3-Leibniz bialgebras (3-Lie bialgebras)

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Abstract

In this paper by use of cohomology complex of 3-Leibniz algebras, the definitions of Leibniz bialgebras (and Lie bialgebras) are extended for the case of 3-Leibniz algebras. Many theorems about Leibniz bialgebras are extended and proved for the case of 3-Leibniz bialgebras (3-Lie bialgebras). Moreover a new theorem on the correspondence between 3-Leibniz bialgebra and its associated Leibniz bialgebra is proved. 3-Lie bialgebra as particular case of the 3-Leibniz bialgebra is investigated. Finally, some simple examples are discussed in detail.

1 Introduction

From historical point of view Kurosh introduced the notion on multilinear operator algebra for the first time in Refs[1, 2]. However, for this algebras one of the most important consequences of Jacobi identity is overlooked i.e., the derivation property of ad_x for an element x of the algebra. Later in [3] Filippov introduced the n-Lie algebra which preserves main properties of Jacobi identity. In Ref[4] the n-Lie modules and representation of n-Lie algebras, generalization of Engel's and Lie's theorems and also Cartan's criterion for solvability of n-Lie algebra have been studied by Kasymov. In the past two decades the study of the n-Leibniz, its cohomology [5], their classifications [6, 7] and deformation of n-Leibniz algebras (see for instance [8]) are under investigation (for a review see [9]). Recently the application of 3-Lie algebra in the M theory [10, 11] has led this branch of

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mathematics to receive the most attention among physicists [9]. One of the most applicable objects in mathematical physics especially in integrable systems is the Lie bialgebra. In this manner the generalization of the concept of Lie bialgebra to the n-Lie bialgebra (in general) and especially 3-Lie bialgebra is a good problem from the abstract point of view. Indeed there are some attempts in this directions from the co-algebra point of view (see [13] and [14]). Here we will study this facts by use of the cohomology of n-Leibniz algebra [5] for 3-Leibniz algebra in general and then for 3-Lie algebra in particular[1]. The outline of the paper is as follows.

In section 2 for self containing of the paper we review the basic definitions and theorems about n-Leibniz [18], n-Lie, its associated Leibniz algebra [19] and Leibniz bialgebra [20].

In section 3 after the separation of first, second and third 3-Leibniz algebra (acording to related identity and related their actions with $\mathcal{A}^{\otimes 3}$ be 3-Leibniz module and related cohomology complex) we give the definition of 3-Leibniz bialgebra $(\mathcal{A}, \gamma)$ for different ith 3-Leibniz algebra $\mathcal{A}$. Then as a proposition we show that the dual space $\mathcal{A}^*$ with μ^t is a 3-Leibiz bialgebra. The investigation of 3-Leibniz bialgebra $(\mathcal{A}, \mathcal{A}^*)$ in terms of structure constants of 3-Leibniz algebra $\mathcal{A}$ and $\mathcal{A}^*$ are given in section 4; and at the end of this section some examples of 3-Leibniz bialgebra are obtained by using matrix calculations. As a theorem the correspondence between 3-Leibniz bialgebra and its associated Leibniz bialgebra is given in section 5. In section 6 the definition of 3-Lie bialgebra as a especial case and the reformulation of this definition in terms of structure constants of 3-Lie algebras $\mathcal{A}$ and $\mathcal{A}^*$ is provided. The matrix form of this reformulation is applied for calculation of some low dimensional 3-Lie bialgebras at the end of section 6. Concluding remarks are given in section 7.

2 Basic definitions and theorems

For self containing of the paper let us recall some basic definitions and theorems about n-Leibniz, n-Lie algebras and also Leibniz bialgebra [20].

Definition 2.1 *[18] An n-Leibniz algebra, is a vector space $\mathcal{A}$ equipped with the an n-linear operation $[.,...,.] : \mathcal{A}^{\otimes n} \longrightarrow \mathcal{A}$ such that for all $x_1, \cdots, x_{n-1}$ the map $ad_{(x_1,\cdots,x_{n-1})} : \mathcal{A} \longrightarrow \mathcal{A}$ given by*

$$ad_{(x_1,\cdots,x_{n-1})}(x) = [x, x_1, \cdots, x_{n-1}], \tag{2.1}$$

is a derivation with respect to $[., \cdots, .]$ i.e.

$$[[y_1, \cdots, y_n], x_1, \cdots, x_{n-1}] = \sum_{i=1}^{n} [y_1, \cdots, y_{i-1}, [y_i, x_1, \cdots, x_{n-1}], y_{i+1}, \cdots, y_n], \tag{2.2}$$

[1]Note that the Lie algebra is a special case of Leibniz algebra [17].

above identity is called fundamental identity for n-Leibniz algebra.

Definition 2.2 *[18] A representation of the n-Leibniz algebra $\mathcal{A}$ is a vector space M equipped with n actions of*

$$\rho_j : \mathcal{A}^{\otimes(j-1)} \otimes M \otimes \mathcal{A}^{\otimes(n-j)} \longrightarrow M \qquad j = 1, 2, \cdots, n,$$

satisfying $2n-1$ equations, which are obtained from (2.2) by letting exactly one of the variables $x_1, \cdots, x_{n-1}, y_1, \cdots y_n$ be in M and all the others in $\mathcal{A}$. In other word M is an n-Leibniz module. The notion of representation of an n-Leibniz algebra for $n = 2$ coincides with the corresponding notion representation of Leibniz algebra in [17].

Theorem 2.3 *[19] Let $\mathcal{A}$ be an n-Leibniz algebra and set $\mathcal{G} := \mathcal{A}^{\otimes(n-1)}$ then there is a Leibniz algebra structure on the space $\mathcal{G}$ with the following bracket:*

$$[x_1 \otimes \cdots \otimes x_{n-1}, y_1 \otimes \cdots \otimes y_n] = \sum_{i=1}^{n-1} y_1 \otimes \cdots \otimes y_{i-1} \otimes [x_1, \cdots, x_{n-1}, y_i] \otimes y_{i+1} \otimes \cdots \otimes y_{n-1}, \tag{2.3}$$

where it is called associated Leibniz algebra.

Definition 2.4 *[5, 19] Let $\mathcal{A}$ be an n-Leibniz algebra and $\mathcal{G} := \mathcal{A}^{\otimes(n-1)}$ be its associated Leibniz algebra. The p-cochain of $\mathcal{A}$ $(p \geq 1)$ with coefficients in $\mathcal{A}$ is a linear map from $\mathcal{G}^{\otimes(p-1)} \otimes \mathcal{A}$ to $\mathcal{A}$. Set $\Gamma L^0(\mathcal{A}, \mathcal{A}) := \mathcal{G}$ for the space of 0-cochains and $\Gamma L^p(\mathcal{A}, \mathcal{A})$ for the space of p-cochains. The coboundary map is given by[5]*

$$d^p : \Gamma L^p(\mathcal{A}, \mathcal{A}) \longrightarrow \Gamma L^{p+1}(\mathcal{A}, \mathcal{A})$$
$$(d^0(x_1 \otimes \cdots \otimes x_{n-1}))(x) = -[x_1, \cdots, x_{n-1}, x],$$

$$\begin{aligned}(d^p(\alpha)(X_1, \cdots, X_{p-1}, Y) \;&=\; \sum_{i=1}^{p-2} \sum_{j=i+1}^{p-1} (-1)^i \alpha(X_1, \cdots, \widehat{X_i}, \cdots, X_{j-1}, [X_i, X_j], X_{j+1}, \cdots, X_{p-1}, Y) \\ &+\; \sum_{i=1}^{p-1} (-1)^i \alpha(X_1, \cdots, \widehat{X_i}, \cdots, X_{p-1}, \{X_i, Y\}) \\ &+\; (-1)^p \alpha(X_1, \cdots, X_{p-1}, [y_1, \cdots .y_n]) \\ &+\; \sum_{i=1}^{p-1} (-1)^{i+1} \{X_i, \alpha(X_1, \cdots, \widehat{X_i}, \cdots, X_{p-1}, Y)\} \\ &+\; (-1)^{p+1} \sum_{i=1}^{n} [y_1, \cdots, y_{i-1}, \alpha(X_1, \cdots, X_{p-1}, y_i), \cdots, y_n], \end{aligned} \tag{2.4}$$

where $\alpha \in \Gamma L^p(\mathcal{A}, \mathcal{A})$, $X_i \in \mathcal{G}$ for $i = 1, \cdots, p-1, Y = y_1 \otimes \cdots \otimes y_n \in \mathcal{A}^{\otimes n}$ and for $X \in \mathcal{G}$ of the form $X = x_1 \otimes \cdots \otimes x_{n-1}$ we set $\{X, Y\} := \sum_{i=1}^{n} y_1 \otimes \cdots \otimes y_{i-1} \otimes [x_1, \cdots, x_{n-1}, y_i] \otimes \cdots \otimes y_n$.

Definition 2.5 *[20] A Leibniz bialgebra $(\mathcal{G}, \delta)$ is a (right or left) Leibniz algebra $\mathcal{G}$ with a linear map (cocommutator) $\delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$ such that*

- *δ is a 1-cocycle on $\mathcal{G}$ with values in $\mathcal{G} \otimes \mathcal{G}$*

$$[X, \delta(Y)]_L + [\delta(X), Y]_R - \delta([X, Y]) = 0, \tag{2.5}$$

  where $[.,.]_L$ and $[.,.]_R$ represent the left and right action of $\mathcal{G}$ on $\mathcal{G} \otimes \mathcal{G}$ respectively such that $\mathcal{G} \otimes \mathcal{G}$ becomes a $\mathcal{G}$-module.

- *$\delta^t : \mathcal{G}^* \otimes \mathcal{G}^* \longrightarrow \mathcal{G}^*$ defines a Leibniz bracket on $\mathcal{G}^*$. If we use the notation $[\xi, \eta]_* = \delta^t(\xi \otimes \eta)$, then $\forall \xi, \eta \in \mathcal{G}^*$ and $\forall X \in \mathcal{G}$ we have*

$$\prec [\xi, \eta]_*, X \succ = \prec \delta^t(\xi \otimes \eta), X \succ = \prec \xi \otimes \eta, \delta(X), \succ, \tag{2.6}$$

  where $\prec, \succ$ is the natural pairing between $\mathcal{G}$ and $\mathcal{G}^*$.

Note that with respect to the type of the Leibniz algebra $\mathcal{G}$ and also its actions on the $\mathcal{G} \otimes \mathcal{G}$; the 1-cocycle condition (2.5) can be rewritten in the following forms:

$$\delta([X, Y]) = [X, \delta(Y)]_L + [\delta(X), Y]_R := (ad_X^{(l)} \otimes 1)(\delta(Y)) + (ad_Y^{(r)} \otimes 1)(\delta(X)), \tag{2.7}$$

$$\delta([X, Y]) = [X, \delta(Y)]_L + [\delta(X), Y]_R := (1 \otimes ad_Y^{(r)} + ad_Y^{(r)} \otimes 1)(\delta(X)), \tag{2.8}$$

$$\delta([X, Y]) = [X, \delta(Y)]_L + [\delta(X), Y]_R := (1 \otimes ad_X^{(l)} + ad_X^{(l)} \otimes 1)(\delta(Y)), \tag{2.9}$$

$$\delta([X, Y]) = [X, \delta(Y)]_L + [\delta(X), Y]_R := (1 \otimes ad_X^{(l)})(\delta(Y)) + (1 \otimes ad_Y^{(r)})(\delta(X)). \tag{2.10}$$

that in cases (2.7) and (2.10) $\mathcal{G}$ can be left or right Leibniz algebra and in case (2.8) ((2.9)) $\mathcal{G}$ is a right (left) Leibniz algebra.

Definition 2.6 *[3] An n-Lie algebra $(\mathcal{A}, [., ..., .])$ is a vector space over a field F together with a skew-symmetric n-linear map $[., ..., .] : \mathcal{A}^{\otimes n} \longrightarrow \mathcal{A}$ such that*

$$[x_1, ..., x_{n-1}, [y_1, ..., y_n]] = \sum_{i=1}^{n} [y_1, ..., [x_1, ..., x_{n-1}, y_i], y_{i+1}, ..., y_n],$$

for all $x_1, ..., x_{n-1}, y_1, ..., y_n \in \mathcal{A}$. This condition is called the fundamental identity or the Filippov identity.

Definition 2.7 *[4] If $\mathcal{A}$ is an n-Lie algebra over a field F and V is a vector space over F then a polylinear mapping $\rho : \mathcal{A}^{\otimes n-1} \longrightarrow End(V)$ is said to be a representation of $\mathcal{A}$ in V if the operators $\rho(x_1, ..., x_{n-1}), \forall x_i \in \mathcal{A}$ be skew-symmetric functions of their arguments and satisfy the identities*

$$[\rho(x_1, ..., x_{n-1}), \rho(y_1, ..., y_{n-1})] = \sum_{i=1}^{n-1} \rho(y_1, ..., y_{i-1}, [x_1, ..., x_{n-1}, y_i], y_{i+1}, ...y_{n-1}), \tag{2.11}$$

$$\rho(x_1, ..., x_{n-2}, [y_1, ..., y_n]) = \sum_{i=1}^{n} (-1)^{i+1} \rho(y_1, ..., \hat{y}_i, ..., y_n) \rho(x_1, ..., x_{n-2}, y_i), \tag{2.12}$$

where $x_1, ..., x_{n-1}, y_1, ..., y_{n-1} \in \mathcal{A}$. In this case, V is said to be an (n-Lie) $\mathcal{A}$-module. For example for $n = 3$ we have

$$[\rho(x_1, x_2), \rho(y_1, y_2)] = \rho([x_1, x_2, y_1], y_2) + \rho(y_1, [x_1, x_2, y_2]), \tag{2.13}$$

$$\rho(x_1, [y_1, y_2, y_3]) = \rho(y_2, y_3)\rho(x_1, y_2) - \rho(y_1, y_3)\rho(x_1, y_2) + \rho(y_1, y_2)\rho(x_1, y_3). \tag{2.14}$$

Definition 2.8 *[19] Let $\mathcal{A}$ be a 3-Lie algebra, an $\mathcal{A}$-valued p-cochain is a linear map $\psi : (\mathcal{A} \otimes \mathcal{A})^{\otimes(p-1)} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ and the coboundary operator is given by:*

$$\begin{aligned} d^p\psi(x_1, ...x_{2p+1}) &= \sum_{j=1}^{p} \sum_{k=2j+1}^{2p+1} (-1)^j \psi(x_1, ..., \hat{x}_{j-1}, \hat{x}_j, ..., [x_{2j-1}, x_{2j}, x_k], ..., x_{2p+1}) \\ &+ \sum_{k=1}^{p} [x_{2k-1}, x_{2k}, \psi(x_1, ..., \hat{x}_{2k-1}, \hat{x}_{2k}, ..., x_{2p+1})] \\ &+ (-1)^{p+1}[x_{2p-1}, \psi(x_1, ..., x_{2p-2}, x_{2p}), x_{2p+1}] \\ &+ (-1)^{p+1}[\psi(x_1, ..., x_{2p-1}), x_{2p}, x_{2p+1}] \end{aligned}$$

3 3-Leibniz bialgebras

Since $[., ..., .]$ in n-Leibniz algebra is not antisymmetric hence we define for all $x_1, \cdots, x_n$ in $\mathcal{A}$ the map $ad_{(x_1, \cdots, \widehat{x_i}, \cdots, x_n)} : \mathcal{A} \longrightarrow \mathcal{A}$ as follows:

$$ad_{(x_1, \cdots, \widehat{x_i}, \cdots, x_n)}(x) = [x_1, \cdots, x_{i-1}, x, x_{i+1}, \cdots, x_n], \quad \text{for } i = 1, \cdots, n \tag{3.1}$$

Definition 3.1 *An i-th n-Leibniz algebra, is a vector space $\mathcal{A}$ equipped with an n-linear operation $[., ..., .] : \mathcal{A}^{\otimes n} \longrightarrow \mathcal{A}$ such that map $ad_{(x_1, \cdots, \widehat{x_i}, \cdots, x_n)}$ is a derivation with respect to $[., \cdots, .]$ i.e.*

$$[x_1, \cdots, x_{i-1}, [y_1, \cdots y_n], x_{i+1}, \cdots, x_n] = \sum_{j=1}^{n} [y_1, \cdots, y_{j-1}, [x_1, \cdots, x_{i-1}, y_j, x_{i+1}, \cdots, x_n], y_{j+1}, \cdots, y_n], \tag{3.2}$$

therefore, for any i we have an identity and n-Leibniz algebra that it is called i-th n-Leibniz algebra.

