\} + \{\mathcal{E}_{1,[g]}, V_{[g]}, \mathcal{E}_{1,[g]}\} + \{V_{[g]}, \mathcal{E}_{1,[g]}, \mathcal{E}_{1,[g]}\} \subset V_{[g]}.$$

It is easy to verify that

$$\{\mathcal{E}_{1,[g]}, \mathcal{E}_{1,[g]}, \mathcal{E}_{1,[g]}\} \subset \mathcal{E}_{1,[g]}.$$

Indeed, $\{\mathcal{E}_{1,[g]}, \mathcal{E}_{1,[g]}, \mathcal{E}_{1,[g]}\} \subset \{\mathcal{E}_{1,[g]}, \mathcal{E}_1, \mathcal{E}_1\} \subset \mathcal{E}_{1,[g]}$. Moreover, we also have

$$\{\mathcal{E}_{1,[g]}, V_{[g]}, V_{[g]}\} \subset \mathcal{E}_{[g]}.$$

In fact, if $\{\mathcal{E}_1, \mathcal{E}_h, \mathcal{E}_k\} \neq 0$ for some $h, k \in [g]$ then $h \in \sum^0$ and $hk \in \sum^1 \cup \{1\}$. From here, if $hk \neq 1$ and $\{g_1, \cdots, g_{2n+1}\}$ is a connection from g to h then $\{g_1, \cdots, g_{2n+1}, 1, k\}$ is a connection from g to hk in case $g_1g_2\cdots g_{2n+1} = h$ and $\{g_1, \cdots, g_{2n+1}, 1, k^{-1}\}$ in case $g_1g_2\cdots g_{2n+1} = h^{-1}$ being so $hk \in [g]$. If $hk = 1$ clearly $\{\mathcal{E}_1, \mathcal{E}_h, \mathcal{E}_{h^{-1}}\} \subset \mathcal{E}_{1,[g]}$. We have showed $\{\mathcal{E}_1, \mathcal{E}_h, \mathcal{E}_k\} \subset \mathcal{E}_{[g]}$ and so $\{\mathcal{E}_{1,[g]}, V_{[g]}, V_{[g]}\} \subset \mathcal{E}_{[g]}$.

Next, let us show that

$$\{V_{[g]}, \mathcal{E}_{1,[g]}, V_{[g]}\} \subset \mathcal{E}_{[g]}.$$

if $\{\mathcal{E}_h, \mathcal{E}_1, \mathcal{E}_k\} \neq 0$ for some $h, k \in [g]$ then $h \in \sum^0$ and $hk \in \sum^1 \cup \{1\}$. From here, if $hk \neq 1$ and $\{g_1, \cdots, g_{2n+1}\}$ is a connection from g to h then $\{g_1, \cdots, g_{2n+1}, 1, k\}$ is a connection from g to hk in case $g_1g_2\cdots g_{2n+1} = h$ and $\{g_1, \cdots, g_{2n+1}, 1, k^{-1}\}$ in case $g_1g_2\cdots g_{2n+1} = h^{-1}$ being so $hk \in [g]$. If $hk = 1$ clearly $\{\mathcal{E}_h, \mathcal{E}_1, \mathcal{E}_{h^{-1}}\} \subset \mathcal{E}_{1,[g]}$. We have showed $\{\mathcal{E}_h, \mathcal{E}_1, \mathcal{E}_k\} \subset \mathcal{E}_{[g]}$.

Next, let us show that

$$\{V_{[g]}, V_{[g]}, \mathcal{E}_{1,[g]}\} \subset \mathcal{E}_{[g]}.$$

if $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_1\} \neq 0$ for some $h, k \in [g]$ then $hk \in \sum^0$ and $hk \in \sum^1 \cup \{1\}$. From here, if $hk \neq 1$ and $\{g_1, \cdots, g_{2n+1}\}$ is a connection from g to h then $\{g_1, \cdots, g_{2n+1}, k, 1\}$ is a connection from g to hk in case $g_1g_2\cdots g_{2n+1} = h$ and $\{g_1, \cdots, g_{2n+1}, k^{-1}, 1\}$ in case $g_1g_2\cdots g_{2n+1} = h^{-1}$ being so $hk \in [g]$. If $hk = 1$ clearly $\{\mathcal{E}_h, \mathcal{E}_{h^{-1}}, \mathcal{E}_1\} \subset \mathcal{E}_{1,[g]}$. We have showed $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_1\} \subset \mathcal{E}_{[g]}$.

Finally, let us show

$$\{V_{[g]}, V_{[g]}, V_{[g]}\} \subset \mathcal{E}_{[g]}.$$

Suppose $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_l\} \neq 0$ for some $h, k, l \in [g]$ being so $hk \in \sum^0 \cup \{1\}$ and $hkl \in \sum^1 \cup \{1\}$. If either $hk = 1$ or $hkl = 1$ then $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_l\} = \mathcal{E}_l \subset V_{[g]}$ or $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_l\} \subset \mathcal{E}_{1,[g]}$

respectively. Hence let us consider $hk \in \sum^0$ and $hkl \in \sum^1$, and take a connection $\{g_1, \cdots, g_{2n+1}\}$ from g to h. We clearly have $\{g_1, \cdots, g_{2n+1}, k, l\}$ is a connection from g to hkl in case $g_1 \cdots g_{2n+1} = h$ and $\{g_1 \cdots g_{2n+1}, k^{-1}, l^{-1}\}$ it is in case $g_1 \cdots g_{2n+1} = h^{-1}$. We have showed $hkl \in [g]$ and so $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_l\} \subset V_{[g]}$, which concludes the proof. □

Definition 3.4. *With the above notation, we call $\mathcal{E}_{[g]}$ the* ***subtriple*** *of $\mathcal{E}$ associated to $[g]$.*

4 Decompositions

We begin this section by showing that for any $g \in \sum^1$, the subtriple $I_{[g]}$ is actually an ideal of $\mathcal{E}$. We need to state some preliminary results.

