On nilpotency in Leibniz algebras

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Abstract

The main result of this paper is to prove that if a (right) Leibniz algebra L is *right nilpotent* of degree n, then L is *strongly nilpotent* of degree less or equal to $4n^2 - 2n + 1$.

Résumé

Nous prouvons que toute algèbre de Leibniz (droite) L *nilpotente à droite* d'indice n est *fortement nilpotente* d'un indice inférieur ou égal à $4n^2 - 2n + 1$.

Keywords. *Leibniz algebra, right nilpotency, left nilpotency, nilpotency, strong nilpotency, index.*

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1 Introduction

In [1] it is proved that a Malcev algebra is strongly nilpotent if and only if it is right nilpotent. So for Malcev algebras right nilpotency, left nilpotency and strong nilpotency are equivalent to nilpotency. Since Malcev algebra is anti-commutative, right nilpotency and left nilpotency are equivalent. This result fails for Leibniz algebras, see for example [4, Exemple 3.3], which is left nilpotent and not right nilpotent.

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Using the notion of Es_k-right nil (or Es_k-left nil) we prove that if an ideal B of a (right) Leibniz algebra L is *right nilpotent* of degree n, then B is *strongly nilpotent* of degree less or equal to $4n^2 - 2n + 1$.

In section 2, we give some definitions that we will used along the paper, then in the section 3, we prove some results on right products of length n. The section 4 is devoted to right products of weight n in the ideal B and in the section 5, we give the main results.

2 Preliminairies

Throughout this paper, F will be a field of characteristic not 2. All vector spaces and algebras will be finite dimensional over F. Let n be a nonnegative integer, and let us denote the set $\{1, 2, \cdots, n\}$ by $\mathbb{I}(n)$.

Definition 2.1 *(Leibniz algebra) [5, 4]*
A Leibniz algebra is a vector space L equipped with a bilinear map $[-,-] : L \times L \longrightarrow L$, satisfying the Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y] \text{ for any } x, y, z \in L. \tag{1}$$

If the condition $[x, x] = 0$ is fulfilled, the Leibniz identity is equivalent to the so-called Jacobi identity. Therefore Lie algebras are particular cases of Leibniz algebras. Algebras which satisfy (1) are also call right Leibniz algebras and left Leibniz algebras are defined as followed:

Definition 2.2 *(left Leibniz algebra) [4]*
A left Leibniz algebra is a vector space L equipped with a bilinear map $[-,-] : L \times L \longrightarrow L$, satisfying the left Leibniz identity:

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \text{ for any } x, y, z \in L. \tag{2}$$

It follows from the Leibniz identity (1) that in any Leibniz algebra one has

$$[y, [x, x]] = 0, \ [z, [x, y]] + [z, [y, x]] = 0, \text{ for all } x, y, z \in L.$$

Definition 2.3 *A subspace H of a Leibniz algebra L is called left (respectively right) ideal if for $a \in H$ and $x \in L$ one has $[x, a] \in H$ (respectively $[a, x] \in H$). If H is both left and right ideal, then H is called (two-sided) ideal.*

Let us denote the product $[a, b]$ by ab for all a, b in L. $Ess(L)$ will be the ideal generated by all the squares of the elements of L.

For an ideal B of a Leibniz algebra L, we introduce the notations and following terminologies:

Let P be a product of m factors $s_m, s_{m-1}, \cdots, s_1$, that have been associated in an arbitrary way. We suppose that n or more factors belong to B. We say that the product P is of length m and of weight n with respect to the ideal B or more simply that P is of length m and of weight n in B. The length m of P will be noted $\#(P)$ and its weight n will be noted $\#_B(P)$.

When $P = ((\cdots((s_m s_{m-1}) s_{m-2} \cdots) s_3) s_2) s_1$ where the association is made always right, we say that P is a *right product* and we write $P = s_m s_{m-1} s_{m-2} \cdots s_1$. Similarly, if $P = s_1(s_2(s_3(\cdots s_{m-2}(s_{m-1}s_m))\cdots))$ where the association is made always left, we say that P is a *left product*.

