

LEIBNIZ ALGEBRAS CONSTRUCTED BY REPRESENTATIONS OF GENERAL DIAMOND LIE ALGEBRAS

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ABSTRACT. In this paper we construct a minimal faithful representation of the $(2m + 2)$ -dimensional complex general Diamond Lie algebra, $\mathfrak{D}_m(\mathbb{C})$, which is isomorphic to a subalgebra of the special linear Lie algebra $\mathfrak{sl}(m + 2, \mathbb{C})$. We also construct a faithful representation of the general Diamond Lie algebra $\mathfrak{D}_m$ which is isomorphic to a subalgebra of the special symplectic Lie algebra $\mathfrak{sp}(2m + 2, \mathbb{R})$. Furthermore, we describe Leibniz algebras with corresponding $(2m + 2)$ -dimensional general Diamond Lie algebra $\mathfrak{D}_m$ and ideal generated by the squares of elements giving rise to a faithful representation of $\mathfrak{D}_m$.

1. INTRODUCTION

Leibniz algebras are a non-antisymmetric generalization of Lie algebras. They were introduced in 1965 by Bloh in [5], who called them D -algebras, and in 1993 Loday [9] made them popular and studied their (co)homology.

Definition 1. An algebra $(L, [-, -])$ over a field $\mathbb{F}$ is called a Leibniz algebra if for any $x, y, z \in L$, the so-called Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds.

Since first works about Leibniz algebras around 1993 several researchers have tried to find analogs of important theorems in Lie algebras. For instance, the classical results on Cartan subalgebras [1, 10], Engel's theorem [3], Levi's decomposition [4], properties of solvable algebras with given nilradical [8] and others from the theory of Lie algebras are also true for Leibniz algebras.

Namely, an analogue of Levi's decomposition for Leibniz algebras asserts that any Leibniz algebra is decomposed into a semidirect sum of its solvable radical and a semisimple Lie algebra. Therefore, the main problem of the description of finite-dimensional Leibniz algebras consists of the study of solvable Leibniz algebras.

In fact, each non-Lie Leibniz algebra L contains a non-trivial ideal (later denoted by I), which is the subspace spanned by the squares of the elements of the algebra L . Moreover, it is easy to see that this ideal belongs to the right annihilator of L , that is $[L, I] = 0$. Note also that the ideal I is

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the minimal ideal with the property that the quotient algebra L/I is a Lie algebra (the quotient algebra is said to be the corresponding Lie algebra to the Leibniz algebra L).

One of the approaches to the investigation of Leibniz algebras is a description of such algebras whose quotient algebra with respect to the ideal I is a given Lie algebra [2, 7, 11, 12].

The map $I \times (L/I) \rightarrow I$ defined as $(v, \bar{x}) \mapsto [v, x]$, $v \in I$, $x \in L$, endows I with a structure of (L/I) -module. If we consider the direct sum of vector spaces $Q(L) = (L/I) \oplus I$, then the operation $(-, -)$ defines a Leibniz algebra structure on $Q(L)$ with multiplication

$$[\bar{x}, \bar{y}] = \overline{[x, y]}, \quad [\bar{x}, v] = [x, v], \quad [v, \bar{x}] = 0, \quad [v, w] = 0, \quad x, y \in L, \quad v, w \in I.$$

Therefore, for given a Lie algebra G and a G -module M , we can construct a Leibniz algebra $L = G \oplus M$ by the above construction.