Lemma 4.1. *The following assertions hold.*

1. *If $g, h \in \sum^1$ with $gh \in \pm \sum^0 \cup \{1\}$, then $h \in [g]$.*
2. *If $g, h \in \sum^1$ and $g \in \pm \sum^0$ with $gh \in \pm \sum^1 \cup \{1\}$, then $h \in [g]$.*
3. *If $g, h \in \sum^1$ and $g, h \in \pm \sum^0$ with $gh \in \pm \sum^0 \cup \{1\}$, then $h \in [g]$.*
4. *If $g, \overline{h} \in \sum^1$ such that $\overline{h} \notin [g]$, then $[\mathcal{E}_g, \mathcal{E}_{\overline{h}}] = [L^0_g, \mathcal{E}_{\overline{h}}] = [L^0_g, L^0_{\overline{h}}] = 0$.*

Proof. 1. If $gh = 1$, then $h = g^{-1}$ and so $h \sim g$. Suppose $gh \neq 1$. Since $gh \in \pm \sum^0$, we have $\{g, h, g^{-1}\}$ is a connection from g to h.

2. We can argue similarly with the connection $\{g, 1, (gh)^{-1}\}$.

3. If $gh = 1$, then $h = g^{-1}$ and so $h \sim g$. Suppose $gh \neq 1$. Since $gh \in \pm \sum^0$, we have $\{g, h, g^{-1}\}$ is a connection from g to h.

4. Consequence of 1, 2 and 3. □

Lemma 4.2. *If $g, \overline{h} \in \sum^1$ are not connected, then $\{\mathcal{E}_g, \mathcal{E}_{g^{-1}}, \mathcal{E}_{\overline{h}}\} = 0$.*

Proof. If $[\mathcal{E}_g, \mathcal{E}_{g^{-1}}] = 0$ it is clear. Suppose then $[\mathcal{E}_g, \mathcal{E}_{g^{-1}}] \neq 0$ and $\{\mathcal{E}_g, \mathcal{E}_{g^{-1}}, \mathcal{E}_{\overline{h}}\} \neq 0$. By Leibniz identity, one gets

$$\{\mathcal{E}_g, \mathcal{E}_{g^{-1}}, \mathcal{E}_{\overline{h}}\} = [[\mathcal{E}_g, \mathcal{E}_{g^{-1}}], \mathcal{E}_{\overline{h}}] \subset [\mathcal{E}_g, [\mathcal{E}_{g^{-1}}, \mathcal{E}_{\overline{h}}]] + [[\mathcal{E}_g, \mathcal{E}_{\overline{h}}], \mathcal{E}_{g^{-1}}].$$

So either $[\mathcal{E}_g, [\mathcal{E}_{g^{-1}}, \mathcal{E}_{\overline{h}}]] \neq 0$ or $[[\mathcal{E}_g, \mathcal{E}_{\overline{h}}], \mathcal{E}_{g^{-1}}] \neq 0$, contradicting Lemma 4.1-4. From here, $\{\mathcal{E}_g, \mathcal{E}_{g^{-1}}, \mathcal{E}_{\overline{h}}\} = 0$. □

Lemma 4.3. *For any $g_0 \in \sum^1$, if $g \in [g_0]$ and $h, k \in \sum^1 \cup \{1\}$, the following assertions hold.*

1. *If $\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\} \neq 0$, then $h, k, ghk \in [g_0] \cup \{1\}$.*
2. *If $\{\mathcal{E}_h, \mathcal{E}_g, \mathcal{E}_k\} \neq 0$, then $h, k, hgk \in [g_0] \cup \{1\}$.*
3. *If $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_g\} \neq 0$, then $h, k, hkg \in [g_0] \cup \{1\}$.*

Proof. 1. It is easy to see that $[\mathcal{E}_g, \mathcal{E}_h] \neq 0$ for $g \in [g_0]$ and $h \in \sum^1 \cup \{1\}$. By Lemma 4.1-1, one gets $h \sim g$ in the case $h \neq 1$. From here, $h \in [g_0] \cup \{1\}$. To show $k, ghk \in [g_0] \cup \{1\}$. We distinguish two cases.

Case 1. Suppose $ghk = 1$. It is clear that $ghk \in [g_0] \cup \{1\}$. If we have $k \neq 1$ then $gh \in \sum^0$. As $(gh)^{-1} = k$, then $\{g, h, 1\}$ would be a connection from g to k and we conclude that $k \in [g_0] \cup \{1\}$.

Case 2. Suppose $ghk \neq 1$. We treat separately two cases. If $gh \neq 1$, then $gh \in \sum^0$ and so $\{g, h, k\}$ is a connection from g to ghk. Hence $ghk \in [g_0]$. In the case $k \neq 1$, we have $\{g, h, (ghk)^{-1}\}$ is a connection from g to k. So $k \in [g_0]$. Finally, if $gh = 1$, then necessarily $k \in [g_0]$. Indeed, if k is not connected to g, we would have by Lemma 4.2 that $\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\} = \{\mathcal{E}_g, \mathcal{E}_{g^{-1}}, \mathcal{E}_k\} = 0$, a contradiction. From here $ghk = k \in [g_0]$.

2. This can be proved completely analogously to 1.

3. By Leibniz identity, $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_g\} = [[\mathcal{E}_h, \mathcal{E}_k], \mathcal{E}_g] \subset [\mathcal{E}_h, [\mathcal{E}_k, \mathcal{E}_g]] + [[\mathcal{E}_h, \mathcal{E}_g], \mathcal{E}_k]$. From $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_g\} \neq 0$, we obtain either $[\mathcal{E}_h, [\mathcal{E}_k, \mathcal{E}_g]] \neq 0$ or $[[\mathcal{E}_h, \mathcal{E}_g], \mathcal{E}_k] \neq 0$. We treat separately two cases.

