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MINIMAL LINEAR REPRESENTATIONS OF FILIFORM LIE ALGEBRAS AND THEIR APPLICATION FOR CONSTRUCTION OF LEIBNIZ ALGEBRAS

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ABSTRACT. In this paper we find minimal faithful representations of several classes filiform Lie algebras by means of strictly upper-triangular matrices. We investigate Leibniz algebras whose corresponding Lie algebras are filiform Lie algebras such that the action $I \times L \to I$ gives rise to a minimal faithful representation of a filiform Lie algebra. The classification up to isomorphism of such Leibniz algebras is given for low-dimensional cases.

1. INTRODUCTION

According Ado's Theorem, given any finite-dimensional complex Lie algebra $\mathfrak{g}$, there exists a matrix algebra isomorphic to $\mathfrak{g}$. In this way, every finite-dimensional complex Lie algebra can be represented as a Lie subalgebra of the complex general linear algebra $\mathfrak{gl}(n,\mathbb{C})$, formed by all the complex $n \times n$ matrices, for some $n \in \mathbb{N}$. We consider the following integer valued invariant of $\mathfrak{g}$:

$$\mu(\mathfrak{g}) = \min\{\dim(M) \mid M \text{ is a faithful } \mathfrak{g}\text{-module}\}$$

It follows from the proof of Ado's Theorem that $\mu(\mathfrak{g})$ can be bounded by a function depending on only n. This value is also equal to the minimal value n such that $\mathfrak{gl}(\mathbb{C}, n)$ contains a subalgebra isomorphic to $\mathfrak{g}$.

Given a Lie algebra $\mathfrak{g}$, a representation of $\mathfrak{g}$ in $\mathbb{C}^n$ is a homomorphism of Lie algebras $f\colon \mathfrak{g} \to \mathfrak{gl}(\mathbb{C}^n) = \mathfrak{gl}(n,\mathbb{C})$. The natural integer n is called the dimension of this representation. We consider faithful representations because such representations allow us to identify a given Lie algebra with its image under the representation, which is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})$. Representations can be also defined by using arbitrary n-dimensional vector spaces V (see [8]). In such a case, a representation would be a homomorphism of Lie algebras from $\mathfrak{g}$ to the Lie algebra of the endomorphisms of the vector space V, $\mathfrak{gl}(V)$, which is called a $\mathfrak{g}$-module. However, it is sufficient to consider representations on $\mathbb{C}^n$ because there always exists a unique $n \in \mathbb{N}$ such that V is isomorphic to $\mathbb{C}^n$.

Many works are devoted to find the value $\mu(\mathfrak{g})$ of several finite-dimensional Lie algebras. In [5], the value of $\mu(\mathfrak{g})$ for abelian Lie algebras and Heisenberg algebras is found, and moreover the estimated value of $\mu(\mathfrak{g})$ for filiform Lie algebras is given. In the works [3, 7, 9] the authors find the matrix representation of some low-dimensional Lie algebras.

In paper [7] the minimal faithful representation of the filiform Lie algebra $\mathcal{L}_n$ is shown and the authors denote

$$\overline{\mu}(\mathfrak{g}) = \min\{n \in \mathbb{N} \mid \text{subalgebra of } g_n \text{ isomorphic to } \mathfrak{g}\},$$

where g_n is an upper triangular square matrix of dimension n. Moreover, they prove the next proposition

Proposition 1 ([7]). *Let $\mathfrak{g}$ be an n-dimensional filiform Lie algebra. Then $\overline{\mu}(\mathfrak{g}) \geq n$.*

The paper is devoted to find minimal linear representations of some classes of filiform Lie algebras of dimension n. Exactly we find a minimal faithful representation of the filiform Lie algebras $\mathcal{Q}_{2n}, \mathcal{R}_n$ and $\mathcal{W}_n$. Moreover we construct Leibniz algebras using these representations of filiform Lie algebras.

Leibniz algebras, which are a non-antisymmetric generalization of Lie algebras, were introduced in 1965 by Bloh in [4], who called them D-algebras and in 1993 Loday [10] made them popular and studied their (co)homology.

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Definition 1. *An algebra $(L, [-,-])$ over a field $\mathbb{F}$ is called a Leibniz algebra if for any $x, y, z \in L$, the so-called Leibniz identity*

$$\big[[x,y],z\big] = \big[[x,z],y\big] + \big[x,[y,z]\big]$$

holds.

One of the method of classification Leibniz algebras is the study of Leibniz algebras with given corresponding Lie algebras. In the papers [2,6,11,12], Leibniz algebras whose corresponding Lie algebras are naturally graded filiform Lie algebras L_n, Heisenberg algebras, simple Lie $\mathfrak{sl}_2$ and Diamond Lie algebras are studied. Let L be a Leibniz algebra. The ideal I generated by the squares of elements of the algebra L, that is by the set $\{[x,x] : x \in L\}$, plays an important role in the theory since it determines the (possible) non-Lie character of L. From the Leibniz identity, this ideal satisfies

$$[L, I] = 0.$$

Clearly, the quotient algebra L/I is a Lie algebra, called the *corresponding Lie algebra* of L. The map $I \times L/I \to I$, $(i, \overline{x}) \mapsto [i, x]$ endows I with a structure of L/I-module (see [1]).

Denote by $Q(L) = L/I \oplus I$. Then the operation $(-,-)$ defines a Leibniz algebra structure on $Q(L)$, where

$$(\overline{x}, \overline{y}) = \overline{[x,y]}, \quad (\overline{x}, i) = [x, i], \quad (i, \overline{x}) = 0, \quad (i, j) = 0, \qquad x, y \in L,\ i, j \in I.$$

Therefore, given a Lie algebra G and a G-module M, we can construct a Leibniz algebra (G, M) by the above construction. The main problem which occurs in this connections is a description of a Leibniz algebra L, such that the corresponding Leibniz algebra $Q(L)$ is isomorphic to a priory given algebra (G, M).

Now we give definitions of nilpotent and filiform Lie algebras.

For a Lie algebra L consider the following lower central series:

$$L^1 = L, \qquad L^{k+1} = [L^1, L^k] \qquad k \geq 1.$$

Definition 2. *A Lie algebra L is called nilpotent if there exists $s \in \mathbb{N}$ such that $L^s = 0$.*

Definition 3. *A Lie algebra L is said to be filiform if $\dim L^i = n - i$, where $n = \dim L$ and $2 \leq i \leq n$.*

We list some classes of n-dimension filiform Lie algebras with basis $\{e_1, \dots, e_n\}$.