The real general Diamond Lie algebra $\mathfrak{D}_m$ is a $(2m + 2)$ -dimensional Lie algebra with basis

$$\{J, P_1, P_2, \dots, P_m, Q_1, Q_2, \dots, Q_m, T\}$$

and non-zero relations

$$[J, P_k] = Q_k, \quad [J, Q_k] = -P_k, \quad [P_k, Q_k] = T, \quad 1 \leq k \leq m.$$

The complexification (for which we shall keep the same symbol $\mathfrak{D}_m(\mathbb{C})$) of the Diamond Lie algebra is $\mathfrak{D}_m \otimes_{\mathbb{R}} \mathbb{C}$, and it shows the following (complex) basis:

$$P_k^+ = P_k - iQ_k, \quad Q_k^- = P_k + iQ_k, \quad T, \quad J, \quad 1 \leq k \leq m,$$

where i is the imaginary unit, and whose nonzero commutators are

$$[J, P_k^+] = iP_k^+, \quad [J, Q_k^-] = -iQ_k^-, \quad [P_k^+, Q_k^-] = 2iT, \quad 1 \leq k \leq m. \quad (1)$$

The Ado's theorem in Lie Theory states that every finite-dimensional complex Lie algebra can be represented as a matrix Lie algebra, formed by matrices. However, that result does not specify which is the minimal order of the matrices involved in such representations. In [6], the value of the minimal order of the matrices for abelian Lie algebras and Heisenberg algebras $\mathfrak{h}_m$, defined on a $(2m + 1)$ -dimensional vector space with basis $X_1, \dots, X_m, Y_1, \dots, Y_m, Z$, and brackets $[X_i, Y_i] = Z$, is found. For abelian Lie algebras of dimension n the minimal order is $\lceil 2\sqrt{n-1} \rceil$.

Lemma 1 ([6]). *For the Heisenberg Lie algebras $\mathfrak{h}_m$, the minimal faithful matrix representation has order equal to $m + 2$.*

In this paper we find a minimal faithful representation of the $(2m + 2)$ -dimensional complex general Diamond Lie algebra, $\mathfrak{D}_m(\mathbb{C})$, which is isomorphic to a subalgebra of the special linear Lie algebra $\mathfrak{sl}(m + 2, \mathbb{C})$. Moreover, we find a faithful representation of $\mathfrak{D}_m$ which is isomorphic to a subalgebra of the symplectic Lie algebra $\mathfrak{sp}(2m + 2, \mathbb{R})$. Then we construct Leibniz algebras with corresponding general Diamond Lie algebra and the ideal generated by the squares of elements in these faithful representations.

2. LEIBNIZ ALGEBRAS ASSOCIATED WITH MINIMAL FAITHFUL REPRESENTATION OF GENERAL DIAMOND LIE ALGEBRAS

In this section we are going to study Leibniz algebras L such that $L/I \cong \mathfrak{D}_m(\mathbb{C})$ and the $\mathfrak{D}_m(\mathbb{C})$ -module I is a minimal faithful representation, that is, the action $I \times \mathfrak{D}_m(\mathbb{C}) \rightarrow I$ gives rise to a minimal faithful representation of $\mathfrak{D}_m(\mathbb{C})$. Moreover, this representation factorizes through $\mathfrak{sl}(m + 2, \mathbb{C})$.

Proposition 1. *Let $\mathfrak{D}_m(\mathbb{C})$ be a $(2m + 2)$ -dimensional general Diamond Lie algebra with basis*

$$\{J, P_1^+, P_2^+, \dots, P_m^+, Q_1^-, Q_2^-, \dots, Q_m^-, T\}.$$

Then its minimal faithful representation is given by

$$\theta J + \sum_{k=1}^m \alpha_k P_k^+ + \sum_{k=1}^m \beta_k Q_k^- + \delta T \mapsto \begin{pmatrix} \frac{im}{m+2}\theta & \alpha_m & \alpha_{m-1} & \dots & \alpha_2 & \alpha_1 & -\frac{i}{2}\delta \\ 0 & -\frac{2i}{m+2}\theta & a_1 & \dots & 0 & 0 & \beta_m \\ 0 & 0 & -\frac{2i}{m+2}\theta & \dots & 0 & 0 & \beta_{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{2i}{m+2}\theta & a_1 & \beta_2 \\ 0 & 0 & 0 & \dots & 0 & -\frac{2i}{m+2}\theta & \beta_1 \\ 0 & 0 & 0 & \dots & 0 & 0 & \frac{im}{m+2}\theta \end{pmatrix}.$$