Case 1. Suppose $[\mathcal{E}_h, [\mathcal{E}_k, \mathcal{E}_g]] \neq 0$, we will show $h, k, hkg \in [g_0] \cup \{1\}$. First to show $k \in [g_0] \cup \{1\}$. The fact that $[\mathcal{E}_k, \mathcal{E}_g] \neq 0$ implies by Lemma 4.1-1 that $k \sim g$ in the case $k \neq 1$. From here, $k \in [g_0] \cup \{1\}$. Next to show $h \in [g_0] \cup \{1\}$. Indeed, if $h \neq 1$ and suppose g is not connected with h, then h is not connected with k in the case $k \neq 1$. By Lemma 4.1-1, $[\mathcal{E}_h, \mathcal{E}_k] = 0$ whever $k \neq 1$, contradicting $[\mathcal{E}_h, [\mathcal{E}_k, \mathcal{E}_g]] \neq 0$.

Next to show if $h \neq 1$ and in the case $k = 1$, we also get $h \in [g_0]$. Indeed, suppose g is not connected with h, in the case $k = 1$, one has $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_g\} = \{\mathcal{E}_h, \mathcal{E}_1, \mathcal{E}_g\} = [[\mathcal{E}_h, \mathcal{E}_1], \mathcal{E}_g]$. From $[\mathcal{E}_h, \mathcal{E}_1] \subset L_h^0$ and Lemma 4.1-4, one gets $\{\mathcal{E}_h, \mathcal{E}_1, \mathcal{E}_g\} = 0$, a contradiction.

Finally, to show $hkg \in [g_0] \cup \{1\}$. Suppose $hkg = 1$ and so $hkg \in [g_0] \cup \{1\}$. Suppose $hkg \neq 1$, by $[\mathcal{E}_k, \mathcal{E}_g] \neq 0$, $kg \in \sum^1 \cup \{1\}$. If $kg \neq 1$, then $kg \in \sum^1$ and so $\{g, k, h\}$ is a connection from g to gkh, hence $gkh \in [g_0]$. If $kg = 1$, then necessarily $h \in [g_0] \cup \{1\}$. Indeed, if $h \neq 1$ and g is not connected with h, by Lemma 4.1-4, $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_g\} = \{\mathcal{E}_h, \mathcal{E}_{g^{-1}}, \mathcal{E}_g\} = [[\mathcal{E}_h, \mathcal{E}_{g^{-1}}], \mathcal{E}_g] = 0$, contradicting $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_g\} \neq 0$. Therefore, we also have $hkg = h \in [g_0] \cup \{1\}$.

Case 2. If $[[\mathcal{E}_h, \mathcal{E}_g], \mathcal{E}_k] \neq 0$, we will show $h, k, hgk \in [g_0] \cup \{1\}$. First to show $h \in [g_0] \cup \{1\}$. The fact that $[\mathcal{E}_h, \mathcal{E}_g] \neq 0$ implies by Lemma 4.1-1 that $h \sim g$ in the case $h \neq 1$. From here, $h \in [g_0] \cup \{1\}$.

Next to show $k \in [g_0] \cup \{1\}$. Indeed, if $k \neq 1$ and g is not connected with k, then h is not connected with k in the case $h \neq 1$. By Lemma 4.1-1, $[\mathcal{E}_h, \mathcal{E}_k] = 0$ whenever $h \neq 1$, contradicting $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_g\} \neq 0$. Similarly, it is easy to show if $k \neq 1$ and in the case $h = 1$, we can obtain $k \in [g_0]$.

Finally, to show $hkg \in [g_0] \cup \{1\}$. Suppose $hkg = 1$ and so $hkg \in [g_0] \cup \{1\}$. Suppose $hkg \neq 1$, by $[\mathcal{E}_h, \mathcal{E}_g] \neq 0$, one has $hg \in \sum^0 \cup \{1\}$. If $hg \neq 1$, then $hg \in \sum^0$ and so $\{g, h, k\}$ is a connection from g to hkg. Hence $hkg \in [g_0]$. If $hg = 1$, then necessarily $k \in [g_0] \cup \{1\}$. Indeed, if $k \neq 1$ and g is not connected with k, by Lemma 4.1-4, $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_g\} =$

$\{\mathcal{E}_{g^{-1}}, \mathcal{E}_k, \mathcal{E}_g\} = [[\mathcal{E}_{g^{-1}}, \mathcal{E}_k], \mathcal{E}_g] = 0$, contradicting $\{\mathcal{E}_h, \mathcal{E}_k, \mathcal{E}_g\} \neq 0$. Therefore, we also have $hkg = k \in [g_0] \cup \{1\}$. □

Lemma 4.4. *For any $g_0 \in \sum^1$, if g, $k \in [g_0]$, $h \in [g_0] \cup \{1\}$ with $ghk = 1$ and $l, m \in \sum^1 \cup \{1\}$, the following assertions hold.*

1. *If $\{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\}, \mathcal{E}_l, \mathcal{E}_m\} \neq 0$, then $l, m, lm \in [g_0] \cup \{1\}$.*
2. *If $\{\mathcal{E}_l, \{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\}, \mathcal{E}_m\} \neq 0$, then $l, m, lm \in [g_0] \cup \{1\}$.*
3. *If $\{\mathcal{E}_l, \mathcal{E}_m, \{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\}\} \neq 0$, then $l, m, lm \in [g_0] \cup \{1\}$.*

Proof. 1. By (2.2), one gets

$$\begin{aligned} 0 \neq \{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\}, \mathcal{E}_l, \mathcal{E}_m\} \subset \{\mathcal{E}_g, \mathcal{E}_h, \{\mathcal{E}_k, \mathcal{E}_l, \mathcal{E}_m\}\} + \{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_l\}, \mathcal{E}_k, \mathcal{E}_m\} \\ + \{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_m\}, \mathcal{E}_k, \mathcal{E}_l\} + \{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_m\}, \mathcal{E}_l, \mathcal{E}_k\}, \end{aligned}$$

any of the above four summands is nonzero. In order to prove l, m, $lm \in [g_0] \cup \{1\}$, we will consider four cases.