1. Let $\mathcal{L}_n$ be the Lie algebra defined by

$$[e_1, e_i] = -[e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n-1.$$

2. Let $\mathcal{Q}_{2s}$ $(n = 2s)$ be the nilpotent Lie algebra defined by

$$\begin{aligned} &[e_1, e_i] = -[e_i, e_1] = e_{i+1}, && 2 \leq i \leq 2s-2,\\ &[e_{2s+1-i}, e_i] = -[e_i, e_{2s+1-i}] = (-1)^i\, e_{2s}, && 2 \leq i \leq s. \end{aligned}$$

3. Let $\mathcal{R}_n$ be defined by

$$\begin{aligned} &[e_1, e_i] = -[e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n-1,\\ &[e_2, e_i] = -[e_i, e_2] = e_{i+2}, \quad 3 \leq i \leq n-2. \end{aligned}$$

4. Let $\mathcal{W}_n$ be the Lie algebra whose brackets in the basis are:

$$[e_i, e_j] = -[e_j, e_i] = (j - i)e_{i+j}, \quad i + j \leq n.$$

The algebras $\mathcal{L}_n$ and $\mathcal{Q}_{2n}$ are naturally graded filiform Lie algebras. The algebra $\mathcal{W}_n$ is the finite-dimensional Witt algebra.

2. Minimum linear representation of filiform Lie algebras

Proposition 2. *Let $\mathcal{Q}_{2n}$ be a $2n$-dimensional filiform Lie algebra with basis $\{e_i\}_{i=1}^{2n}$. Then its minimal faithful representation is given by*

$$a_1e_1 + a_2e_2 + \dots + a_{2n}e_{2n} \mapsto \begin{pmatrix} 0 & a_2 & -a_3 & \dots & a_{2n-2} & -a_{2n-1} & -2a_{2n}\\ 0 & 0 & a_1 & \dots & 0 & 0 & a_{2n-1}\\ 0 & 0 & 0 & \dots & 0 & 0 & a_{2n-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 0 & a_1 & a_3\\ 0 & 0 & 0 & \dots & 0 & 0 & a_2\\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

Proof. Consider a bilinear map $\varphi\colon Q_{2n}\to \mathfrak{gl}_{2n}$ given by

$$\varphi(e_1)=\sum_{k=2}^{2n-2}E_{k,k+1},\quad \varphi(e_i)=(-1)^iE_{1,i}+E_{2n-i+1,2n}\quad 2\le i\le 2n-1,\quad \varphi(e_{2n})=-2E_{1,2n},$$

where $E_{i,j}$ is the matrix with (i,j)-th entry equal to 1 and others zero.

Checking $[\varphi(e_i),\varphi(e_j)]=\varphi(e_i)\varphi(e_j)-\varphi(e_j)\varphi(e_i)$ for all $1\le i,j\le 2n$, we verify that φ is an isomorphism of algebras. Then by Proposition 1 we obtain that it is minimal. □

Let us denote by $V=\mathbb{C}^{2n}$ the natural $\varphi(Q_{2n})$-module and endow it with a Q_{2n}-module structure by

$$(x,e)=x\varphi(e).$$

Then we obtain

$$\begin{cases}(x_i,e_1)=x_{i+1}, & 2\le i\le 2n-2,\\ (x_1,e_i)=(-1)^ix_i, & 2\le i\le 2n-1,\\ (x_{2n+1-i},e_i)=x_{2n}, & 2\le i\le 2n-1,\\ (x_1,e_{2n})=-2x_{2n},\end{cases}\tag{1}$$

and the remaining products are zero.

Proposition 3. *Let $\mathcal{R}_n$ be a n-dimensional filiform Lie algebra with basis $\{e_i\}_{i=1}^n$. Then its minimal faithful representation is given by*

$$a_1e_1+a_2e_2+\dots+a_ne_n\mapsto\begin{pmatrix}0&a_1&a_2&0&\dots&0&0&0&a_n\\ 0&0&a_1&a_2&\dots&0&0&0&a_{n-1}\\ 0&0&0&a_1&\dots&0&0&0&a_{n-2}\\ 0&0&0&0&\dots&0&0&0&a_{n-3}\\ \vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots\\ 0&0&0&0&\dots&0&a_1&a_2&a_4\\ 0&0&0&0&\dots&0&0&a_1&a_3\\ 0&0&0&0&\dots&0&0&0&a_2\\ 0&0&0&0&\dots&0&0&0&0\end{pmatrix}.$$

Proof. We take bilinear map $\psi\colon\mathcal{R}_n\to\mathfrak{gl}(n,\mathbb{C})$ given by

$$\psi(e_1)=\sum_{i=1}^{n-2}E_{i,i+1},\quad \psi(e_2)=\sum_{i=1}^{n-3}E_{i,i+2}+E_{n-1,n},\quad \psi(e_i)=E_{n+1-i,n},\quad 3\le i\le n.$$

Checking $[\psi(e_i),\psi(e_j)]=\psi(e_i)\psi(e_j)-\psi(e_j)\psi(e_i)$ for all $1\le i,j\le n$, we verify that ψ is an isomorphism of algebras. Then by Proposition 1 we obtain that it is minimal. □

Now, we construct a module $V\times\mathcal{R}_n\to V$, such that

$$(x,e)=x\varphi(e).$$

Then we obtain

$$\begin{cases}(x_i,e_1)=x_{i+1}, & 1\le i\le n-2,\\ (x_i,e_2)=x_{i+2}, & 1\le i\le n-3,\\ (x_{n+1-j},e_j)=x_n, & 2\le j\le n.\end{cases}$$

the remaining products in the action being zero.

Denote by $C_m^n=\binom{m}{n}$ the binomial coefficient.

Proposition 4. *Let $\mathcal{W}_n$ be an n-dimensional filiform Lie algebra with basis $\{e_i\}_{i=1}^n$. Then $\mathcal{W}_n$ is isomorphic to a subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ by φ :*

$$\varphi(e_1)=\sum_{k=1}^{n-2}E_{k,k+1},\quad \varphi(e_2)=\sum_{k=1}^{n-3}\frac{1}{n-k}E_{k,k+2}+E_{n-1,n},$$

$$\varphi(e_i)=\frac{1}{(i-2)!}\Big(\sum_{k=1}^{n-i-1}\big(\sum_{s=0}^{i-2}\frac{(-1)^{i+s}C_{i-2}^s}{n-k-s}\big)E_{k,k+i}+E_{n+1-i,n}\Big),\quad 3\le i\le n,$$

and this faithful representation is minimal.