Remark 3.2 *In [18] and [5] $ad_{(x_1, \cdots, \widehat{x_i}, \cdots, x_n)}$ is considered as a derivation with respect to $[., \cdots, .]$ and for the cases $i = n$ and $i = 1$ such that the i-th n-Leibniz algebras for $i = 2, \cdots, n-1$ have not been considered.*

For $n = 3$ we have three 3-Leibniz identities

- If $ad_{(\widehat{x_1}, x_2, x_3)}$ is a derivation with respect to $[., ., .]$ then we have the first 3-Leibniz identity as follows:

$$[[y_1, y_2, y_3], x_2, x_3] = [[y_1, x_2, x_3], y_2, y_3] + [y_1, [y_2, x_2, x_3], y_3] + [y_1, y_2, [y_3, x_2, x_3]]. \tag{3.3}$$

- If $ad_{(x_1,\widehat{x_2},x_3)}$ is a derivation with respect to $[.,.,.]$ then we have the second 3-Leibniz identity as follows:

$$[x_1,[y_1,y_2,y_3],x_3]=[[x_1,y_1,x_3],y_2,y_3]+[y_1,[x_1,y_2,x_3],y_3]+[y_1,y_2,[x_1,y_3,x_3]]. \tag{3.4}$$

- If $ad_{(x_1,x_2,\widehat{x_3})}$ is a derivation with respect to $[.,.,.]$ then we have the third 3-Leibniz identity as follows:

$$[x_1,x_2,[y_1,y_2,y_3]]=[[x_1,x_2,y_1],y_2,y_3]+[y_1,[x_1,x_2,y_2],y_3]+[y_1,y_2,[x_1,x_2,y_3]]. \tag{3.5}$$

Before defining the 3-Leibniz bialgebra let us define special actions such that $\mathcal{A}^{\otimes 3}$ be 3-Leibniz module. We define the following cases of the actions such that $\mathcal{A}^{\otimes 3}$ be 3-Leibniz module.

- If $\mathcal{A}$ is the first 3-Leibniz algebra.

$$\begin{aligned}
&\rho_1:\mathcal{A}^{\otimes 3}\otimes\mathcal{A}^{\otimes 2}\longrightarrow\mathcal{A}^{\otimes 3},\\
&\rho_1(y_1\otimes y_2\otimes y_3,x_2,x_3):=ad^{(3)}_{(\widehat{x_1},x_2,x_3)}(y_1\otimes y_2\otimes y_3),\\
&=(ad_{(\widehat{x_1},x_2,x_3)}\otimes 1\otimes 1+1\otimes ad_{(\widehat{x_1},x_2,x_3)}\otimes 1+1\otimes 1\otimes ad_{(\widehat{x_1},x_2,x_3)})(y_1\otimes y_2\otimes y_3)\\
&=[y_1,x_2,x_3]\otimes y_2\otimes y_3+y_1\otimes[y_2,x_2,x_3]\otimes y_3+y_1\otimes y_2\otimes[y_3,x_2,x_3],\\
&\rho_2:\mathcal{A}\otimes\mathcal{A}^{\otimes 3}\otimes\mathcal{A}\longrightarrow\mathcal{A}^{\otimes 3},\\
&\rho_2(x_1,y_1\otimes y_2\otimes y_3,x_3)=0,\\
&\rho_3:\mathcal{A}^{\otimes 2}\otimes\mathcal{A}^{\otimes 3}\longrightarrow\mathcal{A}^{\otimes 3},\\
&\rho_3(x_1,x_2,y_1\otimes y_2\otimes y_3)=0.
\end{aligned}\tag{3.6}$$

$$\begin{aligned}
&\rho_1:\mathcal{A}^{\otimes 3}\otimes\mathcal{A}^{\otimes 2}\longrightarrow\mathcal{A}^{\otimes 3},\\
&\rho_1(y_1\otimes y_2\otimes y_3,x_2,x_3):=(ad_{(\widehat{x_1},x_2,x_3)}\otimes 1\otimes 1)(y_1\otimes y_2\otimes y_3)=[y_1,x_2,x_3]\otimes y_2\otimes y_3,\\
&\rho_2:\mathcal{A}\otimes\mathcal{A}^{\otimes 3}\otimes\mathcal{A}\longrightarrow\mathcal{A}^{\otimes 3},\\
&\rho_2(x_1,y_1\otimes y_2\otimes y_3,x_3)=(ad_{(x_1,\widehat{x_2},x_3)}\otimes 1\otimes 1)(y_1\otimes y_2\otimes y_3)=[x_1,y_1,x_3]\otimes y_2\otimes y_3,\\
&\rho_3:\mathcal{A}^{\otimes 2}\otimes\mathcal{A}^{\otimes 3}\longrightarrow\mathcal{A}^{\otimes 3},\\
&\rho_3(x_1,x_2,y_1\otimes y_2\otimes y_3)=(ad_{(x_1,x_2,\widehat{x_3})}\otimes 1\otimes 1)(y_1\otimes y_2\otimes y_3)=[x_1,x_2,y_1]\otimes y_2\otimes y_3.
\end{aligned}\tag{3.7}$$

$$\begin{aligned}
&\rho_1:\mathcal{A}^{\otimes 3}\otimes\mathcal{A}^{\otimes 2}\longrightarrow\mathcal{A}^{\otimes 3},\\
&\rho_1(y_1\otimes y_2\otimes y_3,x_2,x_3):=(1\otimes ad_{(\widehat{x_1},x_2,x_3)}\otimes 1)(y_1\otimes y_2\otimes y_3)=y_1\otimes[y_2,x_2,x_3]\otimes y_3,\\
&\rho_2:\mathcal{A}\otimes\mathcal{A}^{\otimes 3}\otimes\mathcal{A}\longrightarrow\mathcal{A}^{\otimes 3},\\
&\rho_2(x_1,y_1\otimes y_2\otimes y_3,x_3)=(1\otimes ad_{(x_1,\widehat{x_2},x_3)}\otimes 1)(y_1\otimes y_2\otimes y_3)=y_1\otimes[x_1,y_2,x_3]\otimes y_3,\\
&\rho_3:\mathcal{A}^{\otimes 2}\otimes\mathcal{A}^{\otimes 3}\longrightarrow\mathcal{A}^{\otimes 3},\\
&\rho_3(x_1,x_2,y_1\otimes y_2\otimes y_3)=(1\otimes ad_{(x_1,x_2,\widehat{x_3})}\otimes 1)(y_1\otimes y_2\otimes y_3)=y_1\otimes[x_1,x_2,y_2]\otimes y_3.
\end{aligned}\tag{3.8}$$

$$\rho_1 : \mathcal{A}^{\otimes 3} \otimes \mathcal{A}^{\otimes 2} \longrightarrow \mathcal{A}^{\otimes 3},$$
$$\rho_1(y_1 \otimes y_2 \otimes y_3, x_2, x_3) := (1 \otimes 1 \otimes ad_{(\widehat{x_1}, x_2, x_3)})(y_1 \otimes y_2 \otimes y_3) = y_1 \otimes y_2 \otimes [y_3, x_2, x_3],$$
$$\rho_2 : \mathcal{A} \otimes \mathcal{A}^{\otimes 3} \otimes \mathcal{A} \longrightarrow \mathcal{A}^{\otimes 3},$$
$$\rho_2(x_1, y_1 \otimes y_2 \otimes y_3, x_3) = (1 \otimes 1 \otimes ad_{(x_1, \widehat{x_2}, x_3)})(y_1 \otimes y_2 \otimes y_3) = y_1 \otimes y_2 \otimes [x_1, y_3, x_3],$$
$$\rho_3 : \mathcal{A}^{\otimes 2} \otimes \mathcal{A}^{\otimes 3} \longrightarrow \mathcal{A}^{\otimes 3},$$
$$\rho_3(x_1, x_2, y_1 \otimes y_2 \otimes y_3) = (1 \otimes 1 \otimes ad_{(x_1, x_2, \widehat{x_3})})(y_1 \otimes y_2 \otimes y_3) = y_1 \otimes y_2 \otimes [x_1, x_2, y_3]. \quad (3.9)$$

Above actions for all $z_1 \otimes z_2 \otimes z_3 \in \mathcal{A}^{\otimes 3}$ and $x_2, x_3, y_1, y_2, y_3 \in \mathcal{A}$ satisfy in the following identities:

$$\rho_1(\rho_1(z_1 \otimes z_2 \otimes z_3, y_2, y_3), x_2, x_3) = \rho_1(\rho_1(z_1 \otimes z_2 \otimes z_3, x_2, x_3), y_2, y_3)$$
$$+\rho_1(z_1 \otimes z_2 \otimes z_3, [y_2, x_2, x_3], y_3) + \rho_1(z_1 \otimes z_2 \otimes z_3, y_2, [y_3, x_2, x_3]), \quad (3.10)$$

$$\rho_1(\rho_2(y_1, z_1 \otimes z_2 \otimes z_3, y_3), x_2, x_3) = \rho_2([y_1, x_2, x_3], z_1 \otimes z_2 \otimes z_3, y_3)$$
$$+\rho_2(y_1, \rho_1(z_1 \otimes z_2 \otimes z_3, x_2, x_3), y_3) + \rho_2(y_1, z_1 \otimes z_2 \otimes z_3, [y_3, x_2, x_3]), \quad (3.11)$$

$$\rho_1(\rho_3(y_1, y_2, z_1 \otimes z_2 \otimes z_3), x_2, x_3) = \rho_3([y_1, x_2, x_3], y_2, z_1 \otimes z_2 \otimes z_3)$$
$$+\rho_3(y_1, [y_2, x_2, x_3], z_1 \otimes z_2 \otimes z_3) + \rho_3(y_1, y_2, \rho_1(z_1 \otimes z_2 \otimes z_3, x_2, x_3)), \quad (3.12)$$

$$\rho_2([y_1, y_2, y_3], z_1 \otimes z_2 \otimes z_3, x_3) = \rho_1(\rho_2(y_1, z_1 \otimes z_2 \otimes z_3, x_3), y_2, y_3)$$
$$+\rho_2(y_1, \rho_2(y_2, z_1 \otimes z_2 \otimes z_3, x_3), y_3) + \rho_3(y_1, y_2, \rho_2(y_3, z_1 \otimes z_2 \otimes z_3, x_3)), \quad (3.13)$$

$$\rho_3([y_1, y_2, y_3], x_2, z_1 \otimes z_2 \otimes z_3) = \rho_1(\rho_3(y_1, x_2, z_1 \otimes z_2 \otimes z_3), y_2, y_3)$$
$$+\rho_2(y_1, \rho_3(y_2, x_2, z_1 \otimes z_2 \otimes z_3), y_3) + \rho_3(y_1, y_2, \rho_3(y_3, x_2, z_1 \otimes z_2 \otimes z_3)). \quad (3.14)$$

- If $\mathcal{A}$ is the second 3-Leibniz algebra then we have the actions $(3.7)-(3.9)$ and following action:

$$\rho_1 : \mathcal{A}^{\otimes 3} \otimes \mathcal{A}^{\otimes 2} \longrightarrow \mathcal{A}^{\otimes 3},$$
$$\rho_1(y_1 \otimes y_2 \otimes y_3, x_2, x_3) = 0,$$
$$\rho_2 : \mathcal{A} \otimes \mathcal{A}^{\otimes 3} \otimes \mathcal{A} \longrightarrow \mathcal{A}^{\otimes 3},$$
$$\rho_2(x_1, y_1 \otimes y_2 \otimes y_3, x_3) := ad^{(3)}_{(x_1, \widehat{x_2}, x_3)}(y_1 \otimes y_2 \otimes y_3)$$
$$= (ad_{(x_1, \widehat{x_2}, x_3)} \otimes 1 \otimes 1 + 1 \otimes ad_{(x_1, \widehat{x_2}, x_3)} \otimes 1 + 1 \otimes 1 \otimes ad_{(x_1, \widehat{x_2}, x_3)})(y_1 \otimes y_2 \otimes y_3)$$
$$= [x_1, y_1, x_3] \otimes y_2 \otimes y_3 + y_1 \otimes [x_1, y_2, x_3] \otimes y_3 + y_1 \otimes y_2 \otimes [x_1, y_3, x_3],$$
$$\rho_3 : \mathcal{A}^{\otimes 2} \otimes \mathcal{A}^{\otimes 3} \longrightarrow \mathcal{A}^{\otimes 3},$$
$$\rho_3(x_1, x_2, y_1 \otimes y_2 \otimes y_3) = 0. \quad (3.15)$$

which satisfies in the following identities:

$$\rho_1(z_1\otimes z_2\otimes z_3,[y_1,y_2,y_3],x_3)=\rho_1(\rho_1(z_1\otimes z_2\otimes z_3,y_1,x_3),y_2,y_3)$$
$$+\rho_2(y_1,\rho_1(z_1\otimes z_2\otimes z_3,y_2,x_3),y_3)+\rho_3(y_1,y_2,\rho_1(z_1\otimes z_2\otimes z_3,y_3,x_3)), \tag{3.16}$$

$$\rho_2(x_1,\rho_1(z_1\otimes z_2\otimes z_3,y_2,y_3),x_3)=\rho_1(\rho_2(x_1,z_1\otimes z_2\otimes z_3,x_3),y_2,y_3)$$
$$+\rho_1(z_1\otimes z_2\otimes z_3,[x_1,y_2,x_3],y_3)+\rho_1(z_1\otimes z_2\otimes z_3,y_2,[x_1,y_3,x_3]), \tag{3.17}$$

$$\rho_2(x_1,\rho_2(y_1,z_1\otimes z_2\otimes z_3,y_3),x_3)=\rho_2([x_1,y_1,x_3],z_1\otimes z_2\otimes z_3,y_3)$$
$$+\rho_2(y_1,\rho_2(x_1,z_1\otimes z_2\otimes z_3,x_3),y_3)+\rho_2(y_1,z_1\otimes z_2\otimes z_3,[x_1,y_3,x_3]), \tag{3.18}$$

$$\rho_2(x_1,\rho_3(y_1,y_2,z_1\otimes z_2\otimes z_3),x_3)=\rho_3([x_1,y_1,x_3],y_2,z_1\otimes z_2\otimes z_3)$$
$$+\rho_3(y_1,[x_1,y_2,x_3],z_1\otimes z_2\otimes z_3)+\rho_3(y_1,y_2,\rho_2(x_1,z_1\otimes z_2\otimes z_3,x_3)), \tag{3.19}$$

$$\rho_3(x_1,[y_1,y_2,y_3],z_1\otimes z_2\otimes z_3)=\rho_1(\rho_3(x_1,y_1,z_1\otimes z_2\otimes z_3),y_2,y_3)$$
$$+\rho_2(y_1,\rho_3(x_1,y_2,z_1\otimes z_2\otimes z_3),y_3)+\rho_3(y_1,y_2,\rho_3(x_1,y_3,z_1\otimes z_2\otimes z_3)). \tag{3.20}$$