Let $S_1, S_2, \cdots, S_p$ be right products. One can write the right product (with S_j as factors) $N = S_p S_{p-1} \cdots S_1$. We call N a standard product.

Definitions 2.4

- *A subspace B of the underlying vector space L is right nilpotent if $B^n = \{0\}$ for some $n \geq 1$, where $B^1 = B$ and $B^{n+1} = B^n \cdot B$. By convention, we set $B^0 = L$. Notice that B^n is generated by right products of length n and weight n in B.*
- *A subspace B of the underlying vector space L is left nilpotent if ${}^nB = 0$ for some $n \geq 1$, where ${}^1B = B$ and ${}^{1+n}B = B \cdot ({}^nB)$. By convention, we set ${}^0B = L$. Notice that ${}^nB$ is generated by left products of length n and weight n in B.*

Definitions 2.5

- *Let $B^{\{n\}}$ be the subspace of the underlying vector space L generated by all the products of length n in B, associated in arbitrary way. We say that the ideal B is nilpotent if there exists an integer n such that $B^{\{n\}} = \{0\}$.*
- *Let $B^{\langle n \rangle}$ be the subspace generated by all products of elements in L with at least n elements in B. A subspace B is strongly nilpotent if $B^{\langle n \rangle} = \{0\}$ for some $n \geq 1$.*

Naturally, $B^{\langle n \rangle}$ is an ideal of L and one has
$B \supseteq B^{\langle 1 \rangle} \supseteq B^{\langle 2 \rangle} \supseteq \cdots \supseteq B^{\langle n \rangle} \supseteq \cdots$ and $B^{\langle i \rangle} B^{\langle j \rangle} \subseteq B^{\langle i+j \rangle}$ for all nonnegative integers $i, j \geq 1$.

Definitions 2.6

- *Let D be a subspace of the underlying vector space L. Let k be a nonnegative integer. $D_{(L,k)}$ is the vector subspace generated by all right products: "$da_k \cdots a_3 a_2 a_1$" where d belongs to D and a_i belongs to L for any integer $i \in \mathbb{I}(k)$.*

- *Let D be a subspace of the underlying vector space L. Let k be a a nonnegative integer. ${}_{(L,k)}D$ is the vector subspace generated by all left products: "$a_1(a_2(a_3(\cdots(a_kd)))\cdots)$" where d belongs to D and a_i belongs to L for any integer $i \in \mathbb{I}(k)$.*

Definitions 2.7

- *Let $B \neq \{0\}$ be an ideal of the Leibniz algebra L. If there is an integer $k \geq 1$, such that the ideal $\mathrm{Es}(B) = B \cap \mathrm{Ess}(L)$ satisfies $\mathrm{Es}(B)_{(L,k)} = \mathrm{Es}(B)\underbrace{AAA\cdots A}_{k \text{ times}} = \{0\}$, we will say that B is Es_k-right nil.*

- *Let $B \neq \{0\}$ be an ideal of the Leibniz algebra L. If there is an integer $k \geq 1$, such that the ideal $\mathrm{Es}(B) = B \cap \mathrm{Ess}(L)$ satisfies ${}_{(L,k)}\mathrm{Es}(B) = \underbrace{A\cdots AAA}_{k \text{ times}}\mathrm{Es}(B) = \{0\}$, we will say that B is Es_k-left nil.*

Definition 2.8 *Let L be a Leibniz algebra and B an ideal in L. Let a, b ly in L. If $a - b \in B$, we will say that $a \equiv b$ (modulo B).*

3 Right products in the Leibniz algebra L

Lemma 3.1 *Let L be a Leibniz algebra and B an ideal of L. For all $a \in L$ and for all $b \in B$, $ab + ba \in \mathrm{Es}(B) = B \cap \mathrm{Ess}(L)$.*