Proof. We take the isomorphism $\varphi\colon \mathcal{W}_n \to \mathfrak{gl}(n,\mathbb{C})$ such that

$$\varphi(e_1)=\sum_{k=1}^{n-2}E_{k,k+1},\quad \varphi(e_2)=\sum_{s=1}^{n-3}\alpha_s E_{s,s+2}+E_{n-1,n}.$$

Now we consider

$$\varphi(e_3)=[\varphi(e_1),\varphi(e_2)]=\varphi(e_1)\varphi(e_2)-\varphi(e_2)\varphi(e_1)=\big(\sum_{k=1}^{n-2}E_{k,k+1}\big)\ \big(\sum_{s=1}^{n-3}\alpha_s E_{s,s+2}+E_{n-1,n}\big)-$$

$$\big(\sum_{s=1}^{n-3}\alpha_s E_{s,s+2}+E_{n-1,n}\big)\ \big(\sum_{k=1}^{n-2}E_{k,k+1}\big)=\sum_{k=1}^{n-4}(\alpha_{k+1}-\alpha_k)E_{k,k+3}+E_{n-2,n},$$

$$\varphi(e_4)=\frac{1}{2}[\varphi(e_1),\varphi(e_3)]=\frac{1}{2}(\varphi(e_1)\varphi(e_3)-\varphi(e_3)\varphi(e_1))=\frac{1}{2}\Big(\big(\sum_{s=1}^{n-2}E_{s,s+1}\big)\ \big(\sum_{k=1}^{n-4}(\alpha_{k+1}-\alpha_k)E_{k,k+3}+E_{n-2,n}\big)$$
$$-\big(\sum_{k=1}^{n-4}(\alpha_{k+1}-\alpha_k)E_{k,k+3}+E_{n-2,n}\big)\ \big(\sum_{s=1}^{n-2}E_{s,s+1}\big)\Big)=\frac{1}{2}\big(\sum_{k=1}^{n-5}(\alpha_k-2\alpha_{k+1}+\alpha_{k+2})E_{k,k+4}+E_{n-3,n}\big),$$

Let us suppose that

$$\varphi(e_i)=\frac{1}{(i-2)!}\Big(\sum_{p=1}^{n-i-1}\ \big(\sum_{s=0}^{i-2}(-1)^{i+s}C_{i-2}^{s}\alpha_{p+s}\big)E_{p,p+i}+E_{n+1-i,n}\Big),\quad 3\le i\le n.$$

Let us we suppose the previous equality true for $i=k$ and we will consider for $i=k+1$.

$$\begin{aligned}
\varphi(e_{k+1})&=\frac{1}{k-1}[\varphi(e_1),\varphi(e_k)]=\frac{1}{k-1}\big(\varphi(e_1)\varphi(e_k)-\varphi(e_k)\varphi(e_1)\big)\\
&=\frac{1}{k-1}\Big(\frac{1}{(k-2)!}\big(\sum_{t=1}^{n-2}E_{t,t+1}\big)\big(\sum_{p=1}^{n-k-1}\ \big(\sum_{s=0}^{k-2}(-1)^{k+s}C_{k-2}^{s}\alpha_{p+s}\big)E_{p,p+k}+E_{n+1-k,n}\big)\\
&\quad-\frac{1}{(k-2)!}\big(\sum_{p=1}^{n-k-1}\ \big(\sum_{s=0}^{k-2}(-1)^{k+s}C_{k-2}^{s}\alpha_{p+s}\big)E_{p,p+k}+E_{n+1-k,n}\big)\big(\sum_{t=1}^{n-2}E_{t,t+1}\big)\Big)\\
&=\frac{1}{(k-1)!}\Big(\sum_{p=1}^{n-k-2}\ \big(\sum_{s=0}^{k-2}(-1)^{k+s}C_{k-2}^{s}(\alpha_{p+s+1}-\alpha_{p+s})\big)E_{p,p+k+1}+E_{n-k,n}\Big)\\
&=\frac{1}{(k-1)!}\Big(\sum_{p=1}^{n-k-2}\ \big(\sum_{s=0}^{k-1}(-1)^{k+s+1}C_{k-1}^{s}\alpha_{p+s}\big)E_{p,p+k+1}+E_{n-k,n}\Big).
\end{aligned}$$

From the multiplications, where $i+j\le n$

$$[\varphi(e_i),\varphi(e_j)]=\varphi(e_i)\varphi(e_j)-\varphi(e_j)\varphi(e_i)$$
$$=\frac{1}{(i-2)!\,(j-2)!}\Big(\Big(\sum_{p=1}^{n-i-1}\big(\sum_{s=0}^{i-2}(-1)^{i+s}C_{i-2}^{s}\alpha_{p+s}\big)E_{p,p+i}+E_{n+1-i,n}\Big)$$
$$\Big(\sum_{q=1}^{n-j-1}\big(\sum_{r=0}^{j-2}(-1)^{j+r}C_{j-2}^{r}\alpha_{q+r}\big)E_{q,q+j}+E_{n+1-j,n}\Big)$$
$$-\Big(\sum_{q=1}^{n-j-1}\big(\sum_{r=0}^{j-2}(-1)^{j+r}C_{j-2}^{r}\alpha_{q+r}\big)E_{q,q+j}+E_{n+1-j,n}\Big)$$
$$\Big(\sum_{p=1}^{n-i-1}\big(\sum_{s=0}^{i-2}(-1)^{i+s}C_{i-2}^{s}\alpha_{p+s}\big)E_{p,p+i}+E_{n+1-i,n}\Big)\Big)$$
$$=\frac{1}{(i-2)!\,(j-2)!}\Big(\sum_{p=1}^{n-i-j-1}\Big(\big(\sum_{s=0}^{i-2}(-1)^{i+s}C_{i-2}^{s}\alpha_{p+s}\big)\big(\sum_{r=0}^{j-2}(-1)^{j+r}C_{j-2}^{r}\alpha_{p+i+r}\big)$$
$$-\big(\sum_{r=0}^{j-2}(-1)^{j+r}C_{j-2}^{r}\alpha_{p+r}\big)\big(\sum_{s=0}^{i-2}(-1)^{i+s}C_{i-2}^{s}\alpha_{p+j+s}\big)\Big)E_{p,p+i+j}$$
$$+\Big(\sum_{s=0}^{i-2}(-1)^{i+s}C_{i-2}^{s}\alpha_{n+s+1-i-j}-\sum_{r=0}^{j-2}(-1)^{j+r}C_{j-2}^{r}\alpha_{n+r+1-i-j}\Big)E_{n+1-i-j,n}\Big).$$