Proof. Consider the bilinear map $\varphi: \mathfrak{D}_m(\mathbb{C}) \rightarrow \mathfrak{sl}(m+2, \mathbb{C})$ given by

$$\begin{aligned} \varphi(J) &= \frac{im}{m+2}e_{1,1} - \sum_{s=2}^{m+1} \frac{2i}{m+2}e_{s,s} + \frac{im}{m+2}e_{m+2,m+2}, & \varphi(T) &= -\frac{i}{2}e_{1,m+2}, \\ \varphi(P_k^+) &= e_{1,m+2-k}, & \varphi(Q_k^-) &= e_{m+2-k,m+2}, & 1 \leq k \leq m, \end{aligned}$$

where $e_{i,j}$ is the matrix whose (i, j) -th entry is a 1 and all others 0's.

By checking $[\varphi(x), \varphi(y)] = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$ for all $x, y \in \mathfrak{D}_m(\mathbb{C})$, we verify that φ is an injective homomorphism of algebras. It is easy to see that $\mathfrak{D}_m \setminus J \cong \mathfrak{h}_m$. By Lemma 1 we obtain that it is minimal. $\square$

Let us denote by $V = \mathbb{C}^{m+2}$ the natural $\varphi(\mathfrak{D}_m(\mathbb{C}))$ -module and endow it with a $\mathfrak{D}_m(\mathbb{C})$ -module structure, $V \times \mathfrak{D}_m(\mathbb{C}) \rightarrow V$, given by

$$(x, e) := x\varphi(e),$$

where $x \in V$ and $e \in \mathfrak{D}_m(\mathbb{C})$.

Then we obtain

$$\begin{cases} (X_1, J) = \frac{im}{m+2}X_1, \\ (X_k, J) = -\frac{2i}{m+2}X_k, & 2 \leq k \leq m+1, \\ (X_{m+2}, J) = \frac{im}{m+2}X_{m+2}, \\ (X_1, P_k^+) = X_{m+2-k}, & 1 \leq k \leq m, \\ (X_{m+2-k}, Q_k^-) = X_{m+2}, & 1 \leq k \leq m, \\ (X_1, T) = -\frac{i}{2}X_{m+2}, \end{cases} \quad (2)$$

and the remaining products in the action being zero.

Now we investigate Leibniz algebras L such that $L/I \cong \mathfrak{D}_m(\mathbb{C})$ and $I = V$ as a $\mathfrak{D}_m(\mathbb{C})$ -module.

Theorem 1. *Let L be an arbitrary Leibniz algebra with corresponding Lie algebra $\mathfrak{D}_m(\mathbb{C})$ and I associated with $\mathfrak{D}_m(\mathbb{C})$ -module defined by (2). Then there exists a basis*

$$\{J, P_1^+, P_2^+, \dots, P_m^+, Q_1^-, Q_2^-, \dots, Q_m^-, T, X_1, X_2, \dots, X_{m+2}\}$$

of L such that

$$[\mathfrak{D}_m(\mathbb{C}), \mathfrak{D}_m(\mathbb{C})] \subseteq \mathfrak{D}_m(\mathbb{C}).$$

Proof. Here we shall use the multiplication table (1) of the complex Diamond Lie algebra. Let us assume that

$$[J, J] = \sum_{k=1}^{m+2} \delta_k X_k.$$

Then by setting

$$J' := J + \frac{i(m+2)\delta_1}{m}X_1 - \sum_{k=2}^{m+1} \frac{i(m+2)\delta_i}{2}X_i + \frac{i(m+2)\delta_{m+2}}{m}X_{m+2},$$

we can assume that $[J, J] = 0$.