- If $\mathcal{A}$ is the third 3-Leibniz algebra then we have the actions $(3.7)-(3.9)$ and following action:

$$\rho_1:\mathcal{A}^{\otimes 3}\otimes\mathcal{A}^{\otimes 2}\longrightarrow\mathcal{A}^{\otimes 3},$$
$$\rho_1(y_1\otimes y_2\otimes y_3,x_2,x_3)=0,$$
$$\rho_2:\mathcal{A}\otimes\mathcal{A}^{\otimes 3}\otimes\mathcal{A}\longrightarrow\mathcal{A}^{\otimes 3},$$
$$\rho_2(x_1,y_1\otimes y_2\otimes y_3,x_3)=0,$$
$$\rho_3(x_1,y_1\otimes y_2\otimes y_3,x_3):=ad^{(3)}_{(x_1,x_2,\widehat{x_3})}(y_1\otimes y_2\otimes y_3)$$
$$=(ad_{(x_1,x_2,\widehat{x_3})}\otimes 1\otimes 1+1\otimes ad_{(x_1,x_2,\widehat{x_3})}\otimes 1+1\otimes 1\otimes ad_{(x_1,x_2,\widehat{x_3})})(y_1\otimes y_2\otimes y_3)$$
$$=[x_1,x_2,y_1]\otimes y_2\otimes y_3+y_1\otimes[x_1,x_2,y_2]\otimes y_3+y_1\otimes y_2\otimes[x_1,x_2,y_3]. \tag{3.21}$$

which satisfies in the following identities:

$$\rho_1(z_1\otimes z_2\otimes z_3,x_2,[y_1,y_2,y_3])=\rho_1(\rho_1(z_1\otimes z_2\otimes z_3,x_2,y_1),y_2,y_3)$$
$$+\rho_2(y_1,\rho_1(z_1\otimes z_2\otimes z_3,x_2,y_2),y_3)+\rho_3(y_1,y_2,\rho_1(z_1\otimes z_2\otimes z_3,x_2,y_3)), \tag{3.22}$$

$$\rho_2(x_1,z_1\otimes z_2\otimes z_3,[y_1,y_2,y_3])=\rho_1(\rho_2(x_1,z_1\otimes z_2\otimes z_3,y_1),y_2,y_3)$$
$$+\rho_2(y_1,\rho_2(x_1,z_1\otimes z_2\otimes z_3,y_2),y_3)+\rho_3(y_1,y_2,\rho_2(x_1,z_1\otimes z_2\otimes z_3,y_3)), \tag{3.23}$$

$$\rho_3(x_1,x_2,\rho_1(z_1\otimes z_2\otimes z_3,y_2,y_3))=\rho_1(\rho_3(x_1,x_2,z_1\otimes z_2\otimes z_3),y_2,y_3)$$
$$+\rho_1(z_1\otimes z_2\otimes z_3,[x_1,x_2,y_2],y_3)+\rho_1(z_1\otimes z_2\otimes z_3,y_2,[x_1,x_2,y_3]), \tag{3.24}$$

$$\rho_3(x_1,x_2,\rho_2(y_1,z_1\otimes z_2\otimes z_3,y_3))=\rho_2([x_1,x_2,y_1],z_1\otimes z_2\otimes z_3,y_3)$$
$$+\rho_2(y_1,\rho_3(x_1,x_2,z_1\otimes z_2\otimes z_3),y_3)+\rho_2(y_1,z_1\otimes z_2\otimes z_3,[x_1,x_2,y_3]), \tag{3.25}$$

$$\rho_3(x_1,x_2,\rho_3(y_1,y_2,z_1\otimes z_2\otimes z_3))=\rho_3([x_1,x_2,y_1],y_2,z_1\otimes z_2\otimes z_3)$$
$$+\rho_3(y_1,[x_1,x_2,y_2],z_1\otimes z_2\otimes z_3)+\rho_3(y_1,y_2,\rho_3(x_1,x_2,z_1\otimes z_2\otimes z_3)). \tag{3.26}$$

In the following definition we suppose that $\mathcal{A}$ be a 3-Leibniz algebra and $\mathcal{G}=\mathcal{A}\otimes\mathcal{A}$ be its associated Leibniz algebra and M is a representation of $\mathcal{A}$. We generalize the p-cochain of $\mathcal{A}$ ($p\geq 1$) with coefficients in $\mathcal{A}$ to p-cochain of $\mathcal{A}$ ($p\geq 1$) with coefficients in M and also the corresponding coboundary map is defined.

Definition 3.3 *Since we have three types of 3-Leibniz algebra we define cohomology complex for them separately.*

1. *If $\mathcal{A}$ be the first 3-Leibniz algebra then $\mathcal{G}=\mathcal{A}\otimes\mathcal{A}$ with the following bracket*

$$[x_1\otimes x_2,y_1\otimes y_2]=[x_1,y_1,y_2]\otimes x_2+x_1\otimes[x_2,y_1,y_2],$$

is a right Leibniz algebra. The p-cochain of $\mathcal{A}$ ($p\geq 1$) with coefficients in M is a linear map from $\mathcal{A}\otimes\mathcal{G}^{\otimes(p-1)}$ to M. Set also $\Gamma L^0(\mathcal{A},M):=\mathcal{A}\otimes M$. The space of p-cochains is denoted by $\Gamma L^p(\mathcal{A},M)$. The coboundary map is given by

$$d^p:\Gamma L^p(\mathcal{A},M)\longrightarrow\Gamma L^{p+1}(\mathcal{A},M)$$
$$(d^0(x\otimes m))(y)=-\rho_3(y,x,m),\quad \forall x,y\in\mathcal{A},\forall m\in M$$

$$\begin{aligned}(d^p(\alpha)(Y,X_1,\cdots,X_{p-1}) &= \sum_{i=1}^{p-2}\sum_{j=i+1}^{p-1}(-1)^i\alpha(Y,X_1,\cdots,\widehat{X_i},\cdots,X_{j-1},[X_i,X_j],X_{j+1},\cdots,X_{p-1})\\ &+\sum_{i=1}^{p-1}(-1)^i\alpha(\{X_i,Y\},X_1,\cdots,\widehat{X_i},\cdots,X_{p-1})\\ &+(-1)^p\alpha([y_1,y_2,y_3],X_1,\cdots,X_{p-1})\\ &+\sum_{i=1}^{p-1}(-1)^{i+1}\rho_1(\alpha(Y,X_1,\cdots,\widehat{X_i},\cdots,X_{p-1}),X_i)\\ &+(-1)^{p+1}\sum_{i=1}^{3}\rho_i(y_1,\cdots,y_{i-1},\alpha(y_i,X_1,\cdots,X_{p-1}),\cdots,y_3),\end{aligned} \tag{3.27}$$

where $X_i\in\mathcal{G}$ for $i=1,\cdots,p-1,Y=y_1\otimes y_2\otimes y_3\in\mathcal{A}^{\otimes 3}$ and for $X\in\mathcal{G}$ of the form $X=x_1\otimes x_2$ we set $\{X,Y\}:=\sum_{i=1}^{3}y_1\otimes\cdots\otimes y_{i-1}\otimes[y_i,x_1,x_2]\otimes\cdots\otimes y_3$. In this case we

have

$$d^1 : \Gamma L^1(\mathcal{A}, M) \longrightarrow \Gamma L^2(\mathcal{A}, M)$$
$$(d^1\alpha)(y_1 \otimes y_2 \otimes y_3) = -\alpha([y_1, y_2, y_3]) + \rho_1(\alpha(y_1), y_2, y_3) + \rho_2(y_1, \alpha(y_2), y_3) + \rho_3(y_1, y_2, \alpha(y_3)) \tag{3.28}$$

$$d^2 : \Gamma L^2(\mathcal{A}, M) \longrightarrow \Gamma L^3(\mathcal{A}, M)$$
$$d^2(\alpha)(y_1 \otimes y_2 \otimes y_3 \otimes x_1 \otimes x_2) = -\alpha([y_1, x_1, x_2], y_2, y_3) - \alpha(y_1, [y_2, x_1, x_2], y_3)$$
$$-\alpha(y_1, y_2, [y_3, x_1, x_2]) + \alpha([y_1, y_2, y_3], x_1, x_2) + \rho_1(\alpha(y_1, y_2, y_3), x_1, x_2) - \rho_1(\alpha(y_1, x_1, x_2), y_2, y_3)$$
$$-\rho_2(y_1, \alpha(y_2, x_1, x_2), y_3) - \rho_3(y_1, y_2, \alpha(y_3, x_1, x_2)). \tag{3.29}$$

2. *If $\mathcal{A}$ be the second 3-Leibniz algebra then $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ with the following bracket*

$$[x_1 \otimes x_2, y_1 \otimes y_2] = [x_1, y_1, x_2] \otimes y_2 + y_1 \otimes [x_1, y_2, x_2],$$

is a left Leibniz algebra and with the following bracket is right Leibniz algebra.

$$[x_1 \otimes x_2, y_1 \otimes y_2] = [y_1, x_1, y_2] \otimes x_2 + x_1 \otimes [y_1, x_2, y_2].$$

The p-cochain of $\mathcal{A}$ ($p \geq 2$) with coefficients in M is a linear map from $\mathcal{A}^{\otimes 3} \otimes \mathcal{G}^{\otimes(p-2)}$ to M. Set also $\Gamma L^0(\mathcal{A}, M) := M \otimes \mathcal{A}$ and $\Gamma L^1(\mathcal{A}, M)$ is a linear map from $\mathcal{A}$ to M. The space of p-cochains is denoted by $\Gamma L^p(\mathcal{A}, M)$. The coboundary map is given by

$$d^p : \Gamma L^p(\mathcal{A}, M) \longrightarrow \Gamma L^{p+1}(\mathcal{A}, M)$$
$$(d^0(m \otimes x))(y) = -\rho_1(m, y, x), \quad \forall x, y \in \mathcal{A}, \forall m \in M$$

$$\begin{aligned}(d^p(\alpha)(Y, X_1, \cdots, X_{p-1}) \;=\; & \sum_{i=1}^{p-2}\sum_{j=i+1}^{p-1}(-1)^i\alpha(Y, X_1, \cdots, \widehat{X_i}, \cdots, X_{j-1}, [X_i, X_j], X_{j+1}, \cdots, X_{p-1}) \\ & + \sum_{i=1}^{p-1}(-1)^i\alpha(\{X_i, Y\}, X_1, \cdots, \widehat{X_i}, \cdots, X_{p-1}) \\ & + (-1)^p\alpha(x_1^1, [y_1, y_2, y_3], x_1^2, X_2, \cdots, X_{p-1}) \\ & + \sum_{i=1}^{p-1}(-1)^{i+1}\rho_2(x_i^1, \alpha(Y, X_1, \cdots, \widehat{X_i}, \cdots, X_{p-1}), x_i^2) \\ & + (-1)^{p+1}\sum_{i=1}^{3}\rho_i(y_1, \cdots, y_{i-1}, \alpha(x_1^1, y_i, x_1^2, \cdots, X_{p-1}), \cdots, y_3),\end{aligned} \tag{3.30}$$

where $X_i \in \mathcal{G}$ for $i = 1, \cdots, p-1, Y = y_1 \otimes y_2 \otimes y_3 \in \mathcal{A}^{\otimes 3}$ and for $X_i \in \mathcal{G}$ of the form $X_i = x_i^1 \otimes x_i^2$ we set $\{X_i, Y\} := \sum_{j=1}^{3} y_1 \otimes \cdots \otimes y_{j-1} \otimes [x_i^1, y_j, x_i^2] \otimes \cdots \otimes y_3$. In this case we

have

$$d^1 : \Gamma L^1(\mathcal{A}, M) \longrightarrow \Gamma L^2(\mathcal{A}, M)$$
$$(d^1\alpha)(y_1 \otimes y_2 \otimes y_3) = -\alpha([y_1, y_2, y_3]) + \rho_1(\alpha(y_1), y_2, y_3) + \rho_2(y_1, \alpha(y_2), y_3) + \rho_3(y_1, y_2, \alpha(y_3)) \tag{3.31}$$

$$d^2 : \Gamma L^2(\mathcal{A}, M) \longrightarrow \Gamma L^3(\mathcal{A}, M)$$
$$\begin{aligned} &d^2(\alpha)(y_1 \otimes y_2 \otimes y_3 \otimes x_1 \otimes x_2) = -\alpha([x_1, y_1, x_2], y_2, y_3) - \alpha(y_1, [x_1, y_2, x_2], y_3) \\ &-\alpha(y_1, y_2, [x_1, y_3, x_2]) + \alpha(x_1, [y_1, y_2, y_3], x_2) + \rho_2(x_1, \alpha(y_1, y_2, y_3), x_2) - \rho_1(\alpha(x_1, y_1, x_2), y_2, y_3) \\ &-\rho_2(y_1, \alpha(x_1, y_2, x_2), y_3) - \rho_3(y_1, y_2, \alpha(x_1, y_3, x_2)). \end{aligned} \tag{3.32}$$