On the other hand

$$[\varphi(e_i),\varphi(e_j)]=(j-i)\varphi(e_{i+j})=\frac{(j-i)}{(i+j-2)!}\Big(\sum_{p=1}^{n-i-j-1}\big(\sum_{s=0}^{i+j-2}(-1)^{i+j+s}C_{i+j-2}^{s}\alpha_{p+s}\big)E_{p,p+i+j}+E_{n+1-i-j,n}\Big).$$

Next, we have the following system of equations

$$\big(\sum_{s=0}^{i-2}(-1)^{i+s}C_{i-2}^{s}\alpha_{p+s}\big)\big(\sum_{r=0}^{j-2}(-1)^{j+r}C_{j-2}^{r}\alpha_{p+i+r}\big)-\big(\sum_{r=0}^{j-2}(-1)^{j+r}C_{j-2}^{r}\alpha_{p+r}\big)$$
$$\big(\sum_{s=0}^{i-2}(-1)^{i+s}C_{i-2}^{s}\alpha_{p+j+s}\big)+\frac{(i-j)(i-2)!\,(j-2)!}{(i+j-2)!}\sum_{s=0}^{i+j-2}(-1)^{i+j+s}C_{i+j-2}^{s}\alpha_{p+s}=0, \tag{2}$$

where $1\le p\le n-i-j-1,\quad i+j\le n-2.$

$$\sum_{r=0}^{j-2}(-1)^{j+r}C_{j-2}^{r}\alpha_{n+r+1-i-j}-\sum_{s=0}^{i-2}(-1)^{i+s}C_{i-2}^{s}\alpha_{n+s+1-i-j}=\frac{(i-j)(i-2)!\,(j-2)!}{(i+j-2)!},\quad i+j\le n. \tag{3}$$

One of the solutions of the system of equations (2) and (3) is

$$\alpha_i=\frac{1}{n-i},\quad 1\le i\le n-3.$$

Now we will check it. We using the next property of binomial coefficients

$$\sum_{k=0}^{m}\frac{(-1)^kC_m^k}{x+k}=\frac{m!}{x(x+1)\cdots(x+m)},\qquad x\notin\{0,-1,\dots,-m\}. \tag{4}$$

By putting all the values of α_i in the system (2)–(3), and by using the property (4), we get

$$\frac{(i-2)!}{(n-i-p+2)(n-i-p+3)\cdots(n-p)}\cdot\frac{(j-2)!}{(n-i-j-p+2)(n-i-j-p+3)\cdots(n-i-p)}$$

$$-\frac{(j-2)!}{(n-j-p+2)(n-j-p+3)\cdots(n-p)}\cdot\frac{(i-2)!}{(n-i-j-p+2)(n-i-j-p+3)\cdots(n-j-p)}$$

$$+\frac{(i-j)(i-2)!\,(j-2)!}{(n-i-j-p+2)(n-i-j-p+3)\cdots(n-p)}=0,\quad 1\le p\le n-i-j-1,\quad i+j\le n-2,$$

and

$$\frac{(j-2)!}{(i+1)(i+2)\cdots(i+j-1)}-\frac{(i-2)!}{(j+1)(j+2)\cdots(i+j-1)}=\frac{(i-j)(i-2)!\,(j-2)!}{(i+j-2)!},\quad i+j\le n.$$

So, the values of α_i satisfy the system of equations (2)–(3).

From Proposition 1 we get that this representation is minimal. □

Now, we construct a module $V\times\mathcal{W}_n\to V$, such that

$$(x,e)=x\varphi(e).$$

Then we obtain

$$\begin{cases}(x_i,e_1)=x_{i+1}, & 1\le i\le n-2,\\ (x_i,e_2)=\frac{1}{n-i}x_{i+2}, & 1\le i\le n-3,\\ (x_i,e_j)=\frac{1}{(j-2)!}\sum\limits_{s=0}^{j-2}\frac{(-1)^{j+s}C_{j-2}^s}{n-i-s}x_{i+j}, & 3\le j\le n-2,\quad 1\le i\le n-j-1,\\ (x_{n+1-j},e_j)=\frac{1}{(j-2)!}x_n, & 2\le j\le n,\end{cases}$$

and the remaining products in the action are zero.

3. Leibniz algebras constructed by minimal faithful representations of Lie algebra

Now we investigate Leibniz algebras L such that $L/I\cong\mathcal{Q}_{2n}$ and $I=V$ as a $\mathcal{Q}_{2n}$-module.

Further we define the multiplications $[e_i,e_j]$ for $1\le i,j\le 2n$. We put

$$[e_i,e_j]=\begin{cases}e_{i+1}+\sum\limits_{k=1}^{2n}\alpha_{i,1}^k x_k, & i=1,\ 2\le j\le 2n-2,\\ -e_{j+1}+\sum\limits_{k=1}^{2n}\alpha_{1,j}^k x_k, & j=1,\ 2\le i\le 2n-2,\\ (-1)^i e_{2n}+\sum\limits_{k=1}^{2n}\alpha_{i,j}^k x_k, & i=2n-j+1,\ 2\le j\le n,\\ (-1)^{i+1}e_{2n}+\sum\limits_{k=1}^{2n}\alpha_{i,j}^k x_k, & j=2n-i+1,\ 2\le i\le n,\\ \sum\limits_{k=1}^{2n}\alpha_{i,j}^k x_k, & \text{otherwise.}\end{cases}\tag{5}$$

In the multiplication (5), by taking the basis transformation

$$\begin{aligned}&e_1'=e_1-\sum_{k=2}^{2n-2}\alpha_{1,1}^{k+1}x_k-(\alpha_{1,2}^{2n}+\alpha_{2,1}^{2n})x_{2n-1}, && e_2'=e_2-\sum_{k=2}^{2n-2}(\alpha_{1,2}^{k+1}+\alpha_{2,1}^{k+1})x_k,\\ &e_i'=[e_1',e_{i-1}'],\quad 3\le i\le 2n-1, && e_{2n}'=[e_{2n-1}',e_2'],\end{aligned}$$

we obtain

$$\begin{aligned}&[e_1,e_1]=\alpha_{1,1}^1x_1+\alpha_{1,1}^2x_2+\alpha_{1,1}^{2n}x_{2n}, && [e_2,e_1]=-e_3+\alpha_{2,1}^1x_1+\alpha_{2,1}^2x_2,\\ &[e_1,e_i]=e_{i+1},\ 2\le i\le 2n-2, && [e_{2n-1},e_2]=e_{2n}.\end{aligned}\tag{6}$$

There are difficult to classify the general case, therefore we classify low-dimensional Leibniz algebras of such type. It is well known that $\mathcal{L}_4\cong\mathcal{Q}_4$, therefore we start classifying Leibniz algebras such that $L/I\cong\mathcal{Q}_6$.