Proof: Obvious. □

Lemma 3.2 *For any integer $n \geq 1$, $B^n \subseteq {}^nB + \mathrm{Es}(B)$.*

Proof: For $n = 1$ or 2, the result is obvious. Let $n = 3$ and a, b, c be elements of B, we have $a(bc) + (bc)a \in Es(B)$ and so $(bc)a$ equals $a(bc)$ modulo $Es(B)$. So, we can write $B^3 \subseteq {}^3B + Es(B)$. Let us set by hypothesis that $B^p \subseteq {}^pB + Es(B)$ for all integer $p \leq n$ and prove that $B^{n+1} \subseteq {}^{1+n}B + Es(B)$. We have

$$\begin{aligned} B^{n+1} &= \left(B^{n-1} \cdot B\right) \cdot B \subseteq B \cdot \left(B^{n-1} \cdot B\right) + Es(B) \\ &\subseteq B \cdot \left[B \cdot B^{n-1} + Es(B)\right] + Es(B) \subseteq B \cdot (B \cdot B^{n-1}) + Es(B) \\ &\subseteq B \cdot (B \cdot \left[{}^{-1+n}B + Es(B)\right]) + Es(B) \\ &\subseteq B \cdot (B \cdot {}^{-1+n}B) + Es(B) \subseteq {}^{1+n}B + Es(B). \end{aligned}$$

□

Lemma 3.3 *Let L be a Leibniz algebra and B an ideal of L. Let us define $B_0 = L$, $B_1 = B$ and $B_k = B^k + \mathrm{Es}(B)$ for all integer $k \geq 2$; B_k is an ideal of L, which satisfies $B_k \supseteq B_{k+1}$.*

Proof: It is known that $Es(B), B_0, B_1$ are ideals. Let us assume that for an integer $k \geq 2$, B_k is an ideal. Then one has $B_k \cdot A \subseteq B^k + Es(B)$ and $A \cdot B_k \subseteq B^k + Es(B)$. Let us show that B_{k+1} is also an ideal. Indeed;

$$
\begin{aligned}
B_{k+1} \cdot A &= \big(B^{k+1} + Es(B)\big) \cdot A \subseteq B^{k+1} \cdot A + Es(B) \\
&\subseteq \big(B^k \cdot B\big) \cdot A + Es(B) \subseteq A \cdot \big(B^k \cdot B\big) + Es(B) \text{ (see Lemma 3.1)} \\
&\subseteq \big(A \cdot B^k\big) \cdot B + (A \cdot B) \cdot B^k + Es(B) \\
&\subseteq \big(B^k + Es(B)\big) \cdot B + B \cdot B^k + Es(B) \\
&\subseteq B^{k+1} + B \cdot B^k + Es(B) \subseteq B^{k+1} + B \cdot \big({}^kB + Es(B)\big) + Es(B) \\
&\subseteq B \cdot {}^kB + B^{k+1} + Es(B) \subseteq {}^{1+k}B + B^{k+1} + Es(B) \\
&\subseteq B^{k+1} + Es(B) = B_{k+1} \text{ (see Lemma 3.2).}
\end{aligned}
$$

Thanks to Lemma 3.1, we have $A \cdot B_{k+1} \subseteq B_{k+1} \cdot A + Es(B)$. So we obtain $A \cdot B_{k+1} \subseteq B^{k+1} + Es(B) = B_{k+1}$. □