Case 1. Suppose $\{\mathcal{E}_g, \mathcal{E}_h, \{\mathcal{E}_k, \mathcal{E}_l, \mathcal{E}_m\}\} \neq 0$. As $k \in [g_0]$ and $\{\mathcal{E}_k, \mathcal{E}_l, \mathcal{E}_m\} \neq 0$, Lemma 4.3-1 shows that l, m, lm are connected with k in the case of being nonzero roots and so $l, m, klm \in [g_0] \cup \{1\}$. If $klm = 1$, then $lm = k^{-1} \in [g_0]$. If $klm \neq 1$, taking into account $0 \neq \{\mathcal{E}_g, \mathcal{E}_h, \{\mathcal{E}_k, \mathcal{E}_l, \mathcal{E}_m\}\} \subset \{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_{klm}\}$, Lemma 4.3-1 gives us that $ghklm = lm \in [g_0]$.

Case 2. Suppose $\{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_l\}, \mathcal{E}_k, \mathcal{E}_m\} \neq 0$. It is clear that $\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_l\} \neq 0$. As $g \in [g_0]$, by Lemma 4.3-1, one gets $l \in [g_0] \cup \{1\}$. It is obvious that $0 \neq \{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_l\}, \mathcal{E}_k, \mathcal{E}_m\} \subset \{\mathcal{E}_{ghl}, \mathcal{E}_k, \mathcal{E}_m\}$. As $k \in [g_0]$, by Lemma 4.3-2, one gets $m \in [g_0] \cup \{1\}$ and $ghlkm = lm \in [g_0] \cup \{1\}$.

Case 3. Suppose $\{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_m\}, \mathcal{E}_k, \mathcal{E}_l\} \neq 0$. It is easy to see that $\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_m\} \neq 0$. As $g \in [g_0]$, by Lemma 4.3-1, one gets $m \in [g_0] \cup \{1\}$. Note that $0 \neq \{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_m\}, \mathcal{E}_k, \mathcal{E}_l\} \subset \{\mathcal{E}_{ghm}, \mathcal{E}_k, \mathcal{E}_l\}$. As $k \in [g_0]$, by Lemma 4.3-2, one gets $l \in [g_0] \cup \{1\}$ and $ghklm = lm \in [g_0] \cup \{1\}$.

Case 4. Suppose $\{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_m\}, \mathcal{E}_l, \mathcal{E}_k\} \neq 0$. It is clear that $\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_m\} \neq 0$. As $g \in [g_0]$, by Lemma 4.3-1, one gets $m \in [g_0] \cup \{1\}$. Note that $0 \neq \{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_m\}, \mathcal{E}_l, \mathcal{E}_k\} \subset \{\mathcal{E}_{ghm}, \mathcal{E}_l, \mathcal{E}_k\}$. As $k \in [g_0]$, by Lemma 4.3-3, one gets $l \in [g_0] \cup \{1\}$ and $ghmlk = lm \in [g_0] \cup \{1\}$.

2. By Proposition 2.5 (3), we obtain that

$$\begin{aligned} & 0 \neq \{\mathcal{E}_l, \{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\}, \mathcal{E}_m\} \\ & \subset \{\{\mathcal{E}_l, \mathcal{E}_k, \mathcal{E}_m\}, \mathcal{E}_h, \mathcal{E}_g\} + \{\{\mathcal{E}_l, \mathcal{E}_k, \mathcal{E}_m\}, \mathcal{E}_g, \mathcal{E}_h\} \\ & + \{\{\mathcal{E}_l, \mathcal{E}_h, \mathcal{E}_g\}, \mathcal{E}_k, \mathcal{E}_m\} + \{\{\mathcal{E}_l, \mathcal{E}_g, \mathcal{E}_h\}, \mathcal{E}_k, \mathcal{E}_m\} + \{\mathcal{E}_l, \mathcal{E}_k, \{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_m\}\}, \end{aligned}$$

any of the above five summands is nonzero. Suppose $\{\{\mathcal{E}_l, \mathcal{E}_k, \mathcal{E}_m\}, \mathcal{E}_h, \mathcal{E}_g\} \neq 0$, it is obvious $\{\mathcal{E}_l, \mathcal{E}_k, \mathcal{E}_m\} \neq 0$. As $k \in [g_0]$, by Lemma 4.3-2, one gets $l, lkm \in [g_0] \cup \{1\}$. Note that $0 \neq \{\{\mathcal{E}_l, \mathcal{E}_k, \mathcal{E}_m\}, \mathcal{E}_h, \mathcal{E}_g\} \subset \{\mathcal{E}_{lkm}, \mathcal{E}_h, \mathcal{E}_g\}$. As $g \in [g_0]$, by Lemma 4.3-3, one gets $lkmhg = lm \in [g_0] \cup \{1\}$. If $\{\{\mathcal{E}_l, \mathcal{E}_k, \mathcal{E}_m\}, \mathcal{E}_g, \mathcal{E}_h\} \neq 0$, $\{\{\mathcal{E}_l, \mathcal{E}_h, \mathcal{E}_g\}, \mathcal{E}_k, \mathcal{E}_m\} \neq 0$,

$\{\{\mathcal{E}_l, \mathcal{E}_g, \mathcal{E}_h\}, \mathcal{E}_k, \mathcal{E}_m\} \neq 0$, $\{\mathcal{E}_l, \mathcal{E}_k, \{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_m\}\} \neq 0$, a similar argument gives us $l, m, lm \in [g_0] \cup \{1\}$.