Using the multiplications (1)–(6), and by checking Leibniz identity, we get the following family of algebras denoted by $\lambda(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9)$:

$$\begin{cases} [e_1, e_1] = \alpha_1 x_6, & [e_1, e_3] = e_4, & [x_1, e_6] = -2x_6, \\ [e_3, e_1] = -e_4, & [e_5, e_3] = \frac{1}{4}\alpha_3 x_6, & [e_2, e_1] = -e_3 + \alpha_2 x_1 + \alpha_3 x_2, \\ [e_4, e_1] = -e_5, & [x_1, e_3] = -x_3, & [e_2, e_2] = \alpha_5 x_3 + \alpha_7 x_4 + \alpha_8 x_5, \\ [e_5, e_1] = -\alpha_4 x_6, & [x_4, e_3] = x_6, & [e_3, e_2] = 4\alpha_2 x_2 - \alpha_6 x_3 - 2\alpha_7 x_5 - \alpha_9 x_6, \\ [e_6, e_1] = -\frac{1}{4}\alpha_3 x_6, & [e_1, e_4] = e_5, & [e_4, e_2] = -2\alpha_2 x_3 + \frac{1}{2}\alpha_6 x_4, \\ [x_2, e_1] = x_3, & [e_3, e_4] = e_6, & [e_2, e_3] = -3\alpha_2 x_2 + \alpha_6 x_3 - \alpha_5 x_4 + \alpha_7 x_5 + \alpha_9 x_6, \\ [x_3, e_1] = x_4, & [x_1, e_4] = x_4, & [e_3, e_3] = -2\alpha_2 x_3 + \frac{1}{2}\alpha_6 x_4, \\ [x_4, e_1] = x_5, & [x_3, e_4] = x_6, & [e_4, e_3] = -e_6 + 2\alpha_2 x_4 - \frac{1}{2}\alpha_6 x_5, \\ [e_1, e_2] = e_3, & [e_1, e_5] = \alpha_4 x_6, & [e_2, e_4] = 4\alpha_2 x_3 - \frac{3}{2}\alpha_6 x_4 + \alpha_5 x_5, \\ [e_5, e_2] = e_6, & [x_1, e_5] = -x_5, & [e_4, e_4] = -2\alpha_2 x_5 - \frac{1}{2}\alpha_3 x_6, \\ [e_6, e_2] = -\alpha_6 x_6, & [x_2, e_5] = x_6, & [e_2, e_5] = -e_6 - 3\alpha_2 x_4 + \frac{3}{2}\alpha_6 x_5, \\ [x_1, e_2] = x_2, & [e_2, e_6] = \frac{5}{2}\alpha_6 x_6, & [e_3, e_5] = 2\alpha_2 x_5 + \frac{3}{4}\alpha_3 x_6, \\ [x_5, e_2] = x_6, & [e_3, e_6] = -2\alpha_2 x_6, & [e_1, e_6] = -2\alpha_2 x_5 - \frac{3}{4}\alpha_3 x_6. \end{cases}$$

Theorem 1. *Let L be a 12-dimensional Leibniz algebra such that $L/I \cong \mathcal{Q}_6$ and I is a natural L/I-module with a minimal faithful representation. Then L is isomorphic to the one of the pairwise non isomorphic algebras given in Appendix A.*

Proof. Let $L(\alpha) := L$ be the 12-dimensional Leibniz algebra given by $\lambda(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9)$. Let $\varphi\colon L(\alpha) \to L(\alpha')$ be the isomorphism of Leibniz algebras:

$$\varphi(e_1) = \sum_{k=1}^{6} A_k e_k + \sum_{k=1}^{6} B_k x_k, \quad \varphi(e_2) = \sum_{k=1}^{6} P_k e_k + \sum_{k=1}^{6} Q_k x_k, \quad \varphi(x_1) = \sum_{k=1}^{6} M_k e_k + \sum_{k=1}^{6} R_k x_k,$$

and the other elements of the new basis are obtained as products of the above elements.

Then, we obtain the following restrictions:

$$A_1 P_2 R_4 \neq 0, \quad A_2 = B_1 = P_1 = M_i = 0, \quad 1 \leq i \leq 6,$$

$$A_6 = \frac{-A_4^2 + 2A_3 A_5}{2A_1}, \quad P_4 = \frac{P_3^2}{2P_2}, \quad R_2 = \frac{A_3 R_1}{A_1}, \quad R_3 = -\frac{A_4 R_1}{A_1}, \quad R_4 = \frac{A_5 R_1}{A_1},$$

$$B_2 = \frac{2A_3^2 \alpha_2}{A_1}, \quad B_3 = \frac{-4\alpha_2 A_3 A_4 - \alpha_6 A_3^2}{2A_1}, \quad B_4 = \frac{2\alpha_2 A_4^2 + \alpha_6 A_3 A_4}{2A_1},$$

$$B_5 = \frac{4\alpha_2 P_2^2 (A_1^2 R_6 - A_4 A_5 R_1 - A_1 A_3 R_5) - \alpha_6 A_4^2 P_2^2 R_1 - 4A_3^2 \alpha_7 P_2^2 R_1}{4A_1 P_2^2 R_1} +$$
$$\frac{\alpha_3 (4A_1 A_5 P_2 P_3 R_1 - 2A_1 A_4 P_3^2 R_1 + 4A_1 A_3 P_2 P_5 R_1 - 4A_1^2 P_2 P_6 R_1 + 4A_1^2 P_2^2 R_5)}{4A_1 P_2^2 R_1},$$

$$Q_1 = -\alpha_2 P_3, \quad Q_2 = \frac{3\alpha_2 (A_3 P_3 - A_4 P_2) - \alpha_3 A_1 P_3}{A_1},$$

$$Q_3 = \frac{2\alpha_2 (4A_5 P_2^2 - A_4 P_2 P_3 - A_3 P_3^2) + \alpha_3 A_1 P_3^2 + 2\alpha_5 A_3 P_2^2 + 2\alpha_6 (A_4 P_2^2 - 2A_3 P_2 P_3)}{2A_1 P_2}$$

$$Q_4 = \frac{\alpha_2 (2A_4 P_3^2 R_1 + A_1 P_2^2 R_5 - 3A_5 P_2 P_3 R_1 + 2A_1 P_2 P_6 R_1 - 2A_3 P_2 P_5 R_1)}{A_1 P_2 R_1} +$$
$$\frac{-4\alpha_3 A_1 P_2 P_5 - 4\alpha_5 A_4 P_2^2 + \alpha_6 (A_3 P_3^2 - 6A_5 P_2^2 + 2A_4 P_2 P_3) + 4\alpha_7 A_3 P_2^2}{4A_1 P_2},$$