Proposition 3.1 *For a given right product $P_0 = a_m a_{m-1} a_{m-2} \cdots a_3 a_2 a_1$ and Q_0 an arbitrary product of length m', let us set recursively, for any integer $i \in \mathbb{I}(m-1)$, $P_i = a_m a_{m-1} \cdots a_{i+1} = \prod_{j=1}^{m-i} a_{m-j+1}$ and $Q_i = -Q_{i-1} a_i$. Then:*

$$
T_m = Q_0 P_0 = \sum_{i=1}^{m-1} Q_{i-1} P_i a_i + Q_{m-1} a_m.
$$

Proof: First of all let us define the following products: For $i \in \mathbb{I}(m-1)$, we set $Q'_{i-1} = Q_i$ and $a'_{m-i+1} = a_{m-i+2}$. It follows that

$$
P'_i = \prod_{j=1}^{m-i} a'_{m-j+1} = \prod_{j=1}^{m-i} a_{m-j+2} = \prod_{j=1}^{m+1-(i+1)} a_{m+1-j+1} = P_{i+1}.
$$

Now if $m = 2$, then
$T_2 = Q_0 (a_2 a_1) = Q_0 a_2 a_1 - Q_0 a_2 a_1 = Q_0 P_1 a_1 + Q_1 a_1,$ and if $m = 3$;

$$
\begin{aligned}
T_3 &= Q_0 (a_3 a_2 a_1) \\
&= Q_0 (a_3 a_2) a_1 - (Q_0 a_1)(a_3 a_2) = Q_0 P_1 a_1 + Q_1 (a_3 a_2) \\
&= Q_0 P_1 a_1 + Q_1 a_3 a_2 - Q_1 a_2 a_3 \\
&= Q_0 P_1 a_1 + Q_1 P_2 a_2 + Q_2 a_3.
\end{aligned}
$$

Assume that

$$T_m = \sum_{i=1}^{m-1} Q_{i-1} P_i a_i + Q_{m-1} a_m. \tag{3}$$

Then for $P = a_{m+1} a_m a_{m-1} \cdots a_3 a_2 a_1$, we have:

$$\begin{aligned} T_{m+1} &= Q_0 \prod_{k=1}^{m+1} a_{m-k+2} = Q_0 \left[\prod_{k=1}^{m} a_{m-k+2} a_1 \right] \\ &= Q_0 \prod_{k=1}^{m} a_{m-k+2} a_1 - Q_0 a_1 \prod_{k=1}^{m} a_{m-k+2} \\ &= Q_0 \prod_{k=1}^{m} a_{m-k+2} a_1 + Q_1 \prod_{k=1}^{m} a_{m-k+2} \end{aligned}$$

It follows that

$$\begin{aligned} T_{m+1} &= Q_0 \left(P_0' a_1 \right) = Q_0 P_0' a_1 + Q_1 P_0' \\ &= Q_0 P_1 a_1 + Q_0' P_0' \end{aligned}$$

Since the length of P_0' is m we can write:

$$\begin{aligned} Q_0' P_0' &= \sum_{i=1}^{m-1} Q_{i-1}' P_i' a_i' + Q_{m-1}' a_m' \\ &= \sum_{i=1}^{m-1} Q_i P_{i+1} a_{i+1} + Q_m a_{m+1}. \end{aligned}$$

Thanks to Equation (3), we obtain:

$$\begin{aligned} T_{m+1} &= Q_0 P_1 a_1 + Q_0' P_0' \\ &= Q_0 P_1 a_1 + \sum_{i=1}^{m-1} Q_i P_{i+1} a_{i+1} + Q_m a_{m+1} \\ &= \sum_{i=1}^{m} Q_{i-1} P_i a_i + Q_m a_{m+1}. \end{aligned}$$

□

Remarque 3.1 *With the hypothesis of Proposition 3.1, note p, p', p'' the respective weight of P, Q_{i-1}, P_i $(1 \leq i \leq m)$ with regard to the ideal B. Let us consider the following table:*

$P_{m-1} = a_m$		$\cdots$	$P_{i-1} = P_i a_i$	$\cdots$
$Q_{m-1} = -Q_{m-2} a_{m-1}$		$\cdots$	$Q_{i-1} = -Q_{i-2} a_{i-1}$	$\cdots$