3. By Proposition 2.5 (3), we obtain that

$$\begin{aligned} 0 &\neq \{\mathcal{E}_l, \mathcal{E}_m, \{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\}\} \\ &\subset \{\{\mathcal{E}_l, \mathcal{E}_m, \mathcal{E}_k\}, \mathcal{E}_h, \mathcal{E}_g\} + \{\{\mathcal{E}_l, \mathcal{E}_m, \mathcal{E}_k\}, \mathcal{E}_g, \mathcal{E}_h\} \\ &+ \{\{\mathcal{E}_l, \mathcal{E}_h, \mathcal{E}_g\}, \mathcal{E}_m, \mathcal{E}_k\} + \{\{\mathcal{E}_l, \mathcal{E}_g, \mathcal{E}_h\}, \mathcal{E}_m, \mathcal{E}_k\} + \{\mathcal{E}_l, \{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_m\}, \mathcal{E}_k\}, \end{aligned}$$

any of the above five summands is nonzero. Suppose $\{\{\mathcal{E}_l, \mathcal{E}_m, \mathcal{E}_k\}, \mathcal{E}_h, \mathcal{E}_g\} \neq 0$, one easily gets $\{\mathcal{E}_l, \mathcal{E}_m, \mathcal{E}_k\} \neq 0$. As $k \in [g_0]$, by Lemma 4.3-3, one gets $l, lmk \in [g_0] \cup \{1\}$. Note that $0 \neq [[\mathcal{E}_l, \mathcal{E}_m, \mathcal{E}_k], \mathcal{E}_h, \mathcal{E}_g] \subset [\mathcal{E}_{lmk}, \mathcal{E}_h, \mathcal{E}_g]$. As $g \in [g_0]$, by Lemma 4.3-3, one gets $lmkhg = lm \in [g_0] \cup \{1\}$. If $\{\{\mathcal{E}_l, \mathcal{E}_m, \mathcal{E}_k\}, \mathcal{E}_g, \mathcal{E}_h\} \neq 0$, $\{\{\mathcal{E}_l, \mathcal{E}_h, \mathcal{E}_g\}, \mathcal{E}_m, \mathcal{E}_k\} \neq 0$, $\{\{\mathcal{E}_l, \mathcal{E}_g, \mathcal{E}_h\}, \mathcal{E}_m, \mathcal{E}_k\} \neq 0$ or $\{\mathcal{E}_l, \{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_m\}, \mathcal{E}_k\} \neq 0$, a similar argument gives us $l, m, lm \in [g_0] \cup \{1\}$. □

Lemma 4.5. *For any $g_0 \in \sum^1$, if g, $k \in [g_0]$, $h \in [g_0] \cup \{1\}$ with $ghk = 1$ and $\overline{h} \notin [g_0]$, the following assertions hold.*

1. $[\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\}, \mathcal{E}_{\overline{h}}] = 0$.
2. $[\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\}, L^0_{\overline{h}}] = 0$.
3. $\{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\}, \mathcal{E}_1, \mathcal{E}_{\overline{h}}\} = 0$.

Proof. 1. By Leibniz identity, we have

$$[\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\}, \mathcal{E}_{\overline{h}}] = [[[\mathcal{E}_g, \mathcal{E}_h], \mathcal{E}_k], \mathcal{E}_{\overline{h}}] \subset [[\mathcal{E}_g, \mathcal{E}_h], [\mathcal{E}_k, \mathcal{E}_{\overline{h}}]] + [[[\mathcal{E}_g, \mathcal{E}_h], \mathcal{E}_{\overline{h}}], \mathcal{E}_k]. \tag{4.3}$$

Let us consider the first summand in (4.3). As $k \in [g_0]$, for $\overline{h} \notin [g_0]$, by Lemma 4.1-4, one gets $[\mathcal{E}_k, \mathcal{E}_{\overline{h}}] = 0$. Therefore $[[\mathcal{E}_g, \mathcal{E}_h], [\mathcal{E}_k, \mathcal{E}_{\overline{h}}]] = 0$. Let us now consider the second summand in (4.3), it is sufficient to verify that

$$[[[\mathcal{E}_g, \mathcal{E}_h], \mathcal{E}_{\overline{h}}], \mathcal{E}_k] = 0.$$

To do so, we first assert that $[[\mathcal{E}_g, \mathcal{E}_h], \mathcal{E}_{\overline{h}}] = 0$. Indeed, by Leibniz identity, we have

$$[[\mathcal{E}_g, \mathcal{E}_h], \mathcal{E}_{\overline{h}}] \subset [\mathcal{E}_g, [\mathcal{E}_h, \mathcal{E}_{\overline{h}}]] + [[\mathcal{E}_g, \mathcal{E}_{\overline{h}}], \mathcal{E}_h], \tag{4.4}$$

where $g \in [g_0]$, $h \in [g_0] \cup \{1\}$, $\overline{h} \notin [g_0]$. Let us consider the first summand in (4.4), if $h \notin \{1\}$, then $h \in [g_0]$. By Lemma 4.1-1, one gets $[\mathcal{E}_h, \mathcal{E}_{\overline{h}}] = 0$. That is, $[\mathcal{E}_g, [\mathcal{E}_h, \mathcal{E}_{\overline{h}}]] = 0$. If $h = 1$, we have $[\mathcal{E}_h, \mathcal{E}_{\overline{h}}] \subset L^0_{\overline{h}}$. By Lemma 4.1-4, one gets $[\mathcal{E}_g, [\mathcal{E}_h, \mathcal{E}_{\overline{h}}]] \subset [\mathcal{E}_g, L^0_{\overline{h}}] = 0$. Therefore $[\mathcal{E}_g, [\mathcal{E}_h, \mathcal{E}_{\overline{h}}]] = 0$. Let us consider the second summand in (4.4), it is sufficient to verify that $[[\mathcal{E}_g, \mathcal{E}_{\overline{h}}], \mathcal{E}_h] = 0$. Indeed, for $g \in [g_0]$ and $\overline{h} \notin [g_0]$. By Lemma 4.1-4, $[[\mathcal{E}_g, \mathcal{E}_{\overline{h}}], \mathcal{E}_h] = 0$.