$$Q_5 = \frac{\alpha_2 (A_5 P_2^2 P_3^2 R_1 - A_4 P_2 P_3^3 R_1 + 2A_3 P_2^2 P_3 P_5 R_1 - 2A_1 P_2^2 P_3 P_6 R_1 - A_1 P_2^3 P_3 R_5)}{A_1 P_2^3 R_1} +$$
$$\frac{8\alpha_5 A_5 P_2^4 R_1 + \alpha_6 (12A_3 P_2^3 P_5 R_1 - 8A_4 P_2^2 P_3^2 R_1 - 12A_1 P_2^3 P_6 R_1 + 12A_5 P_2^3 P_3 R_1)}{8A_1 P_2^3 R_1} +$$

$$\frac{\alpha_3 A_1(P_2^2P_3P_5 - P_3^4) + \alpha_7(A_4P_2^4 - 2A_3P_2^3P_3) + \alpha_8 A_3 P_2^4}{A_1P_2^3},$$

And

$$\alpha'_1 = \frac{\alpha_1}{A_1P_2^2R_1}, \quad \alpha'_2 = \frac{\alpha_2A_1P_2}{R_1}, \quad \alpha'_3 = \frac{\alpha_3A_1}{R_1}, \quad \alpha'_4 = \frac{\alpha_4A_1}{P_2R_1}, \quad \alpha'_5 = \frac{\alpha_5P_2}{A_1R_1}, \quad \alpha'_6 = \frac{\alpha_6P_2}{R_1},$$

$$\alpha'_7 = \frac{2\alpha_7P_2^3 + \alpha_2(P_3^3 - 6P_2^2P_5)}{2A_1^2P_2^2R_1}, \quad \alpha'_8 = \frac{4\alpha_8P_2^3 - \alpha_6(P_3^3 - 6P_2^2P_5)}{4A_1^3P_2^2R_1},$$

$$\alpha'_9 = \frac{\alpha_9}{A_1^2R_1} + \frac{2\alpha_2(2A_5P_2^2P_3R_1 - A_4P_2P_3^2R_1 - 2A_1P_2^2P_6R_1 + 2A_1P_2^3R_5 + 2A_3P_2^2P_5R_1)}{A_1^3P_2^3R_1^2} + \frac{\alpha_3(P_3^3 - 6P_2^2P_5)}{8A_1^2P_2^3R_1}$$

Considering all the possible cases, we obtain the families of algebras listed in the theorem. □

Now we give the classification of Leibniz algebras L such that $L/I \cong \mathcal{W}_5$ and $L/I \cong \mathcal{R}_7$. We denote the next families of algebras by $\mu(\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_7)$ and $\eta(\beta_1,\beta_2,\beta_3,\beta_4)$:

$$\begin{cases}
[e_1,e_1] = \gamma_1x_5, & [e_2,e_1] = -e_3, & [e_3,e_1] = -2e_4,\\
[e_4,e_1] = -3e_5, & [e_5,e_1] = -\gamma_2x_5, & [e_2,e_4] = \frac{1}{2}(\gamma_3x_4 + \gamma_2x_5),\\
[x_2,e_1] = x_3, & [x_3,e_1] = x_4, & [e_1,e_2] = e_3,\\
[x_2,e_2] = \frac{1}{3}x_4, & [x_3,e_3] = x_5, & [e_4,e_2] = -\frac{1}{2}\gamma_2x_5,\\
[e_5,e_2] = -\gamma_5x_5, & [x_1,e_2] = \frac{1}{4}x_3, & [e_2,e_2] = \gamma_3x_2 + \gamma_4x_3 + \gamma_6x_4,\\
[x_4,e_2] = x_5, & [e_1,e_3] = 2e_4, & [e_2,e_3] = e_5 - \gamma_3x_3 + \gamma_4x_4 + \gamma_7x_5,\\
[e_4,e_3] = 3\gamma_5x_5, & [x_1,e_3] = \frac{1}{12}x_4, & [e_3,e_2] = -e_5 - 2\gamma_4x_4 - \gamma_7x_5,\\
[e_1,e_4] = 3e_5, & [x_1,e_1] = x_2, & [e_3,e_4] = -3\gamma_5x_5,\\
[x_2,e_4] = \frac{1}{2}x_5, & [e_1,e_5] = \gamma_2x_5, & [e_2,e_5] = \gamma_5x_5,\\
[x_1,e_5] = \frac{1}{6}x_5. & &
\end{cases}$$

and

$$\begin{cases}
[e_1,e_1] = \beta_1x_7, & [e_1,e_2] = e_3, & [e_1,e_3] = e_4, & [e_1,e_4] = e_5,\\
[e_1,e_5] = e_6, & [e_1,e_6] = e_7, & [e_1,e_7] = \beta_2x_7, & [e_2,e_1] = -e_3,\\
[e_2,e_2] = \beta_3x_4 + \beta_4x_6, & [e_2,e_3] = e_5 - \beta_3x_5, & [e_2,e_4] = e_6 + \beta_3x_6, & [e_2,e_5] = e_7,\\
[e_2,e_6] = \beta_2x_7, & [e_3,e_1] = -e_4, & [e_3,e_2] = -e_5, & [e_4,e_1] = -e_5,\\
[e_4,e_2] = -e_6, & [e_5,e_1] = -e_6, & [e_5,e_2] = -e_7, & [e_6,e_1] = -e_7,\\
[e_6,e_2] = -\beta_2x_7, & [e_7,e_1] = -\beta_2x_7, & [x_1,e_1] = x_2, & [x_1,e_2] = x_3,\\
[x_1,e_7] = x_7, & [x_2,e_1] = x_3, & [x_2,e_2] = x_4, & [x_2,e_6] = x_7,\\
[x_3,e_1] = x_4, & [x_3,e_2] = x_5, & [x_3,e_5] = x_7, & [x_4,e_1] = x_5,\\
[x_4,e_2] = x_6, & [x_4,e_4] = x_7, & [x_5,e_1] = x_6, & [x_5,e_3] = x_7,\\
[x_6,e_2] = x_7. & & &
\end{cases}$$

Theorem 2. *Let L be a 10-dimensional Leibniz algebra such that $L/I \cong \mathcal{W}_5$ and I is a natural L/I-module with a minimal faithful representation. Then L is isomorphic to the one of the pairwise non isomorphic algebras given in Appendix B.*

Theorem 3. *Let L be a 14-dimensional Leibniz algebra such that $L/I \cong \mathcal{R}_7$ and I is a natural L/I-module with a minimal faithful representation. Then L is isomorphic to the one of the following pairwise non isomorphic algebras:*

$$\begin{array}{lllll}
\eta(0,0,0,0,0), & \eta(0,0,0,1), & \eta(0,0,1,0), & \eta(0,1,0,1), & \eta(0,1,\beta_3,0)_{\beta_3\neq 0},\\
\eta(1,0,0,0,0), & \eta(1,0,0,1), & \eta(1,0,1,0), & \eta(1,1,0,\beta_4), & \eta(1,1,\beta_3,0)_{\beta_3\neq 0},
\end{array}$$

with $\beta_3, \beta_4 \in \mathbb{C}$.