Let $\Lambda = \{(a_i)_{1 \leq i \leq m}, (b_j)_{1 \leq j \leq m'}\}$ be the set of all factors of P. It is easy to check that Λ also produces $(Q_{k-1} P_k)\, a_k, Q_{m-1} a_m$ for $k \in \mathbb{I}(m-1)$. Then,

$\cdots$	$P_{i+j} = P_{i+j+1}a_{i+j+1}$	$\cdots$		$P_1 = P_2a_2$	$P_0 = P_1a_1$
$\cdots$	$Q_{i+j} = -Q_{i+j-1}a_{i+j}$	$\cdots$		$Q_1 = -Q_0a_1$	$Q_0 = Q_0$

one has for an integer k in $\mathbb{I}(m-1)$:

$$\begin{aligned} \#(P) &= \#(Q_{k-1}) + \#(P_k) + 1 & (4)\\ \#_B(P) &= \#_B(Q_{k-1}) + \#_B(P_k) + \#_B(a_k) & (5)\\ \#(P) &= \#(Q_{m-1}) + 1 & (6)\\ \#_B(P) &= \#_B(Q_{m-1}) + \#_B(a_m). & (7)\end{aligned}$$

Lemma 3.4 *Any product T with length m in a Leibniz algebra L is a linear combination of right products of length m.*

Proof: By induction on the length m, we have: If m equals 1 or 2, there is nothing to do. If $m = 3$, one can notice that $T = abc$ or $a(bc) = abc - acb$ for all a, b, c in L. The lemma is also obvious.
Let us suppose that the lemma is true for a product which length is strictly less than $m \geq 4$.

Now for a given product (with length m) $T = Q_0P_0$ where P_0 is a right product of length n such that $m > n \geq 1$. Thanks to Proposition 3.1,

$$T = Q_0P_0 = \sum_{i=1}^{n-1} Q_{i-1}P_ia_i + Q_{n-1}a_n.$$

The length of following products $Q_{i-1}P_i$ for $i \in \mathbb{I}(n-1)$ and Q_{n-1} is $m-1$, so they are linear combinations of right products of length $m-1$. Then T is a linear combination of right products of length m. $\square$

4 Right products of weight n in the ideal B

Lemma 4.1 *Any product T with length m and weight n with regard to the ideal B of Leibniz algebra L is a linear combination of right products of length m and weight n.*

Proof: By induction on the length m, we have:
The lemma is obvious if $m \leq 2$. If $m = 3$, then there are a_1, a_2, a_3 elements in L such that P is one of the following linear combinations of right products: $a_3a_2a_1$ and $a_3(a_2a_1) = a_3a_2a_1 - a_3a_1a_2$. So for $m = 3$, P is a linear combination of right products.

Let us suppose that the lemma is true for a product which length is strictly less than $m \geq 4$.

Now for a given product (with length m and weight $n \leq m$) $T = Q_0P_0$ where P_0 is a right product of length m' such that $m > m' \geq 1$. Thanks to Proposition 3.1,

$$T = Q_0P_0 = \sum_{i=1}^{m'-1} Q_{i-1}P_ia_i + Q_{m'-1}a_{m'}.$$

The length of following products $Q_{i-1}P_i$ for $i \in \mathbb{I}(m'-1)$ and $Q_{m'-1}$ is $m-1$, so they are linear combinations of right products of length $m-1$. Then T is a linear combination of right products of length m.