3. *If* $\mathcal{A}$ *be the third* 3*-Leibniz algebra then* $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ *with the following bracket*

$$[x_1 \otimes x_2, y_1 \otimes y_2] = [x_1, x_2, y_1] \otimes y_2 + y_1 \otimes [x_1, x_2, y_2],$$

is a left Leibniz algebra. The p-cochain of $\mathcal{A}$ $(p \geq 1)$ *with coefficients in* M *is a linear map from* $\mathcal{G}^{\otimes(p-1)} \otimes \mathcal{A}$ *to* M*. Set also* $\Gamma L^0(\mathcal{A}, M) := M \otimes \mathcal{A}$*. The space of p-cochains is denoted by* $\Gamma L^p(\mathcal{A}, M)$*. The coboundary map is given by*

$$d^p : \Gamma L^p(\mathcal{A}, M) \longrightarrow \Gamma L^{p+1}(\mathcal{A}, M)$$
$$(d^0(m \otimes x))(y) = -\rho_1(m, x, y), \quad \forall x, y \in \mathcal{A}, \forall m \in M$$

$$\begin{aligned} (d^p(\alpha)(X_1, \cdots, X_{p-1}, Y) &= \sum_{i=1}^{p-2} \sum_{j=i+1}^{p-1} (-1)^i \alpha(X_1, \cdots, \widehat{X_i}, \cdots, X_{j-1}, [X_i, X_j], X_{j+1}, \cdots, X_{p-1}, Y) \\ &+ \sum_{i=1}^{p-1} (-1)^i \alpha(X_1, \cdots, \widehat{X_i}, \cdots, X_{p-1}, \{X_i, Y\}) \\ &+ (-1)^p \alpha(X_1, \cdots, X_{p-1}, [y_1, y_2, y_3]) \\ &+ \sum_{i=1}^{p-1} (-1)^{i+1} \rho_3(X_i, \alpha(X_1, \cdots, \widehat{X_i}, \cdots, X_{p-1}, Y)) \\ &+ (-1)^{p+1} \sum_{i=1}^{3} \rho_i(y_1, \cdots, y_{i-1}, \alpha(X_1, \cdots, X_{p-1}, y_i), \cdots, y_3), \end{aligned} \tag{3.33}$$

where $X_i \in \mathcal{G}$ *for* $i = 1, \cdots, p-1, Y = y_1 \otimes y_2 \otimes y_3 \in \mathcal{A}^{\otimes 3}$ *and for* $X \in \mathcal{G}$ *of the form* $X_i = x_i^1 \otimes x_i^2$ *we set* $\{X_i, Y\} := \sum_{j=1}^{3} y_1 \otimes \cdots \otimes y_{j-1} \otimes [x_i^1, x_i^2, y_j] \otimes \cdots \otimes y_3$*. In this case we*

have

$$d^1 : \Gamma L^1(\mathcal{A}, M) \longrightarrow \Gamma L^2(\mathcal{A}, M)$$

$$(d^1\alpha)(y_1 \otimes y_2 \otimes y_3) = -\alpha([y_1, y_2, y_3]) + \rho_1(\alpha(y_1), y_2, y_3) + \rho_2(y_1, \alpha(y_2), y_3) + \rho_3(y_1, y_2, \alpha(y_3)) \tag{3.34}$$

$$d^2 : \Gamma L^2(\mathcal{A}, M) \longrightarrow \Gamma L^3(\mathcal{A}, M)$$

$$d^2(\alpha)(x_1 \otimes x_2 \otimes y_1 \otimes y_2 \otimes y_3) = -\alpha([x_1, x_2, y_1], y_2, y_3) - \alpha(y_1, [x_1, x_2, y_2], y_3)$$
$$-\alpha(y_1, y_2, [x_1, x_2, y_3]) + \alpha(x_1, x_2 \otimes [y_1, y_2, y_3]) + \rho_3(x_1, x_2, \alpha(y_1, y_2, y_3)) - \rho_1(\alpha(x_1, x_2, y_1), y_2, y_3)$$
$$-\rho_2(y_1, \alpha(x_1, x_2, y_2), y_3) - \rho_3(y_1, y_2, \alpha(x_1, x_2, y_3)). \tag{3.35}$$

In all above cases, since M is a 3-Leibniz module with three actions ρ_1 , ρ_2 and ρ_3 it is easy to see that $d^2 d^1 = 0$ and also the 3-Leibniz identity results that $d^1 d^0 = 0$.

Now, with these actions we define the 3-Leibniz bialgebra.

Definition 3.4 *A 3-Leibniz bialgebra $(\mathcal{A}, \gamma)$ is a (the first or the second or the third) 3-Leibniz algebra $\mathcal{A}$ with a linear map (cocommutator) $\gamma : \mathcal{A} \longrightarrow \mathcal{A}^{\otimes 3}$ such that*

- *γ is a 1-cocycle on $\mathcal{A}$ with values in $\mathcal{A}^{\otimes 3}$ according to (3.6)-(3.9), (3.15) and (3.21).*

$$\gamma[y_1, y_2, y_3] = \rho_1(\gamma(y_1), y_2, y_3) + \rho_2(y_1, \gamma(y_2), y_3) + \rho_3(y_1, y_2, \gamma(y_3)), \tag{3.36}$$

  such that $\mathcal{A}^{\otimes 3}$ be a 3-Leibniz module. In above identity ρ_1,ρ_2 and ρ_3 are three actions that define 3-Leibniz structure on $\mathcal{A}^{\otimes 3}$.

- *$\gamma^t : \mathcal{A}^{*\otimes 3} \longrightarrow \mathcal{A}^*$ defines a 3-Leibniz bracket on $\mathcal{A}^*$.*

If we use the notation

$$[\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3]_* = \gamma^t(\widetilde{x}^1 \otimes \widetilde{x}^2 \otimes \widetilde{x}^3), \quad \forall \widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3 \in \mathcal{A}^* \tag{3.37}$$

then $\forall x \in \mathcal{A}$ we have

$$\prec [\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3]_*, x \succ = \prec \gamma^t(\widetilde{x}^1 \otimes \widetilde{x}^2 \otimes \widetilde{x}^3), X \succ = \prec \widetilde{x}^1 \otimes \widetilde{x}^2 \otimes \widetilde{X}^3, \gamma(x) \succ, \tag{3.38}$$

where $\prec, \succ$ is the natural pairing between $\mathcal{A}$ and $\mathcal{A}^*$.

Note that dependent on the type of the 3-Leibniz algebra $\mathcal{A}$ and also its actions ρ_1 , ρ_2 and ρ_3 such that $\mathcal{A}^{\otimes 3}$ be a 3-Leibniz module, the 1-cocycle condition (3.36) can be rewritten in one of the following forms:

$$\gamma([x_1, x_2, x_3]) = ad^{(3)}_{(\widehat{x_1}, x_2, x_3)}\gamma(x_1), \tag{3.39}$$

$$\gamma([x_1, x_2, x_3]) = ad^{(3)}_{(x_1, \widehat{x_2}, x_3)}\gamma(x_2), \tag{3.40}$$

$$\gamma([x_1, x_2, x_3]) = ad^{(3)}_{(x_1, x_2, \widehat{x_3})}\gamma(x_3), \tag{3.41}$$

$$\begin{aligned}\gamma([x_1, x_2, x_3]) &= (ad_{(\widehat{x_1}, x_2, x_3)} \otimes 1 \otimes 1)(\gamma(x_1)) \\ &+(ad_{(x_1, \widehat{x_2}, x_3)} \otimes 1 \otimes 1)(\gamma(x_2)) + (ad_{(x_1, x_2, \widehat{y_3})} \otimes 1 \otimes 1)(\gamma(x_3)),\end{aligned} \tag{3.42}$$

$$\begin{aligned}\gamma([x_1, x_2, x_3]) &= (1 \otimes ad_{(\widehat{x_1}, x_2, x_3)} \otimes 1)(\gamma(x_1)) \\ &+(1 \otimes ad_{(x_1, \widehat{x_2}, x_3)} \otimes 1)(\gamma(x_2)) + (1 \otimes ad_{(x_1, x_2, \widehat{x_3})} \otimes 1)(\gamma(x_3)),\end{aligned} \tag{3.43}$$

$$\begin{aligned}\gamma([x_1, x_2, x_3]) &= (1 \otimes 1 \otimes ad_{(\widehat{x_1}, x_2, x_3)})(\gamma(x_1)) \\ &+(1 \otimes 1 \otimes ad_{(x_1, \widehat{x_2}, x_3)})(\gamma(x_2)) + (1 \otimes 1 \otimes ad_{(x_1, x_2, \widehat{x_3})})(\gamma(x_3)).\end{aligned} \tag{3.44}$$

In (3.39) , (3.40) ,(3.41) $\mathcal{A}$ is the first and the second and the third 3-Leibniz algebra respectively. In (3.42) , (3.43) , (3.44) $\mathcal{A}$ can be 3-Leibniz algebra from various three types. According to which of the above conditions holds for the 1-cocycle; the 3-Leibniz algebra $\mathcal{A}^*$ can be the first and the second and the third 3-Leibniz algebra. We investigate this subject as follows:

Proposition 3.5 *If $(\mathcal{A}, \gamma)$ is a 3-Leibniz bialgebra, and μ is the 3-Leibniz bracket of $\mathcal{A}$, then $(\mathcal{A}^*, \mu^t)$ is a 3-Leibniz bialgebra, where γ^t is the 3-Leibniz bracket of $\mathcal{A}^*$.*

Proof.

- If we use from (3.39) the value of $\gamma([x_1, x_2, x_3])$, then from (3.38) we have

$$\begin{aligned}\prec [\xi_1, \xi_2, \xi_3]_*, [x_1, x_2, x_3] \succ &= \prec \xi_1 \otimes \xi_2 \otimes \xi_3, \gamma[x_1, x_2, x_3] \succ \\ &= \prec \xi_1 \otimes \xi_2 \otimes \xi_3, (ad_{(\widehat{x_1}, x_2, x_3)} \otimes 1 \otimes 1)(\gamma(x_1)) \succ \\ &\quad + \prec \xi_1 \otimes \xi_2 \otimes \xi_3, (1 \otimes ad_{(\widehat{x_1}, x_2, x_3)} \otimes 1)(\gamma(x_1)) \succ \\ &\quad + \prec \xi_1 \otimes \xi_2 \otimes \xi_3, (1 \otimes 1 \otimes ad_{(\widehat{x_1}, x_2, x_3)})(\gamma(x_1)) \succ .\end{aligned} \tag{3.45}$$

We now define the coadjoint representation of a 3-Leibniz algebra on the dual vector space. Let $\mathcal{A}$ be a 3-Leibniz algebra and let $\mathcal{A}^*$ be its dual vector space, then for $x_1, x_2, x_3 \in \mathcal{A}$ we have

$$ad^*_{(\widehat{x_1},x_2,x_3)} : \mathcal{A}^* \longrightarrow \mathcal{A}^*,$$
$$\prec ad^*_{(\widehat{x_1},x_2,x_3)}\xi, y \succ = - \prec \xi, ad_{(\widehat{x_1},x_2,x_3)}y \succ = - \prec \xi, [y, x_2, x_3] \succ, \tag{3.46}$$
$$ad^*_{(x_1,\widehat{x_2},x_3)} : \mathcal{A}^* \longrightarrow \mathcal{A}^*,$$
$$\prec ad^*_{(x_1,\widehat{x_2},x_3)}\xi, y \succ = - \prec \xi, ad_{(x_1,\widehat{x_2},x_3)}y \succ = - \prec \xi, [x_1, y, x_3] \succ, \tag{3.47}$$
$$ad^*_{(x_1,x_2,\widehat{x_3})} : \mathcal{A}^* \longrightarrow \mathcal{A}^*,$$
$$\prec ad^*_{(x_1,x_2,\widehat{x_3})}\xi, y \succ = - \prec \xi, ad_{(x_1,x_2,\widehat{x_3})}y \succ = - \prec \xi, [x_1, x_2, y] \succ . \tag{3.48}$$

Using these relations, (3.45) can be rewritten as

$$\prec [\xi_1, \xi_2, \xi_3]_*, [x_1, x_2, x_3] \succ = - \prec [ad^*_{(\widehat{x_1},x_2,x_3)}\xi_1, \xi_2, \xi_3]_*, x_1 \succ$$
$$- \prec [\xi_1, ad^*_{(\widehat{x_1},x_2,x_3)}\xi_2, \xi_3]_*, x_1 \succ - \prec [\xi_1, \xi_2, ad^*_{(\widehat{x_1},x_2,x_3)}\xi_3]_*, x_1 \succ . \tag{3.49}$$

In the similar way as above; for any $\xi_1, \xi_2, \xi_3 \in \mathcal{A}^*$ we have

$$ad^*_{(\widehat{\xi_1},\xi_2,\xi_3)} : \mathcal{A} \longrightarrow \mathcal{A} \cong \mathcal{A}^{**},$$
$$\prec ad^*_{(\widehat{\xi_1},\xi_2,\xi_3)}x, \eta \succ = - \prec x, ad_{(\widehat{\xi_1},\xi_2,\xi_3)}\eta \succ = - \prec x, [\eta, \xi_2, \xi_3]_* \succ, \tag{3.50}$$
$$ad^*_{(\xi_1,\widehat{\xi_2},\xi_3)} : \mathcal{A} \longrightarrow \mathcal{A} \cong \mathcal{A}^{**},$$
$$\prec ad^*_{(\xi_1,\widehat{\xi_2},\xi_3)}x, \eta \succ = - \prec x, ad_{(\xi_1,\widehat{\xi_2},\xi_3)}\eta \succ = - \prec x, [\xi_1, \eta, \xi_3]_* \succ, \tag{3.51}$$
$$ad^*_{(\xi_1,\xi_2,\widehat{\xi_3})} : \mathcal{A} \longrightarrow \mathcal{A} \cong \mathcal{A}^{**},$$
$$\prec ad^*_{(\xi_1,\xi_2,\widehat{\xi_3})}x, \eta \succ = - \prec x, ad_{(\xi_1,\xi_2,\widehat{\xi_3})}\eta \succ = - \prec x, [\xi_1, \xi_2, \eta]_* \succ . \tag{3.52}$$

By using these relations, (3.49) can be rewritten as

$$\begin{aligned} \prec [\xi_1, \xi_2, \xi_3]_*, [x_1, x_2, x_3] \succ &= \prec ad^*_{(\widehat{x_1},x_2,x_3)}\xi_1, ad^*_{(\widehat{\xi_1},\xi_2,\xi_3)}x_1 \succ \\ &+ \prec ad^*_{(\widehat{x_1},x_2,x_3)}\xi_2, ad^*_{(\xi_1,\widehat{\xi_2},\xi_3)}x_1 \succ \\ &+ \prec ad^*_{(\widehat{x_1},x_2,x_3)}\xi_3, ad^*_{(\xi_1,\xi_2,\widehat{\xi_3})}x_1 \succ, \end{aligned} \tag{3.53}$$

or

$$\begin{aligned} \prec [\xi_1, \xi_2, \xi_3]_*, \mu(x_1 \otimes x_2 \otimes x_3) \succ &= \prec \xi_1, [ad^*_{(\widehat{\xi_1},\xi_2,\xi_3)}x_1, x_2, x_3] \succ \\ &- \prec \xi_2, [ad^*_{(\xi_1,\widehat{\xi_2},\xi_3)}x_1, x_2, x_3] \succ \\ &- \prec \xi_3, [ad^*_{(\xi_1,\xi_2,\widehat{\xi_3})}x_1, x_2, x_3] \succ, \end{aligned} \tag{3.54}$$

where μ is the 3-Leibniz bracket on $\mathcal{A}$ and μ^t is cocommutator on $\mathcal{A}^*$ i.e. $\mu^t : \mathcal{A}^* \longrightarrow$

$\mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}^*$. Therefore, we have

$$\begin{aligned} \prec \mu^t[\xi_1, \xi_2, \xi_3]_*, x_1 \otimes x_2 \otimes x_3 \succ &= \prec (ad_{(\widehat{\xi_1}, \xi_2, \xi_3)} \otimes 1 \otimes 1)(\mu^t(\xi_1)), x_1 \otimes x_2 \otimes x_3 \succ \\ &+ \prec (ad_{(\xi_1, \widehat{\xi_2}, \xi_3)} \otimes 1 \otimes 1)(\mu^t(\xi_2)), x_1 \otimes x_2 \otimes x_3 \succ \\ &+ \prec (ad_{(\xi_1, \xi_2, \widehat{\xi_3})} \otimes 1 \otimes 1)(\mu^t(\xi_3)), x_1 \otimes x_2 \otimes x_3 \succ, \quad (3.55) \end{aligned}$$

or

$$\begin{aligned} \mu^t[\xi_1, \xi_2, \xi_3]_* &= (ad_{(\widehat{\xi_1}, \xi_2, \xi_3)} \otimes 1 \otimes 1)(\mu^t(\xi_1)) + (ad_{(\xi_1, \widehat{\xi_2}, \xi_3)} \otimes 1 \otimes 1)(\mu^t(\xi_2)) \\ &+ (ad_{(\xi_1, \xi_2, \widehat{\xi_3})} \otimes 1 \otimes 1)(\mu^t(\xi_3)). \quad (3.56) \end{aligned}$$