The proofs of Theorem 2 and Theorem 3 are carried out by applying arguments used in Theorem 1.

APPENDIX A. FIRST APPENDIX

$\lambda(0,0,0,0,0,0,0,0,0)$	$\lambda(0,0,0,0,0,0,0,0,1)$	$\lambda(0,0,0,0,0,0,0,1,0)$	$\lambda(0,0,0,1,1,1,\alpha_7,0,\alpha_9)$
$\lambda(0,0,0,0,0,0,1,0,0)$	$\lambda(0,0,0,0,0,0,1,0,1)$	$\lambda(0,0,0,0,0,0,1,1,0)$	$\lambda(1,0,0,0,1,0,1,\alpha_8,\alpha_9)$
$\lambda(0,0,0,0,0,1,0,0,0)$	$\lambda(0,0,0,0,0,1,0,0,1)$	$\lambda(0,0,0,0,0,1,1,0,0)$	$\lambda(1,0,1,0,1,0,\alpha_7,\alpha_8,0)$
$\lambda(0,0,0,0,1,0,0,0,0)$	$\lambda(0,0,0,0,1,0,0,0,1)$	$\lambda(0,0,0,0,1,0,0,1,0)$	$\lambda(0,0,1,1,0,0,1,\alpha_8,0)$
$\lambda(0,0,0,0,0,0,0,1,1)$	$\lambda(0,0,0,1,0,0,0,1,1)$	$\lambda(0,0,0,0,1,1,\alpha_7,0,0)$	$\lambda(0,0,0,0,1,1,\alpha_7,0,1)$
$\lambda(0,0,0,1,0,0,0,0,0)$	$\lambda(0,0,0,1,0,0,0,0,1)$	$\lambda(0,0,0,1,0,0,0,1,0)$	$\lambda(0,0,0,0,1,0,1,\alpha_8,1)$
$\lambda(0,0,0,1,0,0,1,0,0)$	$\lambda(0,0,0,1,0,0,1,0,1)$	$\lambda(0,0,0,1,0,0,1,1,\alpha_9)$	$\lambda(1,0,0,1,1,0,\alpha_7,\alpha_8,\alpha_9)$
$\lambda(0,0,0,1,0,1,0,0,1)$	$\lambda(0,0,0,1,0,1,0,0,0)$	$\lambda(0,0,0,1,1,0,0,0,0)$	$\lambda(0,0,0,1,1,0,1,\alpha_8,\alpha_9)$
$\lambda(0,0,0,0,0,0,1,1,1)$	$\lambda(0,0,0,1,1,0,0,0,1)$	$\lambda(0,0,0,0,1,0,1,\alpha_8,0)$	$\lambda(0,0,1,0,0,1,1,\alpha_8,0)$
$\lambda(0,0,1,0,0,0,0,1,0)$	$\lambda(0,0,1,0,0,0,1,0,0)$	$\lambda(0,0,1,0,0,0,1,1,0)$	$\lambda(0,0,1,0,1,1,\alpha_7,\alpha_8,0)$
$\lambda(0,0,1,0,0,1,0,1,0)$	$\lambda(0,0,1,0,0,0,0,0,0)$	$\lambda(0,0,1,0,1,0,0,0,0)$	$\lambda(0,0,1,1,0,1,\alpha_7,0,0)$
$\lambda(0,0,0,0,0,1,1,0,1)$	$\lambda(0,0,1,0,0,1,0,0,0)$	$\lambda(0,0,1,1,0,0,0,0,0)$	$\lambda(1,0,1,1,\alpha_5,0,\alpha_7,\alpha_8,0)$
$\lambda(0,0,0,0,1,0,0,1,1)$	$\lambda(0,0,1,0,1,0,0,1,0)$	$\lambda(1,0,0,0,0,0,1,1,\alpha_9)$	$\lambda(0,0,1,1,1,0,\alpha_7,\alpha_8,0)$
$\lambda(0,1,0,0,0,0,0,0,0)$	$\lambda(0,1,0,0,0,0,0,1,0)$	$\lambda(0,1,0,0,0,1,0,\alpha_8,0)$	$\lambda(0,1,0,0,1,\alpha_6,0,\alpha_8,0)$
$\lambda(0,1,0,1,0,0,0,0,0)$	$\lambda(0,1,0,1,0,0,0,1,0)$	$\lambda(0,1,0,1,0,1,0,\alpha_8,0)$	$\lambda(0,1,0,1,1,\alpha_6,0,\alpha_8,0)$
$\lambda(0,1,1,0,0,0,0,0,0)$	$\lambda(0,1,1,0,0,0,0,1,0)$	$\lambda(0,1,1,0,0,1,0,\alpha_8,0)$	$\lambda(0,1,1,0,1,\alpha_6,0,\alpha_8,0)$
$\lambda(0,0,0,1,0,0,0,1,1)$	$\lambda(1,0,0,0,0,0,0,0,0)$	$\lambda(1,0,0,0,1,0,0,1,\alpha_9)$	$\lambda(1,0,1,1,\alpha_5,\alpha_6,\alpha_7,0,0)$
$\lambda(1,0,0,0,0,0,0,1,1)$	$\lambda(1,0,0,0,0,0,1,0,0)$	$\lambda(1,0,1,0,0,1,\alpha_7,0,0)$	$\lambda(0,0,1,1,1,\alpha_6,\alpha_7,0,0)_{\alpha_6\neq 0}$
$\lambda(1,0,0,0,0,1,0,0,0)$	$\lambda(1,0,0,0,0,1,0,0,1)$	$\lambda(1,0,0,0,0,1,1,0,\alpha_9)$	$\lambda(1,1,0,1,\alpha_5,\alpha_6,0,\alpha_8,0)$
$\lambda(1,0,0,0,1,0,0,0,1)$	$\lambda(0,0,1,1,0,0,0,1,0)$	$\lambda(0,0,0,1,1,0,0,1,\alpha_9)$	$\lambda(1,0,0,0,1,1,\alpha_7,0,\alpha_9)$
$\lambda(1,0,0,1,0,0,0,0,0)$	$\lambda(1,0,0,1,0,0,0,0,1)$	$\lambda(1,0,0,1,0,0,0,1,\alpha_9)$	$\lambda(1,0,0,1,0,0,1,\alpha_8,\alpha_9)$
$\lambda(1,0,1,0,0,0,0,0,0)$	$\lambda(1,0,1,0,0,0,0,1,0)$	$\lambda(0,0,0,1,0,1,1,0,\alpha_9)$	$\lambda(1,0,0,1,1,\alpha_6,\alpha_7,0,\alpha_9)_{\alpha_6\neq 0}$
$\lambda(1,0,0,0,1,0,0,0,0)$	$\lambda(1,0,0,0,0,0,1,0,1)$	$\lambda(0,0,1,0,1,0,1,\alpha_8,0)$	$\lambda(1,0,1,0,1,\alpha_6,\alpha_7,0,0)_{\alpha_6\neq 0}$
$\lambda(1,0,0,0,0,0,0,1,0)$	$\lambda(1,0,0,0,0,0,0,0,1)$	$\lambda(1,1,0,0,0,1,0,\alpha_8,0)$	$\lambda(1,1,0,0,1,\alpha_6,,\alpha_8,0)$
$\lambda(1,1,0,0,0,0,0,0,0)$	$\lambda(1,1,0,0,0,0,0,1,0)$	$\lambda(1,0,1,0,0,0,1,\alpha_8,0)$	$\lambda(1,1,1,\alpha_4,\alpha_5,\alpha_6,0,\alpha_8,0)$