Thanks to the equations; Equation (4) to Equation (7), it is clear that for $i \in \mathbb{I}(p-1)$, the weight of the standart product $Q_{i-1}P_ia_i$ and the weight of $Q_{p-1}a_p$ are equal to n. $\square$

Lemma 4.2 *Let L be a Leibniz algebra and $B \neq \{0\}$ an ideal in L. Let $P_0 = a_ma_{m-1}a_{m-2}\cdots a_3a_2a_1$ be a right product with length m and weight $n \geq 1$ in B. Then P_0 belongs to B_n.*

Proof: Let σ be an injective map of $\mathbb{I}(n)$ to dans $\mathbb{I}(m)$ such that $i < j$ implies that $\sigma(i) < \sigma(j)$ and for all $j \in \mathbb{I}(n)$, $a_{\sigma(j)} \in B$. Let us also define for all integer $k \in \mathbb{I}(n-1)$, the following products:

$$\begin{aligned} Q'_0 =& a_ma_{m-1}\cdots a_{\sigma(n)+1}, \\ Q_k =& Q'_{k-1}a_{\sigma(n-k+1)}, \\ Q'_k =& Q_ka_{\sigma(n-k+1)-1}\cdots a_{\sigma(n-k)+1}, \\ Q_n =& Q'_{n-1}a_{\sigma(1)}\cdots a_1. \end{aligned}$$

Clearly, Q_1 belongs to $B \subseteq B^1 + Es(B) = B_1$ and also Q'_1 belongs to B. Since B is an ideal $Q_2 = Q'_1a_{\sigma(n-j+1)}$ belongs to $B \cdot B \subseteq B^2 + Es(B) = B_2$. By induction, let us suppose that for integer $j \in \mathbb{I}(n-1)$, we have Q_j belongs to $B_j = B^j + Es(B)$. Then let us show that Q_{j+1} is an element of $B_{j+1} = B^{j+1} + Es(B)$. Indeed, we have,
$Q'_j = Q_ja_{\sigma(n-j+2)-1}\cdots a_{\sigma(n-j+1)+1}$ is an element of the ideal B_j and so on,
$Q_{j+1} = Q'_ja_{\sigma(n-j+1)}$ belongs to $B_j \cdot B \subseteq B^{j+1} + Es(B) = B_{j+1}$.
Then we have proved that $P_0 = Q_n$ is an element of B_n. $\square$

Lemma 4.3 *Let k, ℓ be integers such that $1 \leq k \leq \ell$ and let L be a Leibniz algebra and B an ideal of L which is Es_k-right nil. A right product $P = a_ma_{m-1}a_{m-2}\cdots a_3a_2a_1$, of length m and weight (in B) n greater or equal to 2ℓ, belongs to $B^{\ell}_{(L,k)}$.*

Proof: The right product $Q = a_m a_{m-1} a_{m-2} \cdots a_{k+1}$ is of weight greater or equal to ℓ, indeed let n' be the weight of Q and n'' be the weight of $a_k \cdots a_3 a_2 a_1$. We have $0 \leq n'' \leq k$ and the equality $P = Q a_k \cdots a_3 a_2 a_1$ implies that $n' \leq n \leq k + n'$. So $n' \geq n - k \geq 2\ell - k \geq \ell$. The Lemma 4.2 tells that $Q \in B_\ell$. And so on, $P = Q a_k \cdots a_3 a_2 a_1 \in (B_\ell)_{(L,k)} = \left(B^\ell + Es(B)\right)_{(L,k)} = B^l_{(L,k)}$ since B est un ideal Es_k-right nil. □

Lemma 4.4 *Let k be an integer such that the ideal B is Es_k-right nil. Let P be product of weight $t \geq 4k^2 - 2k + 1$ with regard to the ideal B. then P is a linear combination of right products Q_j ($P = \sum_{j \text{ fini}} \mu_j Q_j$) such that, for any j, we have Q_j belongs to $\left(B^k\right)_{(L,k)}$ or has at least one factor in $\left(B^k\right)_{(L,k)}$.*

Proof: Let $k > 1$ and $t \geq 4k^2 - 2k + 1$. Thanks to Lemma 4.1, any product P of weight greater or equal to t is a linear combination of right products of weight greater or equal to t. Let $P = \sum_j \mu_j Q_j$ where Q_j is a right product of weight greater or equal to t.

For any j we have $Q_j = s_{j,p} s_{j,p-1} \cdots s_{j,1}$ where $s_{j,i} \in L$ (p is the length of Q_j).