But, this relation is the 1-cocycle condition (3.42) for $(\mathcal{A}^*, \mu^t)$ such that it shows $\mathcal{A}^{*\otimes 3}$ is a 3-Leibniz module on $\mathcal{A}^*$; i.e. $\mathcal{A}^*$ *can be* 3*-Leibniz algebra from various three types.*

- In the same way, if one uses from (3.40) for the value of $\gamma([x_1, x_2, x_3])$, then by assuming that $\mathcal{A}$ *is the second* 3*-Leibniz algebra*, we have

$$\begin{aligned} \mu^t[\xi_1, \xi_2, \xi_3]_* &= (1 \otimes ad_{(\widehat{\xi_1}, \xi_2, \xi_3)} \otimes 1)(\mu^t(\xi_1)) + (1 \otimes ad_{(\xi_1, \widehat{\xi_2}, \xi_3)} \otimes 1)(\mu^t(\xi_2)) \\ &+ (1 \otimes ad_{(\xi_1, \xi_2, \widehat{\xi_3})} \otimes 1)(\mu^t(\xi_3)), \quad (3.57) \end{aligned}$$

instead of (3.56), such that this relation is the 1-cocycle condition (3.43) for $(\mathcal{A}^*, \mu^t)$, where it shows $\mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}^*$ is a 3-Leibniz module on $\mathcal{A}^*$ and $\mathcal{A}^*$ *can be* 3*-Leibniz algebra from various three types.*

On the other hand, for the third 3-Leibniz algebra $(\mathcal{A}, \mu)$ when one uses (3.41) for the value $\gamma([x_1, x_2, x_3])$ we have

$$\begin{aligned} \mu^t[\xi_1, \xi_2, \xi_3]_* &= (1 \otimes 1 \otimes ad_{(\widehat{\xi_1}, \xi_2, \xi_3)})(\mu^t(\xi_1)) + (1 \otimes 1 \otimes ad_{(\xi_1, \widehat{\xi_2}, \xi_3)})(\mu^t(\xi_2)) \\ &+ (1 \otimes 1 \otimes ad_{(\xi_1, \xi_2, \widehat{\xi_3})})(\mu^t(\xi_3)), \quad (3.58) \end{aligned}$$

instead of (3.56), and this shows that μ^t is a 1-cocycle condition (3.44) for $(\mathcal{A}^*, \mu^t)$ such that it shows $\mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}^*$ is a 3-Leibniz module on $\mathcal{A}^*$ and $\mathcal{A}^*$ *can be* 3*-Leibniz algebra from various three types.*

- In the same way, if one uses from (3.42) for the value of $\gamma([x_1, x_2, x_3])$, then by assuming that $\mathcal{A}$ *is* 3*-Leibniz algebra from various three types*, we have

$$\mu^t[\xi_1, \xi_2, \xi_3]_* = ad^{(3)}_{(\widehat{\xi_1}, \xi_2, \xi_3)} \mu^t(x_1), \quad (3.59)$$

and this shows that μ^t is a 1-cocycle condition (3.39) for $(\mathcal{A}^*, \mu^t)$ such that it shows $\mathcal{A}^* \otimes \mathcal{A}^* \otimes \mathcal{A}^*$ is a 3-Leibniz module on $\mathcal{A}^*$ and $\mathcal{A}^*$ *is the first* 3*-Leibniz algebra.*

- Using (3.43) for the value of $\gamma([x_1, x_2, x_3])$, then by assuming that $\mathcal{A}$ *is* 3*-Leibniz algebra from various three types*, we have

$$\mu^t[\xi_1, \xi_2, \xi_3]_* = ad^{(3)}_{(\xi_1, \widehat{\xi_2}, \xi_3)}\mu^t(x_2), \tag{3.60}$$

and this shows that μ^t is a 1-cocycle condition (3.40) for $(\mathcal{A}^*, \mu^t)$ such that it shows $\mathcal{A}^*\otimes\mathcal{A}^*\otimes\mathcal{A}^*$ is a 3-Leibniz module on $\mathcal{A}^*$ and $\mathcal{A}^*$ *is the second* 3*-Leibniz algebra.*

- Finally, if one uses from (3.44) for the value of $\gamma([x_1, x_2, x_3])$, then by assuming that $\mathcal{A}$ *is* 3*-Leibniz algebra from various three types*, we have

$$\mu^t[\xi_1, \xi_2, \xi_3]_* = ad^{(3)}_{(\xi_1, \xi_2, \widehat{\xi_3})}\mu^t(x_3), \tag{3.61}$$

and this shows that μ^t is a 1-cocycle condition (3.41) for $(\mathcal{A}^*, \mu^t)$ such that it shows $\mathcal{A}^*\otimes\mathcal{A}^*\otimes\mathcal{A}^*$ is a 3-Leibniz module on $\mathcal{A}^*$ and $\mathcal{A}^*$ *is the third* 3*-Leibniz algebra.*

Therefore, a 3-Leibniz bialgebra $(\mathcal{A}, \gamma)$ can also be denoted by $(\mathcal{A}, \mathcal{A}^*)$. ∎
There are no Manin triple for 3-Leibniz bialgebras.

4 3-Leibniz bialgebra in terms of structure constants; some examples

In this section, we obtain some examples of 3-Leibniz bialgebras. For these proposes we first rewrite the 1-cocycle conditions (3.39)-(3.41) in terms of structure constants of the 3-Leibniz algebra $\mathcal{A}$ and $\mathcal{A}^*$. If we choose $(\{x_i\}, f_{ijk}{}^m)$ and $(\{\widetilde{x}^i\}, \widetilde{f}^{ijk}{}_m)$ as the basis and structure constants of 3-Leibniz algebra $\mathcal{A}$ and $\mathcal{A}^*$ respectively; then we have the commutation relations as follows

$$[x_i, x_j, x_k] = f_{ijk}{}^m x_m, \qquad [\widetilde{x}^i, \widetilde{x}^j, \widetilde{x}^k]_* = \widetilde{f}^{ijk}{}_m \widetilde{x}^m. \tag{4.1}$$

Using (3.38) we have

$$\begin{aligned} \prec \widetilde{x}^j \otimes \widetilde{x}^k \otimes \widetilde{x}^m, \gamma(x^i) \succ &= \prec \gamma^t(\widetilde{x}^j \otimes \widetilde{x}^k \otimes \widetilde{x}^m), x_i \succ = \prec [\widetilde{x}^j, \widetilde{x}^k, \widetilde{x}^m]_*, x_i \succ \\ &= \prec \widetilde{f}^{jkm}{}_n \widetilde{x}^n, x_i \succ = \widetilde{f}^{jkm}{}_i, \end{aligned} \tag{4.2}$$

namely

$$\gamma(x_i) = \widetilde{f}^{jkm}{}_i x_j \otimes x_k \otimes x_m \tag{4.3}$$

Now using structure constants of $\mathcal{A}$ and (4.3) in the 1-cocycle conditions (3.39)-(3.41) we obtain the following relations respectively:

$$f_{isn}{}^p \widetilde{f}^{jkm}{}_p = \widetilde{f}^{j'km}{}_i f_{j'sn}{}^j + \widetilde{f}^{jk'm}{}_i f_{k'sn}{}^k + \widetilde{f}^{jkm'}{}_i f_{m'sn}{}^m, \tag{4.4}$$

$$f_{isn}{}^p \widetilde{f}^{jkm}{}_p = \widetilde{f}^{j'km}{}_s f_{ij'n}{}^j + \widetilde{f}^{jk'm}{}_s f_{ik'n}{}^k + \widetilde{f}^{jkm'}{}_s f_{im'n}{}^m, \tag{4.5}$$

$$f_{isn}{}^p \widetilde{f}^{jkm}{}_p = \widetilde{f}^{j'km}{}_n f_{isj'}{}^j + \widetilde{f}^{jk'm}{}_n f_{isk'}{}^k + \widetilde{f}^{jkm'}{}_n f_{ism'}{}^m. \tag{4.6}$$

Note that similar to the Lie bialgebras case [15] one can use three relations as a definition of 3-Leibniz bialgebra.

Definition 4.1 *Two 3-Leibniz algebra $\mathcal{A}$ and $\mathcal{A}^*$ constract a 3-Leibniz bialgebra if their structure constants satisfy in relations* $(4.4)-(4.6)$.

To use these relations in the calculations we must first translate the tensor form of these relations to the matrix forms by using the following adjoint representations

$$f_{isn}{}^{p} = (\chi_{is})_n{}^{p} = (Y_i{}^{p})_{sn} = f'_{sin}{}^{p} = (\chi'_{si})_n{}^{p} = (Y'_s{}^{p})_{in} \tag{4.7}$$

$$\widetilde{f}^{jkm}{}_{p} = (\widetilde{\chi}^{jk})^m{}_{p} = (\widetilde{Y}^j{}_{p})^{km} = \widetilde{f'}^{kjm}{}_{p} = (\widetilde{\chi'}^{kj})^m{}_{p} = (\widetilde{Y'}^k{}_{p})^{jm} \tag{4.8}$$

Then, relations $(4.4)-(4.6)$ have the following matrix forms respectively:

$$(\chi_{is})(\widetilde{\chi}^{jk})^t = (Y'_s{}^{j})^t(\widetilde{Y'}^k{}_{i}) + (Y'_s{}^{k})^t(\widetilde{Y}^j{}_{i}) + (\widetilde{\chi}^{jk})^{m'}{}_{i}(\chi_{m's}), \tag{4.9}$$

$$(\chi_{is})(\widetilde{\chi}^{jk})^t = (Y_i{}^{j})^t(\widetilde{Y'}^k{}_{s}) + (Y_i{}^{k})^t(\widetilde{Y}^j{}_{s}) + (\widetilde{\chi}^{jk})^{m'}{}_{s}(\chi_{im'}), \tag{4.10}$$

$$(\chi_{is})(\widetilde{\chi}^{jk})^t = (\chi_{is})_{j'}{}^{j}(\widetilde{\chi}^{j'k})^t + (\chi_{is})_{k'}{}^{k}(\widetilde{\chi}^{j'k})^t + (\widetilde{\chi}^{jk})^t(\chi_{is}), \tag{4.11}$$

where in the above relations t stands for transpose of a matrix. On the other hand, identities (3.3) - (3.5) for 3-Leibniz algebra $\mathcal{A}^*$ in terms of structure constant as follows:

$$\widetilde{f}^{ijk}{}_{p}\widetilde{f}^{psm}{}_{n} = \widetilde{f}^{ism}{}_{p}\widetilde{f}^{pjk}{}_{n} + \widetilde{f}^{jsm}{}_{p}\widetilde{f}^{ipk}{}_{n} + \widetilde{f}^{ksm}{}_{p}\widetilde{f}^{ijp}{}_{n}, \tag{4.12}$$

$$\widetilde{f}^{jks}{}_{p}\widetilde{f}^{ipm}{}_{n} = \widetilde{f}^{ijm}{}_{p}\widetilde{f}^{pks}{}_{n} + \widetilde{f}^{ikm}{}_{p}\widetilde{f}^{jps}{}_{n} + \widetilde{f}^{ism}{}_{p}\widetilde{f}^{jkp}{}_{n}, \tag{4.13}$$

$$\widetilde{f}^{ksm}{}_{p}\widetilde{f}^{ijp}{}_{n} = \widetilde{f}^{ijk}{}_{p}\widetilde{f}^{psm}{}_{n} + \widetilde{f}^{ijs}{}_{p}\widetilde{f}^{kpm}{}_{n} + \widetilde{f}^{ijm}{}_{p}\widetilde{f}^{ksp}{}_{n}, \tag{4.14}$$

where we have the following matrix form for these relations respectively:

$$(\widetilde{\chi}^{ij})(\widetilde{Y'}^s{}_{n}) = (\widetilde{Y'}^j{}_{n})^t(\widetilde{\chi}^{is})^t + (\widetilde{Y}^i{}_{n})^t(\widetilde{\chi}^{js})^t + (\widetilde{\chi}^{ij})^p{}_{n}(\widetilde{Y'}^s{}_{p}), \tag{4.15}$$

$$(\widetilde{\chi}^{jk})(\widetilde{Y}^i{}_{n}) = (\widetilde{Y'}^k{}_{n})^t(\widetilde{\chi}^{ij})^t + (\widetilde{Y}^j{}_{n})^t(\widetilde{\chi}^{ik})^t + (\widetilde{\chi}^{jk})^p{}_{m}(\widetilde{Y}^i{}_{p}), \tag{4.16}$$

$$(\widetilde{\chi}^{ks})(\widetilde{\chi}^{ij}) = (\widetilde{\chi}^{ij})^k{}_{p}(\widetilde{\chi}^{ps}) + (\widetilde{\chi}^{ij})^s{}_{p}(\widetilde{\chi}^{kp}) + (\widetilde{\chi}^{ij})(\widetilde{\chi}^{ks}). \tag{4.17}$$

Now, one can use the relations (4.9)-(4.11) and (4.15)-(4.17) for calculation of the dual 3-Leibniz algebra $\mathcal{A}^*$. According to the type of 3-Leibniz algebras $\mathcal{A}$ and $\mathcal{A}^*$, we must solve the following equations:

- If $\mathcal{A}$ and $\mathcal{A}^*$ are both the first 3-Leibniz algebra

$$\begin{aligned}
&(\chi_{is})(\widetilde{\chi}^{jk})^t - (Y'_s{}^{j})^t(\widetilde{Y'}^k{}_{i}) - (Y'_s{}^{k})^t(\widetilde{Y}^j{}_{i}) - (\widetilde{\chi}^{jk})^{m'}{}_{i}(\chi_{m's}) = 0,\\
&(\widetilde{\chi}^{ij})(\widetilde{Y'}^s{}_{n}) - (\widetilde{Y'}^j{}_{n})^t(\widetilde{\chi}^{is})^t - (\widetilde{Y}^i{}_{n})^t(\widetilde{\chi}^{js})^t - (\widetilde{\chi}^{ij})^p{}_{n}(\widetilde{Y'}^s{}_{p}) = 0.
\end{aligned} \tag{4.18}$$

- If $\mathcal{A}$ and $\mathcal{A}^*$ are the first and the second 3-Leibniz algebra respectively

$$\begin{aligned}
&(\chi_{is})(\widetilde{\chi}^{jk})^t - (Y'_s{}^{j})^t(\widetilde{Y'}^k{}_{i}) - (Y'_s{}^{k})^t(\widetilde{Y}^j{}_{i}) - (\widetilde{\chi}^{jk})^{m'}{}_{i}(\chi_{m's}) = 0,\\
&(\widetilde{\chi}^{jk})(\widetilde{Y}^i{}_{n}) - (\widetilde{Y'}^k{}_{n})^t(\widetilde{\chi}^{ij})^t - (\widetilde{Y}^j{}_{n})^t(\widetilde{\chi}^{ik})^t - (\widetilde{\chi}^{jk})^p{}_{m}(\widetilde{Y}^i{}_{p}) = 0.
\end{aligned} \tag{4.19}$$

- If $\mathcal{A}$ and $\mathcal{A}^*$ are the first and the third 3-Leibniz algebra respectively

$$
\begin{aligned}
&(\chi_{is})(\widetilde{\chi}^{jk})^t - (Y'_s{}^j)^t(\widetilde{Y'}^k{}_i) - (Y'_s{}^k)^t(\widetilde{Y}^j{}_i) - (\widetilde{\chi}^{jk})^{m'}{}_i(\chi_{m's}) = 0,\\
&(\widetilde{\chi}^{ks})(\widetilde{\chi}^{ij}) - (\widetilde{\chi}^{ij})^k{}_p(\widetilde{\chi}^{ps}) - (\widetilde{\chi}^{ij})^s{}_p(\widetilde{\chi}^{kp}) - (\widetilde{\chi}^{ij})(\widetilde{\chi}^{ks}) = 0.
\end{aligned}
\tag{4.20}
$$