Let us denote

$$[J, P_k^+] = iP_k^+ + \sum_{s=1}^{m+2} \alpha_{k,s} X_s, \quad [J, Q_k^-] = -iQ_k^- + \sum_{s=1}^{m+2} \beta_{k,s} X_s, \quad 1 \leq k \leq m.$$

Taking the following basis transformation:

$$J' = J, \quad P_k^{+'} = P_k^+ - \sum_{s=1}^{m+2} i\alpha_{k,s} X_s, \quad Q_k^{-'} = Q_k^- + \sum_{k=2}^{m+2} i\beta_{k,s} X_s, \quad T' = -i/2[P_1^{+'}, Q_1^{-'}], \quad 1 \leq k \leq m,$$

we can assume that

$$[J, P_k^+] = iP_k^+, \quad [J, Q_k^-] = -iQ_k^-, \quad [P_1^+, Q_1^-] = 2iT, \quad 1 \leq k \leq m.$$

By applying the Leibniz identity to the triples $\{J, J, P_k^+\}$, $\{J, J, Q_k^-\}$, we derive

$$[P_k^+, J] = -[J, P_k^+], \quad [Q_k^-, J] = -[J, Q_k^-], \quad 1 \leq k \leq m.$$

By considering Leibniz identity for the triples we have the following constraints.

Leibniz identity	Constraints
$\{P_1^+, J, Q_1^-\}$	$\implies [T, J] = 0,$
$\{J, T, J\}$	$\implies [J, T] = 0,$
$\{J, P_k^+, Q_s^-\}$	$\implies [P_k^+, Q_s^-] = -[Q_s^-, P_k^+], \quad 1 \leq k, s \leq m,$
$\{P_k^+, J, Q_s^-\}$	$\implies [P_k^+, Q_s^-] = 0, \quad 1 \leq k, s \leq m, \quad k \neq s,$
$\{P_k^+, J, Q_k^-\}$	$\implies [P_k^+, Q_k^-] = 2iT, \quad 2 \leq k \leq m,$
$\{P_k^+, J, P_s^+\}$	$\implies [P_k^+, P_s^+] = 0, \quad 1 \leq k, s \leq m,$
$\{Q_k^-, J, Q_s^-\}$	$\implies [Q_k^-, Q_s^-] = 0, \quad 1 \leq k, s \leq m,$
$\{J, P_k^+, T\}$	$\implies [P_k^+, T] = 0, \quad 1 \leq k \leq m,$
$\{J, Q_k^-, T\}$	$\implies [Q_k^-, T] = 0, \quad 1 \leq k \leq m,$
$\{P_k^+, P_k^+, Q_k^-\}$	$\implies [T, P_k^+] = 0, \quad 1 \leq k \leq m,$
$\{Q_k^-, Q_k^-, P_k^+\}$	$\implies [T, Q_k^-] = 0, \quad 1 \leq k \leq m.$

□

3. LEIBNIZ ALGEBRAS CONSTRUCTED BY REPRESENTATION OF GENERAL DIAMOND ALGEBRA WHICH IS ISOMORPHIC TO A SUBALGEBRA OF $\mathfrak{sp}(2m+2, \mathbb{R})$

In this section we are going to study Leibniz algebras L such that $L/I \cong \mathfrak{D}_m$ and the $\mathfrak{D}_m$ -module I is a faithful representation. Moreover, this representation factorizes through $\mathfrak{sp}(2m+2, \mathbb{R})$.