2. Note that

$$[\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\}, L^0_{\overline{h}}] \subset [[\mathcal{E}_g, \mathcal{E}_h], [\mathcal{E}_k, L^0_{\overline{h}}]] + [[[\mathcal{E}_g, \mathcal{E}_h], L^0_{\overline{h}}], \mathcal{E}_k]. \tag{4.5}$$

Let us consider the first summand in (4.5). As $k \neq 1$, one gets $[[\mathcal{E}_g, \mathcal{E}_h], [\mathcal{E}_k, L^0_{\overline{h}}]] = 0$ by Lemma 4.1-4. Let us consider the second summand in (4.5). By Leibniz identity, we obtain $[[\mathcal{E}_g, \mathcal{E}_h], L^0_{\overline{h}}] = 0$, so $[[[\mathcal{E}_g, \mathcal{E}_h], L^0_{\overline{h}}], \mathcal{E}_k] = 0$.

3. It is a consequence of Lemma 4.5-1, 2 and

$$\{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\}, \mathcal{E}_1, \mathcal{E}_{\overline{h}}\} \subset [\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\}, [\mathcal{E}_1, \mathcal{E}_{\overline{h}}]] + [[\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\}, \mathcal{E}_{\overline{h}}], \mathcal{E}_1].$$

Thus the lemma follows. □

Definition 4.6. *A Leibniz triple system $\mathcal{E}$ is said to be* ***simple*** *if its product is nonzreo and its only ideals are $\{0\}$, J and $\mathcal{E}$.*

It should be noted that the above definition agrees with the definition of a simple Lie triple system, since $J = \{0\}$ in this case.

Theorem 4.7. *The following assretions hold.*

1. *For any $g_0 \in \sum^1$, the subtriple $\mathcal{E}_{[g_0]} = \mathcal{E}_{1,[g_0]} \oplus V_{[g_0]}$ of $\mathcal{E}$ associated to $[g_0]$ is an ideal of $\mathcal{E}$.*

2. *If $\mathcal{E}$ is simple, then $\sum^1$ has all of its elements connected and*

$$\mathcal{E}_1 = \sum_{g \in \sum^1,\ h \in \sum^1 \cup \{1\}} \{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_{(gh)^{-1}}\}.$$

Proof. 1. Recall that

$$\mathcal{E}_{1,[g_0]} := span_{\mathbb{K}}\{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_{(gh)^{-1}}\} : g \in [g_0], h \in [g_0] \cup \{1\}\} \subset \mathcal{E}_1. \tag{4.6}$$

In order to complete the proof, it is sufficient to show that

$$\{\mathcal{E}_{[g_0]}, \mathcal{E}, \mathcal{E}\} + \{\mathcal{E}, \mathcal{E}_{[g_0]}, \mathcal{E}\} + \{\mathcal{E}, \mathcal{E}, \mathcal{E}_{[g_0]}\} \subset \mathcal{E}_{[g_0]}.$$

We first check that $\{\mathcal{E}_{[g_0]}, \mathcal{E}, \mathcal{E}\} \subset \mathcal{E}_{[g_0]}$. We easily have $\{\mathcal{E}_{1,[g_0]}, \mathcal{E}, \mathcal{E}\} \subset \mathcal{E}_{[g_0]}$. (4.6) together with Lemma 4.4 imply

$$\{\mathcal{E}_{1,[g_0]}, \mathcal{E}_1, \mathcal{E}_g\} + \{\mathcal{E}_{1,[g_0]}, \mathcal{E}_g, \mathcal{E}_1\} + \{\mathcal{E}_{1,[g_0]}, \mathcal{E}_g, \mathcal{E}_h\} \subset \mathcal{E}_{[g_0]},$$

for any $g, h \in \sum^1$. From here,

$$\{\mathcal{E}_{1,[g_0]}, \mathcal{E}, \mathcal{E}\} = \{\mathcal{E}_{1,[g_0]}, \mathcal{E}_1 \oplus (\oplus_{g \in \sum^1} \mathcal{E}_g), \mathcal{E}_1 \oplus (\oplus_{h \in \sum^1} \mathcal{E}_h)\} \subset \mathcal{E}_{[g_0]}. \tag{4.7}$$

Since $V_{[g_0]} := \oplus_{g \in [g_0]} \mathcal{E}_g$, we have by Lemma 4.3 and (4.6) that

$$\{\oplus_{g \in [g_0]} \mathcal{E}_g, \mathcal{E}_1, \mathcal{E}_1\} + \{\oplus_{g \in [g_0]} \mathcal{E}_g, \mathcal{E}_1, \mathcal{E}_h\} + \{\oplus_{g \in [g_0]} \mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_1\} + \{\oplus_{g \in [g_0]} \mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_k\} \subset \mathcal{E}_{g_0},$$

for any $h, k \in \sum^1$. So

$$\{V_{[g_0]}, \mathcal{E}, \mathcal{E}\} = \{\oplus_{g \in [g_0]} \mathcal{E}_g, \mathcal{E}_1 \oplus (\oplus_{h \in \sum^1} \mathcal{E}_h), \mathcal{E}_1 \oplus (\oplus_{k \in \sum^1} \mathcal{E}_k)\} \subset \mathcal{E}_{[g_0]}. \tag{4.8}$$

From (4.7) and (4.8), we have

$$\{\mathcal{E}_{[g_0]}, \mathcal{E}, \mathcal{E}\} = \{\mathcal{E}_{1,[g_0]} \oplus V_{[g_0]}, \mathcal{E}, \mathcal{E}\} \subset \mathcal{E}_{[g_0]},$$

and so $\mathcal{E}_{[g_0]}$ is an ideal of $\mathcal{E}$.