- If $\mathcal{A}$ and $\mathcal{A}^*$ are the second and the first 3-Leibniz algebra respectively

$$
\begin{aligned}
&(\chi_{is})(\widetilde{\chi}^{jk})^t - (Y_i{}^j)^t(\widetilde{Y'}^k{}_s) - (Y_i{}^k)^t(\widetilde{Y}^j{}_s) - (\widetilde{\chi}^{jk})^{m'}{}_s(\chi_{im'}) = 0,\\
&(\widetilde{\chi}^{ij})(\widetilde{Y'}^s{}_n) - (\widetilde{Y'}^j{}_n)^t(\widetilde{\chi}^{is})^t - (\widetilde{Y}^i{}_n)^t(\widetilde{\chi}^{js})^t - (\widetilde{\chi}^{ij})^p{}_n(\widetilde{Y'}^s{}_p) = 0.
\end{aligned}
\tag{4.21}
$$

- If $\mathcal{A}$ and $\mathcal{A}^*$ are both the second 3-Leibniz algebra

$$
\begin{aligned}
&(\chi_{is})(\widetilde{\chi}^{jk})^t - (Y_i{}^j)^t(\widetilde{Y'}^k{}_s) - (Y_i{}^k)^t(\widetilde{Y}^j{}_s) - (\widetilde{\chi}^{jk})^{m'}{}_s(\chi_{im'}) = 0,\\
&(\widetilde{\chi}^{jk})(\widetilde{Y}^i{}_n) - (\widetilde{Y'}^k{}_n)^t(\widetilde{\chi}^{ij})^t - (\widetilde{Y}^j{}_n)^t(\widetilde{\chi}^{ik})^t - (\widetilde{\chi}^{jk})^p{}_m(\widetilde{Y}^i{}_p) = 0.
\end{aligned}
\tag{4.22}
$$

- If $\mathcal{A}$ and $\mathcal{A}^*$ are the second and the third 3-Leibniz algebra respectively

$$
\begin{aligned}
&(\chi_{is})(\widetilde{\chi}^{jk})^t - (Y_i{}^j)^t(\widetilde{Y'}^k{}_s) - (Y_i{}^k)^t(\widetilde{Y}^j{}_s) - (\widetilde{\chi}^{jk})^{m'}{}_s(\chi_{im'}) = 0,\\
&(\widetilde{\chi}^{ks})(\widetilde{\chi}^{ij}) - (\widetilde{\chi}^{ij})^k{}_p(\widetilde{\chi}^{ps}) - (\widetilde{\chi}^{ij})^s{}_p(\widetilde{\chi}^{kp}) - (\widetilde{\chi}^{ij})(\widetilde{\chi}^{ks}) = 0.
\end{aligned}
\tag{4.23}
$$

- If $\mathcal{A}$ and $\mathcal{A}^*$ are the third and the first 3-Leibniz algebra respectively

$$
\begin{aligned}
&(\chi_{is})(\widetilde{\chi}^{jk})^t - (\chi_{is})_{j'}{}^j(\widetilde{\chi}^{j'k})^t - (\chi_{is})_{k'}{}^k(\widetilde{\chi}^{j'k})^t - (\widetilde{\chi}^{jk})^t(\chi_{is}) = 0,\\
&(\widetilde{\chi}^{ij})(\widetilde{Y'}^s{}_n) - (\widetilde{Y'}^j{}_n)^t(\widetilde{\chi}^{is})^t - (\widetilde{Y}^i{}_n)^t(\widetilde{\chi}^{js})^t - (\widetilde{\chi}^{ij})^p{}_n(\widetilde{Y'}^s{}_p) = 0.
\end{aligned}
\tag{4.24}
$$

- If $\mathcal{A}$ and $\mathcal{A}^*$ are the third and the second 3-Leibniz algebra respectively

$$
\begin{aligned}
&(\chi_{is})(\widetilde{\chi}^{jk})^t - (\chi_{is})_{j'}{}^j(\widetilde{\chi}^{j'k})^t - (\chi_{is})_{k'}{}^k(\widetilde{\chi}^{j'k})^t - (\widetilde{\chi}^{jk})^t(\chi_{is}) = 0,\\
&(\widetilde{\chi}^{jk})(\widetilde{Y}^i{}_n) - (\widetilde{Y'}^k{}_n)^t(\widetilde{\chi}^{ij})^t - (\widetilde{Y}^j{}_n)^t(\widetilde{\chi}^{ik})^t - (\widetilde{\chi}^{jk})^p{}_m(\widetilde{Y}^i{}_p) = 0.
\end{aligned}
\tag{4.25}
$$

- If $\mathcal{A}$ and $\mathcal{A}^*$ are both the third 3-Leibniz algebra

$$
\begin{aligned}
&(\chi_{is})(\widetilde{\chi}^{jk})^t - (\chi_{is})_{j'}{}^j(\widetilde{\chi}^{j'k})^t - (\chi_{is})_{k'}{}^k(\widetilde{\chi}^{j'k})^t - (\widetilde{\chi}^{jk})^t(\chi_{is}) = 0,\\
&(\widetilde{\chi}^{ks})(\widetilde{\chi}^{ij}) - (\widetilde{\chi}^{ij})^k{}_p(\widetilde{\chi}^{ps}) - (\widetilde{\chi}^{ij})^s{}_p(\widetilde{\chi}^{kp}) - (\widetilde{\chi}^{ij})(\widetilde{\chi}^{ks}) = 0.
\end{aligned}
\tag{4.26}
$$

Now, by use of the above relations we obtain some examples as follows.

Example 4.2 *We consider the following three dimensional first 3-Leibniz algebras [21]*

1. $[e_2, e_3, e_3] = e_1 \quad , \quad [e_3, e_3, e_3] = e_2,$

for this example we have

$$\chi_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \chi_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \chi_{12} = \chi_{21} = \chi_{32} = \chi_{11} = \chi_{22} = \chi_{13} = \chi_{31} = 0,$$

$$Y_2{}^1 = Y_3{}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y_1{}^2 = Y_1{}^1 = Y_2{}^2 = Y_2{}^3 = Y_3{}^3 = Y_1{}^3 = Y_3{}^1 = 0,$$

$$\chi'_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \chi'_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \chi'_{12} = \chi'_{21} = \chi'_{23} = \chi'_{11} = \chi'_{22} = \chi'_{13} = \chi'_{31} = 0,$$

$$Y'_3{}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y'_3{}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y'_1{}^2 = Y'_1{}^1 = Y'_2{}^2 = Y'_2{}^3 = Y'_3{}^3 = Y'_1{}^3 = Y_2{}^1 = 0,$$

By solving the system of equations $(4.18) - (4.20)$ *we obtain the following* $\mathcal{A}^*$ *algebras:*

- $\mathcal{A}^*$ *as a second* 3*-Leibniz algebra*

  $$[\widetilde{e}^1, \widetilde{e}^1, \widetilde{e}^1]_* = b\widetilde{e}^2 + a\widetilde{e}^3, \quad [\widetilde{e}^1, \widetilde{e}^2, \widetilde{e}^1]_* = b\widetilde{e}^3,$$

  where a *and* b *are any non zero real numbers.*

- $\mathcal{A}^*$ *as a third* 3*-Leibniz algebra*

  $$[\widetilde{e}^1, \widetilde{e}^1, \widetilde{e}^1]_* = b\widetilde{e}^2 + a\widetilde{e}^3, \quad [\widetilde{e}^1, \widetilde{e}^1, \widetilde{e}^2]_* = b\widetilde{e}^3,$$

  where a *and* b *are any non zero real numbers.*

2. $[e_3, e_2, e_3] = e_2 \quad , \quad [e_3, e_3, e_2] = -e_2, \quad , \quad [e_3, e_3, e_3] = e_1 + e_2,$

we have

$$\chi_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \chi_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad \chi_{12} = \chi_{21} = \chi_{23} = \chi_{11} = \chi_{22} = \chi_{13} = \chi_{31} = 0,$$

$$Y_3{}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \quad Y_3{}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y_1{}^2 = Y_1{}^1 = Y_2{}^2 = Y_2{}^3 = Y_3{}^3 = Y_1{}^3 = Y_2{}^1 = 0,$$

$$\chi'_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \chi'_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad \chi'_{12} = \chi'_{21} = \chi'_{32} = \chi'_{11} = \chi'_{22} = \chi'_{13} = \chi'_{31} = 0,$$

$$Y_2'{}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y_3'{}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad Y_3'{}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$Y_1'{}^2 = Y_1'{}^1 = Y_2'{}^2 = Y_2'{}^3 = Y_3'{}^3 = Y_1'{}^3 = Y_2'{}^1 = 0,$$

In the same way by solving the system of equations $(4.18) - (4.20)$ *we obtain the following* $\mathcal{A}^*$ *algebras:*

- $\mathcal{A}^*$ *as a first* 3*-Leibniz algebra*

  $$[\widetilde{e}^1, \widetilde{e}^1, \widetilde{e}^2]_* = a\widetilde{e}^3, \quad [\widetilde{e}^1, \widetilde{e}^2, \widetilde{e}^2]_* = b\widetilde{e}^3, \quad [\widetilde{e}^2, \widetilde{e}^1, \widetilde{e}^2]_* = c\widetilde{e}^3, \quad [\widetilde{e}^2, \widetilde{e}^2, \widetilde{e}^2]_* = d\widetilde{e}^3,$$

  where a*,* b*,* c*,* d *are any non zero real numbers.*

- $\mathcal{A}^*$ *as second* 3*-Leibniz algebra*

  $$[\widetilde{e}^1, \widetilde{e}^1, \widetilde{e}^1]_* = a\widetilde{e}^3, \quad [\widetilde{e}^1, \widetilde{e}^1, \widetilde{e}^2]_* = b\widetilde{e}^3, \quad [\widetilde{e}^1, \widetilde{e}^2, \widetilde{e}^1]_* = c\widetilde{e}^3, \quad [\widetilde{e}^1, \widetilde{e}^2, \widetilde{e}^2]_* = d\widetilde{e}^3,$$

  where a*,* b*,* c*,* d *are any non zero real numbers.*

- $\mathcal{A}^*$ *as third* 3*-Leibniz algebra*

  $$\begin{aligned} &[\widetilde{e}^1, \widetilde{e}^1, \widetilde{e}^1]_* = a\widetilde{e}^3, \quad [\widetilde{e}^1, \widetilde{e}^1, \widetilde{e}^2]_* = b\widetilde{e}^3, \\ &[\widetilde{e}^1, \widetilde{e}^2, \widetilde{e}^1]_* = c\widetilde{e}^3, \quad [\widetilde{e}^1, \widetilde{e}^2, \widetilde{e}^2]_* = d\widetilde{e}^3, \\ &[\widetilde{e}^2, \widetilde{e}^1, \widetilde{e}^1]_* = m\widetilde{e}^3, \quad [\widetilde{e}^2, \widetilde{e}^1, \widetilde{e}^2]_* = f\widetilde{e}^3, \\ &[\widetilde{e}^2, \widetilde{e}^2, \widetilde{e}^1]_* = g\widetilde{e}^3, \quad [\widetilde{e}^2, \widetilde{e}^2, \widetilde{e}^2]_* = h\widetilde{e}^3, \end{aligned}$$

  where a*,* b*,* c*,* d*,* m*,* f*,* g*,* h *are any non zero real numbers.*

5 Correspondence between 3-Leibniz bialgebra and its associated Leibniz bialgebra

In this section, we determine the type of the associated Leibniz algebra for any three types of 3-Leibniz algebras and prove a theorem about correspondence between 3-Leibniz bialgebra and its associated Leibniz bialgebra [20].

- If $\mathcal{A}$ be the first 3-Leibniz algebra then its associated Leibniz algebra $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ with

$$[x_1 \otimes x_2, y_1 \otimes y_2] = [x_1, y_1, y_2] \otimes x_2 + x_1 \otimes [x_2, y_1, y_2], \tag{5.1}$$

  is right Leibniz algebra.

- If $\mathcal{A}$ be the second 3-Leibniz algebra then its associated Leibniz algebra $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ with following brackets

$$[x_1 \otimes x_2, y_1 \otimes y_2] = [x_1, y_1, x_2] \otimes y_2 + y_1 \otimes [x_1, y_2, x_2], \tag{5.2}$$

$$[x_1 \otimes x_2, y_1 \otimes y_2] = [y_1, x_1, y_2] \otimes x_2 + x_1 \otimes [y_1, x_2, y_2], \tag{5.3}$$

  is left and right Leibniz algebra respectively.

- If $\mathcal{A}$ be the third 3-Leibniz algebra then its associated Leibniz algebra $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ with

$$[x_1 \otimes x_2, y_1 \otimes y_2] = [x_1, x_2, y_1] \otimes y_2 + y_1 \otimes [x_1, x_2, y_2], \tag{5.4}$$

  is a left Leibniz algebra.

Now we prove a theorem about the correspondence between 3-Leibniz bialgebra and its associated Leibniz bialgebra [**?**].

Theorem 5.1 *Let $(\mathcal{A}, \gamma)$ be a 3-Leibniz bialgebra and $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ be its associated Leibniz algebra then there exist a linear map $\delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$ such that it defines a Leibniz bialgebra structure on $\mathcal{G}$. Conversely if $(\mathcal{G}, \delta)$ is a Leibniz bialgebra such that $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ and $\mathcal{A}$ be a 3-Leibniz algebra then there exist a linear map $\gamma : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ such that it defines a 3-Leibniz bialgebra structure on $\mathcal{A}$.*

Proof. Since $\mathcal{A}$ can be 3-Leibniz algebra from various three types the proof of the theorem divided to following three parts[2].

[2]Here we write only the proof of one case. The proof of the other cases are similar.

1. If $(\mathcal{A},\gamma)$ be a 3-Leibniz bialgebra such that (3.39) is valid for $\gamma[x_1,x_2,x_3]$ then $\mathcal{G}=\mathcal{A}\otimes\mathcal{A}$ is a right Leibniz algebra with bracket (5.1). $\mathcal{A}^*$ can be 3-Leibniz algebra from three cases.