- if, there is one element s_{j,i_0} such that it's weight is greater or equal to $2k$, then s_{j,i_0} belongs to $\left(B^k\right)_{(L,k)}$ by application of Lemma 4.3. And so on Q_j has a factor in $\left(B^k\right)_{(L,k)}$.

- else, every factor $s_{j,i}$ has a weight strictly less than $2k$. Let q be the number of factors $s_{j,i}$ with a weight greater or equals to 1. Then one has $q\,(2k-1) \geq t = 4k^2 - 2k + 1$ and then $q > 2k$.
  When the weight of $s_{j,i}$ is greater or equal to 1, we have $s_{j;i}$ belongs to B. So Q_j is of weight greater or equal to q. Since $q \geq 2k$, we have $Q_j \in \left(B^k\right)_{(L,k)}$ thanks to Lemma 4.3.

For language simplification, we will say that Q_j has at least one factor in $\left(B^k\right)_{(L,k)}$. □

5 Main Theorem

Theoreme 5.1 *Let k' be a nonnegative integer, let L be a Leibniz algebra and let B be an ideal of L which is $\mathrm{Es}_{k'}$-right nil. Then the following assertions are equivalents:*

(i) *B is right nilpotent ;*

(ii) *B is nilpotent ;*

(iii) *B is strongly nilpotent. ;*

Proof: Indeed, for any integer $k \geq 1$, the vectors spaces's inclusions $B^k \subseteq B^{\{k\}} \subseteq B^{\langle k \rangle}$ tell us that $(iii) \Rightarrow (ii) \Rightarrow (i)$.
Furthermore, suppose that there is an integer $\ell' \geq 1$ such that $B^{\ell'} = \{0\}$. Let us define $k = \max\{k', \ell'\}$, then for an integer *ell* such that $\ell \geq 4k^2 - 2k + 1$, the Lemma 4.4 tells us that any product P with weight greater or equal to $\ell \geq 4k^2 - 2k + 1$, in B is a linear combination of right products which have at least one factor in $\left(B^k\right)_{(L,k)} \subseteq \left(B^{k'}\right)_{(L,k)} = \{0\}$. And so on $P = 0$. Then $B^{\langle \ell \rangle} = 0$. The implication $(i) \Rightarrow (iii)$ is done. $\square$

Corollary 5.1 *Let L be a Leibniz algebra. The following assertions are equivalents:*

(i) *L is right nilpotent ;*

(ii) *L is nilpotent ;*

(iii) *L is strongly nilpotent.*

Proof: Clearly we have $(iii) \Rightarrow (ii) \Rightarrow (i)$.
Notice that the ideal $Ess(A)$ is a subset of A^2. Assume that we have (i) and then let us show that (iii) is verified.
We know that there is an integer $\ell' > 1$ which satisfies that $A^{\ell'} = \{0\}$. So we have
$Ess(A)\underbrace{AAA\ldots A}_{\ell'-2 \text{ times}} \subseteq (A^2)\underbrace{AAA\ldots A}_{\ell'-2 \text{ times}} \subseteq A^{\ell'} = \{0\}$. So A is $Es_{\ell'}$-right nil. With part of the proof of the theorem 5.1, we can conclude that $(i) \Rightarrow (iii)$. $\square$

Remarque 5.1

- For (right) Leibniz algebra L, the ideal $Ess(L)$ is always Es_1-left nil. But if $Ess(L)$ is not Es_k-right nil for some integer k, L is not nilpotent.
- For (left) Leibniz algebra L, the ideal $Ess(L)$ is always Es_1-right nil. But if $Ess(L)$ is not Es_k-left nil for some integer k, L is not nilpotent (cf. [3]).
- By duality, we have also proved that if a (left) Leibniz algebra L is *left nilpotent* of degree n, then L is *strongly nilpotent* of degree less or equal to $4n^2 - 2n + 1$.

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