By Lemmas 4.3 and 4.4, a similar argument gives us $[\mathcal{E}, \mathcal{E}_{[g_0]}, \mathcal{E}] \subset \mathcal{E}_{[g_0]}$ and $[\mathcal{E}, \mathcal{E}, \mathcal{E}_{[g_0]}] \subset \mathcal{E}_{[g_0]}$. Consequently, this proves $\mathcal{E}_{[g_0]}$ is an ideal of $\mathcal{E}$.

2. The simplicity of $\mathcal{E}$ implies $\mathcal{E}_{[g_0]} \in \{J, \mathcal{E}\}$ for any $g \in [g_0]$. If $g \in [g_0]$ is such that $\mathcal{E}_{[g_0]} = \mathcal{E}$. Then $[g_0] = \sum^1$. Hence, $\mathcal{E}$ has all its nonzero roots connected. Otherwise, if $\mathcal{E}_{[g_0]} = J$ for any $g \in [g_0]$ then $\mathcal{E}_{[g_0]} = \mathcal{E}_{[\alpha_0]}$ for any $g_0, \alpha_0 \in \sum^1$ and so $[g_0] = \sum^1$, we also conclude that $\mathcal{E}$ has all its nonzero roots connected. □

Theorem 4.8. *For a linear complement U of $span_{\mathbb{K}}\{\{\mathcal{E}_g, \mathcal{E}_h, \mathcal{E}_{(gh)^{-1}}\} : g \in \sum^1, h \in \sum^1 \cup \{1\}\}$ in $\mathcal{E}_1$, we have*

$$\mathcal{E} = U + \sum_{[g] \in \sum^1/\sim} I_{[g]},$$

where any $I_{[g]}$ is one of the ideals described in Theorem 4.7, which also satisfy $[I_{[g]}, \mathcal{E}, I_{[h]}] = [I_{[g]}, I_{[h]}, \mathcal{E}] = [\mathcal{E}, I_{[g]}, I_{[h]}] = 0$ if $[g] \neq [h]$.

Proof. By proposition 3.2, we can consider the quotient set $\sum^1/\sim := \{[g] : g \in \sum^1\}$. We have $I_{[g]}$ is well defined and by Theorem 4.7-1 an ideal of $\mathcal{E}$. Therefore

$$\mathcal{E} = U + \sum_{[g] \in \sum^1/\sim} I_{[g]}.$$

Next, it is sufficient to show that $\{I_{[g]}, \mathcal{E}, I_{[h]}\} = 0$ if $[g] \neq [h]$. Note that,

$$\begin{aligned}
\{I_{[g]}, \mathcal{E}, I_{[h]}\} &= \{\mathcal{E}_{1,[g]} \oplus V_{[g]}, \mathcal{E}_1 \oplus (\oplus_{k \in \sum^1} \mathcal{E}_k), \mathcal{E}_{1,[h]} \oplus V_{[h]}\} \\
&= \{\mathcal{E}_{1,[g]}, \mathcal{E}_1, \mathcal{E}_{1,[h]}\} + \{\mathcal{E}_{1,[g]}, \mathcal{E}_1, V_{[h]}\} + \{\mathcal{E}_{1,[g]}, \oplus_{k \in \sum^1} \mathcal{E}_k, \mathcal{E}_{1,[h]}\} \\
&+ \{\mathcal{E}_{1,[g]}, \oplus_{k \in \sum^1} \mathcal{E}_k, V_{[h]}\} + \{V_{[g]}, \mathcal{E}_1, \mathcal{E}_{1,[h]}\} + \{V_{[g]}, \mathcal{E}_1, V_{[h]}\} \\
&+ \{V_{[g]}, \oplus_{k \in \sum^1} \mathcal{E}_k, \mathcal{E}_{1,[h]}\} + \{V_{[g]}, \oplus_{k \in \sum^1} \mathcal{E}_k, V_{[h]}\}.
\end{aligned}$$

Hence, if $[\alpha] \neq [\beta]$, by Lemmas 4.3 and 4.4, it is easy to see

$$\begin{aligned}
\{\mathcal{E}_{1,[g]}, \mathcal{E}_1, V_{[h]}\} &= \{\mathcal{E}_{1,[g]}, \oplus_{k \in \sum^1} \mathcal{E}_k, V_{[h]}\} = \{V_{[g]}, \mathcal{E}_1, \mathcal{E}_{1,[h]}\} = \{V_{[g]}, \mathcal{E}_1, V_{[h]}\} \\
&= \{V_{[g]}, \oplus_{k \in \sum^1} \mathcal{E}_k, \mathcal{E}_{1,[h]}\} = \{V_{[g]}, \oplus_{k \in \sum^1} \mathcal{E}_k, V_{[h]}\} = 0.
\end{aligned}$$