   (a) If $\mathcal{A}^*$ is the first 3-Leibniz algebra then $\mathcal{G}^*=\mathcal{A}^*\otimes\mathcal{A}^*$ with bracket from type (5.1) is a right Leibniz algebra. We want to prove there exist a linear map $\delta:\mathcal{G}\longrightarrow\mathcal{G}\otimes\mathcal{G}$ such that it defines a Leibniz bialgebra structure on $\mathcal{G}$. We can rewrite the bracket $[.,.]_*$ on $\mathcal{G}^*$ with γ^t as follows:

$$[\widetilde{x}^1\otimes\widetilde{x}^2,\widetilde{y}^1\otimes\widetilde{y}^2]_*=\gamma^t(\widetilde{x}^1\otimes\widetilde{y}^1\otimes\widetilde{y}^2)\otimes\widetilde{x}^2+\widetilde{x}^1\otimes\gamma^t(\widetilde{x}^2\otimes\widetilde{y}^1\otimes\widetilde{y}^2),\tag{5.5}$$

using the following flip operators

$$\sigma_{24}:\mathcal{A}^{*\otimes 4}\longrightarrow\mathcal{A}^{*\otimes 4},\ \ \sigma_{24}(\xi_1\otimes\xi_2\otimes\xi_3\otimes\xi_4)=\xi_1\otimes\xi_4\otimes\xi_3\otimes\xi_2\tag{5.6}$$

$$\sigma_{34}:\mathcal{A}^{*\otimes 4}\longrightarrow\mathcal{A}^{*\otimes 4},\ \ \sigma_{34}(\xi_1\otimes\xi_2\otimes\xi_3\otimes\xi_4)=\xi_1\otimes\xi_2\otimes\xi_4\otimes\xi_3,\tag{5.7}$$

(5.5) can be written as

$$[\widetilde{x}^1\otimes\widetilde{x}^2,\widetilde{y}^1\otimes\widetilde{y}^2]_*=\left((\gamma^t\otimes I_{\mathcal{A}^*})\circ\sigma_{24}\circ\sigma_{34}+I_{\mathcal{A}^*}\otimes\gamma^t\right)(\widetilde{x}^1\otimes\widetilde{x}^2\otimes\widetilde{y}^1\otimes\widetilde{y}^2),\tag{5.8}$$

setting

$$\delta^t:=[,]_*,\tag{5.9}$$

then we have

$$\delta^t=(\gamma^t\otimes I_{\mathcal{A}^*})\circ\sigma_{24}\circ\sigma_{34}+I_{\mathcal{A}^*}\otimes\gamma^t,\tag{5.10}$$

and so

$$\delta(x_1\otimes x_2)=\left(\sigma^t_{34}\circ\sigma^t_{24}\circ(\gamma\otimes I_{\mathcal{A}})+I_{\mathcal{A}}\otimes\gamma\right)(x_1\otimes x_2),\tag{5.11}$$

where $\sigma^t_{24},\sigma^t_{34}:\mathcal{A}^{\otimes 4}\longrightarrow\mathcal{A}^{\otimes 4}$ act as follows

$$\sigma^t_{24}(x_1\otimes x_2\otimes x_3\otimes x_4)=x_1\otimes x_4\otimes x_3\otimes x_2,\tag{5.12}$$

$$\sigma^t_{34}(x_1\otimes x_2\otimes x_3\otimes x_4)=x_1\otimes x_2\otimes x_4\otimes x_3,\tag{5.13}$$

then for any $X,Y\in\mathcal{G}$ we have

$$\begin{aligned}
&\delta[X,Y]=\delta[x_1\otimes x_2,y_1\otimes y_2]=\delta([x_1,y_1,y_2]\otimes x_2+x_1\otimes[x_2,y_1,y_2])\\
&=\left(\sigma^t_{34}\circ\sigma^t_{24}\circ(\gamma\otimes I_{\mathcal{A}})+I_{\mathcal{A}}\otimes\gamma\right)([x_1,y_1,y_2]\otimes x_2+x_1\otimes[x_2,y_1,y_2])\\
&=[x_1^1,y_1,y_2]\otimes x_2\otimes x_1^2\otimes x_1^3+x_1^1\otimes x_2\otimes[x_1^2,y_1,y_2]\otimes x_1^3\\
&+x_1^1\otimes x_2\otimes x_1^2\otimes[x_1^3,y_1,y_2]+[x_1,y_1,y_2]\otimes x_2^1\otimes x_2^2\otimes x_2^3\\
&+x_1^1\otimes[x_2,y_1,y_2]\otimes x_1^2\otimes x_1^3+x_1\otimes[x_2^1,y_1,y_2]\otimes x_2^2\otimes x_2^3\\
&+x_1\otimes x_2^1\otimes[x_2^2,y_1,y_2]\otimes x_2^3+x_1\otimes x_2^1\otimes x_2^2\otimes[x_2^3,y_1,y_2]\\
&=\left(1_{\mathcal{G}}\otimes ad^{(r)}_Y+ad^{(r)}_Y\otimes 1_{\mathcal{G}}\right)\delta(X)
\end{aligned}\tag{5.14}$$

where in above identity we use $\gamma(x_1) = x_1^1 \otimes x_1^2 \otimes x_1^3$ and $\gamma(x_2) = x_2^1 \otimes x_2^2 \otimes x_2^3$.

In the same way one can prove the following cases:

(b) If $\mathcal{A}^*$ be the second 3-Leibniz algebra then

i. If $\mathcal{G}^* = \mathcal{A}^* \otimes \mathcal{A}^*$ be a left Leibniz algebra with bracket from type (5.2) then

$$\delta(x_1 \otimes x_2) = \left(\sigma_{23}^t \circ (\gamma \otimes I_{\mathcal{A}}) + \sigma_{34}^t \circ \sigma_{24}^t \circ \sigma_{12}^t \circ (I_{\mathcal{A}} \otimes \gamma)\right)(x_1 \otimes x_2) \tag{5.15}$$

ii. If $\mathcal{G}^* = \mathcal{A}^* \otimes \mathcal{A}^*$ be a right Leibniz algebra with bracket from type (5.3) then

$$\delta(x_1 \otimes x_2) = \left(\sigma_{34}^t \circ \sigma_{24}^t \circ \sigma_{12}^t \circ (\gamma \otimes I_{\mathcal{A}}) + \sigma_{23}^t \circ (I_{\mathcal{A}} \otimes \gamma)\right)(x_1 \otimes x_2) \tag{5.16}$$

(c) If $\mathcal{A}^*$ be the third 3-Leibniz algebra then $\mathcal{G}^* = \mathcal{A}^* \otimes \mathcal{A}^*$ be a left Leibniz algebra with bracket from type (5.4) then we have

$$\delta(x_1 \otimes x_2) = \left(\gamma \otimes I_{\mathcal{A}} + \sigma_{23}^t \circ \sigma_{23}^t \circ (I_{\mathcal{A}} \otimes \gamma)\right)(x_1 \otimes x_2). \tag{5.17}$$

In all above cases 1-cocycle condition is

$$\delta[X, Y] = \left(1_{\mathcal{G}} \otimes ad_Y^{(r)} + ad_Y^{(r)} \otimes 1_{\mathcal{G}}\right)\delta(X) \quad \forall X, Y \in \mathcal{G} \tag{5.18}$$

2. If $(\mathcal{A}, \gamma)$ be a 3-Leibniz bialgebra such that (3.40) is valid for $\gamma[x_1, x_2, x_3]$ then we have the following cases:

(a) If $\mathcal{A}^*$ be the first 3-Leibniz algebra then $\mathcal{G}^* = \mathcal{A}^* \otimes \mathcal{A}^*$ be a right Leibniz algebra with bracket from type (5.1).

i. If $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ be a left Leibniz algebra with bracket (5.2) then we have

$$\delta[X, Y] = \left(1_{\mathcal{G}} \otimes ad_X^{(l)} + ad_X^{(l)} \otimes 1_{\mathcal{G}}\right)\delta(Y) \quad \forall X, Y \in \mathcal{G}$$

ii. If $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ be a right Leibniz algebra with bracket (5.3) then we have

$$\delta[X, Y] = \left(1_{\mathcal{G}} \otimes ad_Y^{(r)} + ad_Y^{(r)} \otimes 1_{\mathcal{G}}\right)\delta(X) \quad \forall X, Y \in \mathcal{G}$$

in above two cases we have

$$\delta(x_1 \otimes x_2) = \left(\sigma_{34}^t \circ \sigma_{24}^t \circ (\gamma \otimes I_{\mathcal{A}}) + I_{\mathcal{A}} \otimes \gamma\right)(x_1 \otimes x_2). \tag{5.19}$$

(b) If $\mathcal{A}^*$ be the second 3-Leibniz algebra then

i. If $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ and $\mathcal{G}^* = \mathcal{A}^* \otimes \mathcal{A}^*$ are both left Leibniz algebra with bracket from type (5.2) then we have

$$\delta[X, Y] = \left(1_{\mathcal{G}} \otimes ad_X^{(l)} + ad_X^{(l)} \otimes 1_{\mathcal{G}}\right)\delta(Y) \quad \forall X, Y \in \mathcal{G}$$

ii. If $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ be a right Leibniz algebra with bracket (5.3) and $\mathcal{G}^* = \mathcal{A}^* \otimes \mathcal{A}^*$ be a left Leibniz algebra with bracket from type (5.2) then we have

$$\delta[X,Y] = \left(1_{\mathcal{G}} \otimes ad_Y^{(r)} + ad_Y^{(r)} \otimes 1_{\mathcal{G}}\right) \delta(X) \quad \forall X, Y \in \mathcal{G},$$

where in the above two cases we have

$$\delta(x_1 \otimes x_2) = \left(\sigma_{23}^t \circ (\gamma \otimes I_{\mathcal{A}}) + \sigma_{34}^t \circ \sigma_{24}^t \circ \sigma_{12}^t \circ (I_{\mathcal{A}} \otimes \gamma)\right)(x_1 \otimes x_2). \tag{5.20}$$

iii. If $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ be a left Leibniz algebra with bracket (5.2) and $\mathcal{G}^* = \mathcal{A}^* \otimes \mathcal{A}^*$ be a right Leibniz algebra with bracket from type (5.3) then we have

$$\delta[X,Y] = \left(1_{\mathcal{G}} \otimes ad_X^{(l)} + ad_X^{(l)} \otimes 1_{\mathcal{G}}\right) \delta(Y) \quad \forall X, Y \in \mathcal{G}$$

iv. If $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ and $\mathcal{G}^* = \mathcal{A}^* \otimes \mathcal{A}^*$ are both right Leibniz algebra with bracket from type (5.3) then we have

$$\delta[X,Y] = \left(1_{\mathcal{G}} \otimes ad_Y^{(r)} + ad_Y^{(r)} \otimes 1_{\mathcal{G}}\right) \delta(X) \quad \forall X, Y \in \mathcal{G},$$

where in the above two cases we have

$$\delta(x_1 \otimes x_2) = \left(\sigma_{34}^t \circ \sigma_{24}^t \circ \sigma_{12}^t \circ (\gamma \otimes \sigma_{23}^t \circ I_{\mathcal{A}}) + \circ(I_{\mathcal{A}} \otimes \gamma)\right)(x_1 \otimes x_2). \tag{5.21}$$

(c) If $\mathcal{A}^*$ be the third 3-Leibniz algebra then we have:

i. If $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ be a left Leibniz algebra with bracket (5.2) and $\mathcal{G}^* = \mathcal{A}^* \otimes \mathcal{A}^*$ be a left Leibniz algebra with bracket from type (5.4) then we have

$$\delta[X,Y] = \left(1_{\mathcal{G}} \otimes ad_X^{(l)} + ad_X^{(l)} \otimes 1_{\mathcal{G}}\right) \delta(Y) \quad \forall X, Y \in \mathcal{G}$$

ii. If $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ be a right Leibniz algebra with bracket (5.3) and $\mathcal{G}^* = \mathcal{A}^* \otimes \mathcal{A}^*$ be a left Leibniz algebra with bracket from type (5.4) then we have

$$\delta[X,Y] = \left(1_{\mathcal{G}} \otimes ad_Y^{(r)} + ad_Y^{(r)} \otimes 1_{\mathcal{G}}\right) \delta(X) \quad \forall X, Y \in \mathcal{G}$$

where in the above two cases we have

$$\delta(x_1 \otimes x_2) = \left(\gamma \otimes I_{\mathcal{A}} + \sigma_{23}^t \circ \sigma_{12}^t \circ (I_{\mathcal{A}} \otimes \gamma)\right)(x_1 \otimes x_2). \tag{5.22}$$

3. If $(\mathcal{A}, \gamma)$ be a 3-Leibniz bialgebra such that (3.41) is valid for $\gamma[x_1, x_2, x_3]$ then $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ with bracket (5.4) is a left Leibniz algebra and also $\mathcal{A}^*$ can be 3-Leibniz algebra from three types

(a) If $\mathcal{A}^*$ is the first 3-Leibniz algebra then $\mathcal{G}^* = \mathcal{A}^* \otimes \mathcal{A}^*$ with bracket from type (5.1) is a right Leibniz algebra then we have

$$\delta(x_1 \otimes x_2) = \left(\sigma^t_{34} \circ \sigma^t_{24} \circ (\gamma \otimes I_{\mathcal{A}}) + I_{\mathcal{A}} \otimes \gamma\right)(x_1 \otimes x_2). \tag{5.23}$$

(b) If $\mathcal{A}^*$ is the second 3-Leibniz algebra then we have

i. If $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ be a left Leibniz algebra with bracket (5.2) then we have

$$\delta(x_1 \otimes x_2) = \left(\sigma^t_{23} \circ (\gamma \otimes I_{\mathcal{A}}) + \sigma^t_{34} \circ \sigma^t_{24} \circ \sigma^t_{12} \circ (I_{\mathcal{A}} \otimes \gamma)\right)(x_1 \otimes x_2). \tag{5.24}$$

ii. If $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ be a right Leibniz algebra with bracket (5.3) then we have

$$\delta(x_1 \otimes x_2) = \left(\sigma^t_{34} \circ \sigma^t_{24} \circ \sigma^t_{12} \circ (\gamma \otimes I_{\mathcal{A}}) + \sigma^t_{23} \circ (I_{\mathcal{A}} \otimes \gamma)\right)(x_1 \otimes x_2). \tag{5.25}$$

(c) If $\mathcal{A}^*$ is the second 3-Leibniz algebra then $\mathcal{G}^* = \mathcal{A}^* \otimes \mathcal{A}^*$ with bracket from type (5.4) is a left Leibniz algebras then we have

$$\delta(x_1 \otimes x_2) = \left(\gamma \otimes I_{\mathcal{A}} + \sigma^t_{23} \circ \sigma^t_{12} \circ (I_{\mathcal{A}} \otimes \gamma)\right)(x_1 \otimes x_2). \tag{5.26}$$

where in the above cases we have

$$\delta[X, Y] = \left(1_{\mathcal{G}} \otimes ad^{(l)}_X + ad^{(l)}_X \otimes 1_{\mathcal{G}}\right)\delta(Y) \quad \forall X, Y \in \mathcal{G}$$

The proof of inverse is clearly. ■

6 3-Lie bialgebras

In this section we suppose $\mathcal{A}$ is a 3-Lie algebra as a special case, then we have the following fundamental identity for $n = 3$

$$[x_1, x_2, [y_1, y_2, y_3]] = [[x_1, x_2, y_1], y_2, y_3] + [y_1, [x_1, x_2, y_2], y_3] + [y_1, y_2, [x_1, x_2, y_3]], \tag{6.1}$$

We want to define bialgebra structure for 3-Lie algebra similar to 3-Leibniz algebra with a little difference.

Remark 6.1 *In Definition 2.7 if ρ satisfies only in identity* (2.11) *we say ρ is a semi-representation of $\mathcal{A}$ in V.*

If $V = \mathcal{A}$ it is clearly that $\rho : \mathcal{A} \otimes \mathcal{A} \longrightarrow End(\mathcal{A})$ with the following relation

$$\rho(x_1, x_2)(z) = ad_{(x_1, x_2, \hat{x_3})}(z) = [x_1, x_2, z],$$

is a representation of $\mathcal{A}$ in $\mathcal{A}$.