with $\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8,\alpha_9\in\mathbb{C}$.

APPENDIX B. SECOND APPENDIX

$\mu(0,0,0,0,0,0,0)$	$\mu(0,0,1,0,1,0,0)$	$\mu(0,1,0,0,0,\gamma_6,0)$	$\mu(0,0,1,1,\gamma_5,0,0)_{\gamma_5\neq 0}$
$\mu(0,0,0,0,0,0,1)$	$\mu(1,0,0,0,0,0,0)$	$\mu(1,0,0,0,1,\gamma_6,0)$	$\mu(0,1,0,1,\gamma_5,\gamma_6,0)_{\gamma_5\neq 0}$
$\mu(0,0,0,0,0,1,0)$	$\mu(1,0,0,0,0,0,1)$	$\mu(0,1,0,1,0,\gamma_6,\gamma_7)$	$\mu(0,1,1,\gamma_4,\gamma_5,0,0)_{\gamma_5\neq 0}$
$\mu(0,0,0,0,0,1,1)$	$\mu(0,0,0,1,0,1,\gamma_7)$	$\mu(1,0,0,1,0,\gamma_6,\gamma_7)$	$\mu(1,0,0,1,\gamma_5,\gamma_6,0)_{\gamma_5\neq 0}$
$\mu(0,0,0,0,1,0,0)$	$\mu(0,0,0,1,\gamma_5,1,0)$	$\mu(1,0,1,\gamma_4,0,0,\gamma_7)$	$\mu(1,0,1,\gamma_4,\gamma_5,0,0)_{\gamma_5\neq 0}$
$\mu(0,0,0,0,1,1,0)$	$\mu(0,0,1,1,0,0,\gamma_7)$	$\mu(0,1,1,\gamma_4,0,0,\gamma_7)$	$\mu(1,1,0,\gamma_4,\gamma_5,\gamma_6,0)_{\gamma_5\neq 0}$
$\mu(0,0,0,1,0,0,1)$	$\mu(0,1,0,0,0,\gamma_6,1)$	$\mu(1,1,0,\gamma_4,0,\gamma_6,\gamma_7)$	$\mu(1,1,\gamma_3,\gamma_4,0,0,\gamma_7)_{\gamma_3\neq 0}$
$\mu(0,0,1,0,0,0,0)$	$\mu(1,0,0,0,0,1,\gamma_7)$	$\mu(0,0,0,1,\gamma_5,0,0)_{\gamma_5\neq 0}$	$\mu(1,1,\gamma_3,\gamma_4,\gamma_5,0,0)_{\gamma_3\neq 0,\gamma_5\neq 0}$
$\mu(0,0,1,0,0,0,1)$	$\mu(0,1,0,0,1,\gamma_6,0)$		

with $\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_7\in\mathbb{C}$.

REFERENCES

[1] S. Albeverio, S. A. Ayupov, B. A. Omirov, On nilpotent and simple Leibniz algebras, Comm. Algebra 33 (1) (2005) 159–172.

[2] S. A. Ayupov, L. M. Camacho, A. K. Khudoyberdiyev, B. A. Omirov, Leibniz algebras associated with representations of filiform Lie algebras, J. Geom. Phys. 98 (2015) 181–195.

[3] J. C. Benjumea, J. Núñez, A. F. Tenorio, Minimal linear representations of the low-dimensional nilpotent Lie algebras, Math. Scand. 102 (1) (2008) 17–26.

[4] A. Bloh, A generalization of the concept of a Lie algebra, Sov. Math., Dokl. 6 (1965) 1450–1452.

[5] D. Burde, On a refinement of Ado's theorem, Arch. Math. (Basel) 70 (2) (1998) 118–127.

[6] A. J. Calderón, L. M. Camacho, B. A. Omirov, Leibniz algebras of Heisenberg type, J. Algebra 452 (2016) 427–447.

[7] M. Ceballos, J. Núñez, A. F. Tenorio, Representing filiform Lie algebras minimally and faithfully by strictly upper-triangular matrices, J. Algebra Appl. 12 (4) (2013) 1250196, 15.

[8] W. Fulton, J. Harris, Representation theory, vol. 129 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1991, a first course, Readings in Mathematics.

[9] R. Ghanam, G. Thompson, Minimal matrix representations of five-dimensional Lie algebras, Extracta Math. 30 (1) (2015) 95–133.

[10] J.-L. Loday, Une version non commutative des algèbres de Lie: les algèbres de Leibniz, Enseign. Math. (2) 39 (3-4) (1993) 269–293.

[11] B. A. Omirov, I. S. Rakhimov, R. M. Turdibaev, On description of Leibniz algebras corresponding to sl_2, Algebr. Represent. Theory 16 (5) (2013) 1507–1519.

[12] S. Uguz, I. A. Karimjanov, B. A. Omirov, Leibniz algebras associated with representations of the Diamond Lie algebra, arXiv:1507.01349 (2015).

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