Proposition 2. *Let $\mathfrak{D}_m$ be a $(2m+2)$ -dimensional general Diamond Lie algebra with basis $\{J, P_1, P_2, \dots, P_m, Q_1, Q_2, \dots, Q_m, T\}$. Then it is isomorphic to a subalgebra of $\mathfrak{sp}(2m+2, \mathbb{R})$ by*

$$aJ + \sum_{k=1}^m b_k P_k + \sum_{k=1}^m c_k Q_k + dT \mapsto \begin{pmatrix} 0 & b_1 & b_2 & \dots & b_m & c_m & \dots & c_2 & c_1 & 2d \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & -a & c_1 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & -a & 0 & c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -a & \dots & 0 & 0 & c_m \\ \hline 0 & 0 & 0 & \dots & a & 0 & \dots & 0 & 0 & -b_m \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & a & \dots & 0 & 0 & \dots & 0 & 0 & -b_2 \\ 0 & a & 0 & \dots & 0 & 0 & \dots & 0 & 0 & -b_1 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

Proof. Consider the bilinear map $\varphi: \mathfrak{D}_m \rightarrow \mathfrak{sp}(2m+2, \mathbb{R})$ given by

$$\begin{aligned} \varphi(J) &= -\sum_{s=2}^{m+1} e_{k,2m+3-k} + \sum_{s=m+2}^{2m+1} e_{k,2m+3-k}, & \varphi(T) &= 2e_{1,2m+2}, \\ \varphi(P_k) &= e_{1,1+k} - e_{2m+2-k,2m+2}, & \varphi(Q_k) &= e_{1,2m+2-k} + e_{k+1,2m+2}, \quad 1 \leq k \leq m. \end{aligned}$$

By checking $[\varphi(x), \varphi(y)] = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$ for all $x, y \in \mathfrak{D}_m$, we verify that φ is an injective homomorphism of algebras. $\square$

Let us denote by $V = \mathbb{R}^{2m+2}$ the natural $\varphi(\mathfrak{D}_m)$ -module and endow it with a $\mathfrak{D}_m$ -module structure, $V \times \mathfrak{D}_m \rightarrow V$, given by

$$(x, e) := x\varphi(e),$$

where $x \in V$ and $e \in \mathfrak{D}_m$.

Then we obtain

$$\left\{ \begin{array}{ll} (X_k, J) = -X_{2m+3-k}, & 2 \leq k \leq m+1, \\ (X_k, J) = X_{2m+3-k}, & m+2 \leq k \leq 2m+1, \\ (X_1, P_k) = X_{k+1}, & 1 \leq k \leq m, \\ (X_{2m+2-k}, P_k) = -X_{2m+2}, & 1 \leq k \leq m, \\ (X_1, Q_k) = X_{2m+2-k}, & 1 \leq k \leq m, \\ (X_{k+1}, Q_k) = X_{2m+2}, & 1 \leq k \leq m, \\ (X_1, T) = 2X_{2m+2}, & 1 \leq k \leq m, \end{array} \right. \quad (3)$$

and the remaining products in the action being zero.

Theorem 2. *An arbitrary Leibniz algebra with corresponding Lie algebra $\mathfrak{D}_m$ and I associated with $\mathfrak{D}_m$ -module defined by (3) admits a basis $\{J, P_1, P_2, \dots, P_m, Q_1, Q_2, \dots, Q_m, T, X_1, X_2, \dots, X_{2m+2}\}$ such that the multiplication table $[\mathfrak{D}_m, \mathfrak{D}_m]$ has the following form:*

$$\left\{ \begin{array}{ll} [J, J] = a_1 X_{2m+2}, & [J, P_k] = -[P_k, J] = Q_k, \\ [J, Q_k] = -[Q_k, J] = -P_k, & [P_k, Q_k] = -[Q_k, P_k] = T, \\ [P_k, P_s] = [Q_k, Q_s] = b_{k,s} X_{2m+2}, & [P_k, Q_s] = [Q_k, P_s] = c_{k,s} X_{2m+2}, \end{array} \right.$$

with the restrictions

$$b_{k,s} = -b_{s,k}, \quad c_{k,s} = c_{s,k},$$

where $1 \leq k, s \leq m, k \neq s$.