Remark 6.2 *If $\rho_i : \mathcal{A}\otimes\mathcal{A} \longrightarrow End(V_i), i = 1,2,3$ are three representation of $\mathcal{A}$ in vector spaces V_i then $\rho : \mathcal{A}\otimes\mathcal{A} \longrightarrow End(V_1\otimes V_2\otimes V_3)$ with the following identities*

$$\begin{aligned}\rho(x_1,x_2)(y_1\otimes y_2\otimes y_3) &= \rho_1(x_1,x_2)(y_1)\otimes y_2\otimes y_3 + y_1\otimes\rho_2(x_1,x_2)(y_2)\otimes y_3\\ &\quad + y_1\otimes y_2\otimes\rho_3(x_1,x_2)(y_3), \qquad (6.2)\end{aligned}$$

$\forall x_1,x_2\in\mathcal{A}, \forall y_i\in V_i, i = 1,2,3$ is not a representation of $\mathcal{A}$ in $V_1\otimes V_2\otimes V_3$ but it is a semi-representation of $\mathcal{A}$ in $V_1\otimes V_2\otimes V_3$. Note that in Lie algebra case the tensor product of representations of Lie algebra in vector spaces $V_i, i = 1,...,n$ is a representation of Lie algebra in $V_1\otimes V_2\otimes...\otimes V_n$. If $V_1 = V_2 = V_3 = \mathcal{A}$ then $\rho : \mathcal{A}\otimes\mathcal{A} \longrightarrow End(\mathcal{A}^{\otimes 3})$ with the following relation

$$\begin{aligned}\rho(y_1,y_2)(z_1\otimes z_2\otimes z_3) &= (ad_{(y_1,y_2,\hat{y}_3)}\otimes 1\otimes 1 + 1\otimes ad_{(y_1,y_2,\hat{y}_3)}\otimes 1 + 1\otimes 1\otimes ad_{(y_1,y_2,\hat{y}_3)})(z_1\otimes z_2\otimes z_3)\\ &:= ad^{(3)}_{(y_1,y_2,\hat{y}_3)}(z_1\otimes z_2\otimes z_3), \qquad (6.3)\end{aligned}$$

is a semi-representation of $\mathcal{A}$ in $\mathcal{A}^{\otimes 3}$.

We need to generalize definition 2.8 for any representation of $\mathcal{A}$ in any vector space V.

Definition 6.3 *Let $\mathcal{A}$ be a 3-Lie algebra, an V-valued p-cochain is a linear map $\psi : (\mathcal{A}\otimes\mathcal{A})^{\otimes(p-1)}\otimes\mathcal{A} \longrightarrow V$ and denote the space of V-valued p-cochains with $\Gamma^p(\mathcal{A}, V)$, the coboundary operator is given by:*

$$\begin{aligned}d^p\psi(x_1,...x_{2p+1}) &= \sum_{j=1}^{p}\sum_{k=2j+1}^{2p+1}(-1)^j\psi(x_1,...,\hat{x}_{j-1},\hat{x}_j,...,[x_{2j-1},x_{2j},x_k],...,x_{2p+1})\\ &\quad+\sum_{k=1}^{p}\rho(x_{2k-1},x_{2k},\psi(x_1,...,\hat{x}_{2k-1},\hat{x}_{2k},...,x_{2p+1}))\\ &\quad-(-1)^{p+1}\rho(x_{2p-1},x_{2p+1},\psi(x_1,...,x_{2p-2},x_{2p}),)\\ &\quad+(-1)^{p+1}\rho(x_{2p},x_{2p+1},\psi(x_1,...,x_{2p-1})),\end{aligned}$$

where $\rho : \mathcal{A}\otimes\mathcal{A} \longrightarrow End(V)$ is a representation of $\mathcal{A}$ in V.
For $p = 1$ we have

$$d^1\psi(x_1,x_2,x_3) = -\psi([x_1,x_2,x_3]) + \rho(x_1,x_2,\psi(x_3)) - \rho(x_1,x_3,\psi(x_2)) + \rho(x_2,x_3,\psi(x_1))$$

Definition 6.4 *A 3-Lie bialgebra $(\mathcal{A},\gamma)$ ia a 3-Lie algebra $\mathcal{A}$ with a linear map (cocommutator) $\gamma : \mathcal{A} \longrightarrow \mathcal{A}^{\otimes 3}$ such that*

- *γ is a 1-cocycle on $\mathcal{A}$ with values in $\mathcal{A}^{\otimes 3}$,*

$$\gamma[y_1,y_2,y_3] = \rho(y_2,y_3,\gamma(y_1)) + \rho(y_3,y_1,\gamma(y_2)) + \rho(y_1,y_2,\gamma(y_3)), \qquad (6.4)$$

  where ρ is a semi-representation of $\mathcal{A}$ in $\mathcal{A}^{\otimes 3}$.

- $\gamma^t : \mathcal{A}^{*\otimes 3} \longrightarrow \mathcal{A}$ *defines a* 3*-Lie bracket on* $\mathcal{A}^*$.

If we use the (3.37) *then we have relation* (3.38) *similar to* 3*-Leibniz bialgebra.*

By use of (6.3) 1-cocycle condition (6.4) is rewritten as follows

$$\gamma[y_1, y_2, y_3] = ad^{(3)}_{(y_2,y_3,\hat{y}_1)}\gamma(y_1) + ad^{(3)}_{(y_3,y_1,\hat{y}_2)}\gamma(y_2) + ad^{(3)}_{(y_1,y_2,\hat{y}_3)}\gamma(y_3). \tag{6.5}$$

Proposition 6.5 *If* $(\mathcal{A}, \gamma)$ *is a* 3*-Lie bialgebra, and* μ *is the* 3*-Lie bracket of* $\mathcal{A}$*, then* $(\mathcal{A}, \mu^t)$ *is a* 3*-Lie bialgebra, where* γ^t *is the* 3*-Lie bracket of* $\mathcal{A}^*$*.*

Proof. By use of (3.38) and (6.5) the proof is clearly. ■

Theorem 6.6 *Let* $(\mathcal{A}, \gamma)$ *be a* 3*-Lie bialgebra and* $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ *be its associated Leibniz algebra then there exist a linear map* $\delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$ *such that it defines a Lie bialgebra structure on* $\mathcal{G}$*. Conversely if* $(\mathcal{G}, \delta)$ *is a Lie bialgebra such that* $\mathcal{G} = \mathcal{A} \otimes \mathcal{A}$ *and* $\mathcal{A}$ *be a* 3*-Lie algebra then there exist a linear map* $\gamma : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ *such that it defines a* 3*-Lie bialgebra structure on* $\mathcal{A}$*.*

Proof. From the theorem 5.1 the proof is clearly. ■

6.1 3-Lie bialgebra in terms of structure constants; some examples

Here we rewrite the 1-cocycle condition (6.5) in terms of structure constants of the 3-Lie algebra $\mathcal{A}$ and $\mathcal{A}^*$. Note that in this case we have (4.1) and (4.3) same as 3-Leibniz algebra with difference that the structure constants are antisymmetric.

$$\gamma[x_i, x_s, x_n] = ad^{(3)}_{(x_s,x_n,\hat{x}_i)}\gamma(x_i) + ad^{(3)}_{(x_n,x_i,\hat{x}_s)}\gamma(x_s) + ad^{(3)}_{(x_i,x_s,\hat{x}_n)}\gamma(x_n). \tag{6.6}$$

Using (3.38) and (4.1) in (6.6) we have

$$\begin{aligned} f_{isn}{}^{p}\tilde{f}^{jkm}{}_{p} \;=\;& f_{j'sn}{}^{j}\tilde{f}^{j'km}{}_{i} + f_{k'sn}{}^{k}\tilde{f}^{jk'm}{}_{i} + f_{m'sn}{}^{m}\tilde{f}^{jkm'}{}_{i} \\ &+ f_{ij'n}{}^{j}\tilde{f}^{j'km}{}_{s} + f_{ik'n}{}^{k}\tilde{f}^{jk'm}{}_{s} + f_{im'n}{}^{m}\tilde{f}^{jkm'}{}_{s} \\ &+ f_{isj'}{}^{j}\tilde{f}^{j'km}{}_{n} + f_{isk'}{}^{k}\tilde{f}^{jk'm}{}_{n} + f_{ism'}{}^{m}\tilde{f}^{jkm'}{}_{n}. \end{aligned} \tag{6.7}$$

Definition 6.7 *Two* 3*-Lie algebra* $\mathcal{A}$ *and* $\mathcal{A}^*$ *constract a* 3*-Lie bialgebra if their structure constants satisfy in relations* (6.7)*.*

To use this relation in the calculations, one can rewrite t in terms of the matrix form using the following adjoint representations:

$$
\begin{aligned}
& f_{isn}{}^{p} = (\chi_{is})_n{}^{p} = (Y_i{}^{p})_{sn} = f_{sni}{}^{p} = (\chi_{sn})_i{}^{p} = (Y_s{}^{p})_{ni} \\
& = f_{nis}{}^{p} = (\chi_{ni})_s{}^{p} = (Y_n{}^{p})_{is} = -f_{sin}{}^{p} = -(\chi_{si})_n{}^{p} = -(Y_s{}^{p})_{in} \\
& = -f_{nsi}{}^{p} = -(\chi_{ns})_i{}^{p} = -(Y_n{}^{p})_{si} = f_{ins}{}^{p} = -(\chi_{in})_s{}^{p} = -(Y_i{}^{p})_{ns},
\end{aligned}
\tag{6.8}
$$

$$
\begin{aligned}
& \widetilde{f}^{jkm}{}_{p} = (\widetilde{\chi}^{jk})^{m}{}_{p} = (\widetilde{Y}^{j}{}_{p})^{km} = \widetilde{f}^{kmj}{}_{p} = (\widetilde{\chi}^{km})^{j}{}_{p} = (\widetilde{Y}^{k}{}_{p})^{mj} \\
& = \widetilde{f}^{mjk}{}_{p} = (\widetilde{\chi}^{mj})^{k}{}_{p} = (\widetilde{Y}^{m}{}_{p})^{jk} = -\widetilde{f}^{kjm}{}_{p} = -(\widetilde{\chi}^{kj})^{m}{}_{p} = -(\widetilde{Y}^{k}{}_{p})^{jm} \\
& = -\widetilde{f}^{mkj}{}_{p} = -(\widetilde{\chi}^{mk})^{j}{}_{p} = -(\widetilde{Y}^{m}{}_{p})^{kj} = -\widetilde{f}^{jmk}{}_{p} = -(\widetilde{\chi}^{jm})^{k}{}_{p} = -(\widetilde{Y}^{j}{}_{p})^{mk}.
\end{aligned}
\tag{6.9}
$$

Then relation (6.7) has the following matrix form

$$
\begin{aligned}
\chi_{ns}(\widetilde{\chi}^{mk})^{t} \;=\;& (\widetilde{\chi}^{mk})^{t}(\chi_{ns}) - (\chi_{k's})_n{}^{k}(\widetilde{\chi}^{mk'})^{t} - (\widetilde{\chi}^{m'k})^{t}(\chi_{m's})_n{}^{m} \\
& -(\widetilde{\chi}^{j'k})^{m}{}_{s}(\chi_{nj'}) + Y_n{}^{k}(\widetilde{Y}^{m}{}_{s})^{t} + Y_n{}^{p}\widetilde{Y}^{k}{}_{s} \\
& -(\widetilde{\chi}^{j'k})^{m}{}_{n}\chi_{j's} - Y_s{}^{k}(\widetilde{Y}^{m}{}_{n})^{t} + Y_s{}^{m}(\widetilde{Y}^{k}{}_{s})^{t},
\end{aligned}
\tag{6.10}
$$

where in the above relation t stands for transpose of a matrix. Fundamental identity for 3-Lie algebra $\mathcal{A}^*$ in terms of structure constant has the following form

$$
\widetilde{f}^{nsj}{}_{p}\widetilde{f}^{mkp}{}_{i} = \widetilde{f}^{mkn}{}_{p}\widetilde{f}^{psj}{}_{i} + \widetilde{f}^{mks}{}_{p}\widetilde{f}^{npj}{}_{i} + \widetilde{f}^{mkj}{}_{p}\widetilde{f}^{nsp}{}_{i},
\tag{6.11}
$$

with the matrix form as

$$
\widetilde{\chi}^{ns}\widetilde{\chi}^{mk} = (\widetilde{\chi}^{mk})^{n}{}_{p}\widetilde{\chi}^{ps} + (\widetilde{\chi}^{mk})^{s}{}_{p}\widetilde{\chi}^{np} + \widetilde{\chi}^{mk}\widetilde{\chi}^{ns}.
\tag{6.12}
$$

Now for obtaining the dual of $\mathcal{A}$ one must solve the following equations

$$
\begin{aligned}
& (\widetilde{\chi}^{mk})^{t}(\chi_{ns}) - (\chi_{k's})_n{}^{k}(\widetilde{\chi}^{mk'})^{t} - (\widetilde{\chi}^{m'k})^{t}(\chi_{m's})_n{}^{m} - (\widetilde{\chi}^{j'k})^{m}{}_{s}(\chi_{nj'}) + Y_n{}^{k}(\widetilde{Y}^{m}{}_{s})^{t} + Y_n{}^{p}\widetilde{Y}^{k}{}_{s} \\
& -(\widetilde{\chi}^{j'k})^{m}{}_{n}\chi_{j's} - Y_s{}^{k}(\widetilde{Y}^{m}{}_{n})^{t} + Y_s{}^{m}(\widetilde{Y}^{k}{}_{s})^{t} - \chi_{ns}(\widetilde{\chi}^{mk})^{t} = 0, \\
& (\widetilde{\chi}^{mk})^{n}{}_{p}(\widetilde{\chi}^{ps}) + (\widetilde{\chi}^{mk})^{s}{}_{p}(\widetilde{\chi}^{np}) + (\widetilde{\chi}^{mk})(\widetilde{\chi}^{ns}) - (\widetilde{\chi}^{ns})(\widetilde{\chi}^{mk}) = 0.
\end{aligned}
\tag{6.13}
$$

Example 6.8 *We consider the following four dimentional* 3*-Lie algebra [**?**]*

$$
[e_2, e_3, e_4] = e_1, \qquad [e_1, e_3, e_4] = e_2,
$$

for this example we have

$$\chi_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \chi_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \chi_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\chi_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \chi_{34} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_1{}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad Y_2{}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad Y_3{}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$Y_3{}^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Y_4{}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_4{}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By solving the system of equations (6.13) *we obtain the following* 3*-Lie algebras as dual of* $\mathcal{A}$

$$[\tilde{e}^1, \tilde{e}^2, \tilde{e}^4]_* = b\tilde{e}^1, \quad [\tilde{e}^3, \tilde{e}^2, \tilde{e}^4]_* = b\tilde{e}^3,$$

$$[\tilde{e}^2, \tilde{e}^3, \tilde{e}^1]_* = b\tilde{e}^2, \quad [\tilde{e}^4, \tilde{e}^3, \tilde{e}^1]_* = b\tilde{e}^4,$$

where b *is any non zero real number.*

7 Conclusion

In this paper we defined the 3-Leibniz and 3-Lie bialgebras using cohomology of 3-Leibniz and 3-Lie algebras. Many theorems have been given , in particular, we have proven the correspondence between 3-Leibniz bialgebra and its associated Leibniz bialgebras. There are some open problems related to this work. The definition of r-matrix and Yang-Baxter equation were related to 3-Leibniz and 3-Lie bialgebra. Applying the definition of 3-Lie bialgebra in M theory [10, 11, 12] as a physical application is our future[22]. We know that for the Nambu-Lie group G [23] on the dual space $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$ we have an n-Lie algebra structure. One can also investigate the concept of Nambu-Poisson-Lie group and the relation between 3-Lie bialgebra and Lie bialgebra on the space $\mathcal{G}$ [24].

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