Next, we will show that $\{\mathcal{E}_{1,[g]}, \oplus_{k \in \sum^1} \mathcal{E}_k, \mathcal{E}_{1,[h]}\} = 0$. Indeed, for $\{\mathcal{E}_{\alpha_1}, \mathcal{E}_{\alpha_2}, \mathcal{E}_{(\alpha_1\alpha_2)^{-1}}\} \in \mathcal{E}_{1,[g]}$ with $\alpha_1 \in [g]$, $\alpha_2 \in [g] \cup \{1\}$ and for $\{\mathcal{E}_{\beta_1}, \mathcal{E}_{\beta_2}, \mathcal{E}_{(\beta_1\beta_2)^{-1}}\} \in \mathcal{E}_{1,[h]}$ with $\beta_1 \in [h]$, $\beta_2 \in [h] \cup \{1\}$, by Proposition 2.5 (3) and Lemma 4.4, one gets

$$\begin{aligned}
&\{\{\mathcal{E}_{\alpha_1}, \mathcal{E}_{\alpha_2}, \mathcal{E}_{(\alpha_1\alpha_2)^{-1}}\}, \oplus_{k \in \sum^1} \mathcal{E}_k, \{\mathcal{E}_{\beta_1}, \mathcal{E}_{\beta_2}, \mathcal{E}_{(\beta_1\beta_2)^{-1}}\}\} \\
&\subset \{\{\{\mathcal{E}_{\alpha_1}, \mathcal{E}_{\alpha_2}, \mathcal{E}_{(\alpha_1\alpha_2)^{-1}}\}, \oplus_{k \in \sum^1} \mathcal{E}_k, \mathcal{E}_{(\beta_1\beta_2)^{-1}}\}, \mathcal{E}_{\beta_2}, \mathcal{E}_{\beta_1}\}
\end{aligned}$$

$$
\begin{aligned}
&+\{\{\{\mathcal{E}_{\alpha_1},\mathcal{E}_{\alpha_2},\mathcal{E}_{(\alpha_1\alpha_2)^{-1}}\},\oplus_{k\in\sum^1}\mathcal{E}_k,\mathcal{E}_{(\beta_1\beta_2)^{-1}}\},\mathcal{E}_{\beta_1},\mathcal{E}_{\beta_2}\}\\
&+\{\{\{\mathcal{E}_{\alpha_1},\mathcal{E}_{\alpha_2},\mathcal{E}_{(\alpha_1\alpha_2)^{-1}}\},\mathcal{E}_{\beta_2}\mathcal{E}_{\beta_1}\},\oplus_{k\in\sum^1}\mathcal{E}_k,\mathcal{E}_{(\beta_1\beta_2)^{-1}}\},\\
&+\{\{\{\mathcal{E}_{\alpha_1},\mathcal{E}_{\alpha_2},\mathcal{E}_{(\alpha_1\alpha_2)^{-1}}\},\mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\},\oplus_{k\in\sum^1}\mathcal{E}_k,\mathcal{E}_{(\beta_1\beta_2)^{-1}}\},\\
&+\{\{\mathcal{E}_{\alpha_1},\mathcal{E}_{\alpha_2},\mathcal{E}_{(\alpha_1\alpha_2)^{-1}}\},\{\mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2},\oplus_{k\in\sum^1}\mathcal{E}_k\},\mathcal{E}_{(\beta_1\beta_2)^{-1}}\}\\
&=0.
\end{aligned}
$$

A similar method gives that $\{\mathcal{E}_{1,[g]},\mathcal{E}_1,\mathcal{E}_{1,[h]}\}=0$. So we prove that $\{I_{[g]},\mathcal{E},I_{[h]}\}=0$ if $[g]\neq[h]$. A similar argument gives that $\{I_{[g]},I_{[h]},\mathcal{E}\}=\{\mathcal{E},I_{[g]},I_{[h]}\}=0$ if $[g]\neq[h]$. $\square$

Definition 4.9. *The* ***annihilator*** *of a Leibniz triple system $\mathcal{E}$ is the set* $\mathrm{Ann}(\mathcal{E})=\{x\in\mathcal{E}:\{x,\mathcal{E},\mathcal{E}\}+\{\mathcal{E},x,\mathcal{E}\}+\{\mathcal{E},\mathcal{E},x\}=0\}$.

Definition 4.10. *We will say that $\mathcal{E}_1$ is* ***tight*** *if* $\mathcal{E}_1=\mathrm{span}_{\mathbb{K}}\{\{\mathcal{E}_g,\mathcal{E}_h,\mathcal{E}_{(gh)^{-1}}\}:g\in\sum^1,h\in\sum^1\cup\{1\}\}$.

Corollary 4.11. *If $\mathrm{Ann}(\mathcal{E})=0$ and $\mathcal{E}_1$ is tight, then $\mathcal{E}$ is the direct sum of the ideals given in Theorem 4.7-1,*

$$\mathcal{E}=\oplus_{[g]\in\sum^1/\sim}I_{[g]}.$$

Proof. From the fact that $\mathcal{E}_1$ is tight, we clearly have

$$\mathcal{E}=\oplus_{[g]\in\sum^1/\sim}I_{[g]}.$$

To finish, we show the direct character of the sum. Given $x\in I_{[g]}\cap(\sum_{[h]\in(\sum^1/\sim)\setminus[g]}I_{[h]})$ we have from the fact $\{I_{[g]},\mathcal{E},I_{[h]}\}=0$ that

$$\{x,\mathcal{E},I_{[g]}\}+\{x,\mathcal{E},\sum_{[h]\in(\sum^1/\sim)\setminus[g]}I_{[h]}\}=0.$$

It implies $\{x,\mathcal{E},\mathcal{E}\}=0$. Using the equations $\{I_{[g]},I_{[h]},\mathcal{E}\}=\{\mathcal{E},I_{[g]},I_{[h]}\}=0$ for $[g]\neq[h]$, one gets $\{\mathcal{E},x,\mathcal{E}\}=\{\mathcal{E},\mathcal{E},x\}=0$. That is, $x\in\mathrm{Ann}(\mathcal{E})=0$. Thus $x=0$. $\square$

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