Proof. Let us assume that

$$[J, J] = \sum_{k=1}^{m+2} \delta_k X_k.$$

Then by setting

$$J' = J + \sum_{k=2}^{m+1} \delta_{2m+3-k} X_k - \sum_{k=m+2}^{2m+1} \delta_{2m+3-k} X_k,$$

we can assume that

$$[J, J] = \delta_1 X_1 + \delta_{2m+2} X_{2m+2}.$$

Let us suppose that $[J, T] = \sum_{k=1}^{2m+2} \rho_k X_k$ and considering the Leibniz identity to $\{J, T, J\}$, we get

$$\delta_1 = 0, \quad [J, T] = \rho_1 X_1 + \rho_{2m+2} X_{2m+2}.$$

By making the change of basis element $J' = J - \frac{1}{2}\rho_{2m+2} X_1$ we get the

$$[J, T] = \rho_1 X_1.$$

Let us suppose

$$[J, P_k] = Q_k + \sum_{s=1}^{2m+2} \lambda_{k,s} X_s, \quad [J, Q_k] = -P_k + \sum_{s=1}^{2m+2} \mu_{k,s} X_s, \quad 1 \leq k \leq m.$$

Taking the following basis transformation:

$$J' = J, \quad P'_k = P_k - \sum_{s=1}^{2m+2} \mu_{k,s} X_s, \quad Q'_k = Q_k + \sum_{k=1}^{2m+2} \lambda_{k,s} X_s, \quad T' = [P'_1, Q'_1], \quad 1 \leq k \leq m,$$

we can assume that

$$[J, P_k] = Q_k, \quad [J, Q_k] = -P_k, \quad [P_1, Q_1] = T, \quad 1 \leq k \leq m.$$

By applying the Leibniz identity to the triples $\{J, J, P_k\}$, $\{J, J, Q_k\}$ we derive

$$[P_k, J] = -[J, P_k], \quad [Q_k, J] = -[J, Q_k], \quad 1 \leq k \leq m.$$

By verifying the Leibniz identity on elements, we have the following the restrictions.

Leibniz identity	Constraints
$\{J, P_k, T\}$	$\implies [Q_k, T] = \rho_1 X_{k+1}, \quad 1 \leq k \leq m,$
$\{J, Q_k, T\}$	$\implies [P_k, T] = -\rho_1 X_{2m+2-k}, \quad 1 \leq k \leq m,$
$\{P_1, T, Q_1\}$	$\implies [T, T] = 0,$

We set

$$\begin{cases} [P_j, Q_j] = T + \sum_{t=1}^{2m+2} \beta_{j,t} X_t, & [Q_k, P_k] = -T + \sum_{t=1}^{2m+2} \gamma_{k,t} X_t, \\ [P_k, P_s] = \sum_{t=1}^{2m+2} \eta_{k,s,t} X_t, & [Q_k, Q_s] = \sum_{t=1}^{2m+2} \theta_{k,s,t} X_t, \\ [P_k, Q_s] = \sum_{t=1}^{2m+2} \nu_{k,s,t} X_t, \quad k \neq s, & [Q_k, P_s] = \sum_{t=1}^{2m+2} \xi_{k,s,t} X_t, \quad k \neq s, \end{cases}$$

where $2 \leq j \leq m$, $1 \leq k, s \leq m$.

By applying the Leibniz identity to $\{P_k, P_s, T\}$ and $\{Q_k, Q_s, T\}$, we obtain

$$\eta_{k,k,1} = \theta_{k,k,1} = \frac{1}{2} \rho_1, \quad \eta_{k,s,1} = \theta_{k,s,1} = 0, \quad 1 \leq k, s \leq m, \quad k \neq s.$$

By considering the next equality

$$\begin{aligned} \rho_1 X_1 &= [J, T] = [J, [P_k, Q_k]] = [[J, P_k], Q_k] - [[J, Q_k], P_k] \\ &= [Q_k, Q_k] + [P_k, P_k] = \rho_1 X_1 + \sum_{s=2}^{2m+2} (\eta_{k,k,s} + \theta_{k,k,s}) X_s, \end{aligned}$$

we get

$$\theta_{k,k,s} = -\eta_{k,k,s}, \quad 2 \leq s \leq 2m+2, \quad 1 \leq k \leq m.$$

Analogously, by applying the Leibniz identity to $\{J, P_k, Q_s\}$, $\{J, Q_k, Q_s\}$ and $\{J, P_k, P_s\}$, we get

$$[P_k, P_s] = -[Q_s, Q_k], \quad [P_k, Q_s] = [P_s, Q_k], \quad [Q_k, P_s] = [Q_s, P_k], \quad 1 \leq k, s \leq m, \quad k \neq s. \quad (4)$$

By applying the Leibniz identity to $\{P_1, J, Q_1\}$ and $\{P_1, P_1, Q_1\}$, we have

$$[T, J] = \sum_{s=2}^{2m+2} 2\eta_{1,1,s} X_s, \quad [T, P_1] = \frac{3}{2} \rho_1 X_{2m+1} + \eta_{1,1,2} X_{2m+2}.$$

By the next identity

$$\begin{aligned} [Q_1, [J, P_1]] &= [[Q_1, J], P_1] - [[Q_1, P_1], J] = [P_1, P_1] - [-T + \sum_{s=1}^{2m+2} \gamma_{1,s} X_s, J] \\ &= \frac{1}{2} \rho_1 X_1 + \sum_{s=1}^{2m+2} \eta_{1,1,s} X_s + \sum_{s=2}^{2m+2} 2\eta_{1,1,s} X_s + \sum_{s=2}^{m+1} \gamma_{1,s} X_{2m+3-s} - \sum_{s=m+2}^{2m+1} \gamma_{1,s} X_{2m+3-s} \\ &= \frac{1}{2} \rho_1 X_1 + \sum_{s=1}^{2m+2} 3\eta_{1,1,s} X_s + \sum_{s=2}^{m+1} \gamma_{1,s} X_{2m+3-s} - \sum_{s=m+2}^{2m+1} \gamma_{1,s} X_{2m+3-s}. \end{aligned}$$

On the other hand

$$[Q_1, [J, P_1]] = [Q_1, Q_1] = \frac{1}{2}\rho_1 X_1 - \sum_{s=1}^{2m+2} \eta_{1,1,s} X_s,$$

and from this we deduce

$$\gamma_{1,s} = -4\eta_{1,1,2m+3-s}, \quad \gamma_{1,k} = 4\eta_{1,1,2m+3-k}, \quad \eta_{1,1,2m+2} = 0,$$

with $2 \leq s \leq m+1$, $m+2 \leq k \leq 2m+1$.

Now by considering the identity

$$\begin{aligned} T &= [P_1, Q_1] = [P_1, [J, P_1]] = [[P_1, J], P_1] - [[P_1, P_1], J] \\ &= -[Q_1, P_1] - \left[\frac{1}{2}\rho_1 X_1 + \sum_{k=2}^{2m+1} \eta_{1,1,k} X_k, J \right] \\ &= T - \gamma_{1,1} X_1 + \sum_{k=2}^{m+1} 4\eta_{1,1,2m+3-k} X_k - \sum_{k=m+2}^{2m+1} 4\eta_{1,1,2m+3-k} X_k \\ &\quad + \gamma_{1,2m+2} X_{2m+2} + \sum_{k=2}^{m+1} \eta_{1,1,k} X_{2m+3-k} - \sum_{k=m+2}^{2m+1} \eta_{1,1,k} X_{2m+3-k}, \end{aligned}$$

we get

$$\gamma_{1,1} = \gamma_{1,2m+2} = \eta_{1,1,k} = 0, \quad 2 \leq k \leq 2m+1.$$

By the next Leibniz identity

$$\begin{aligned} \rho_1 X_2 &= [Q_1, T] = [Q_1, [P_1, Q_1]] = [[Q_1, P_1], Q_1] - [[Q_1, Q_1], P_1] \\ &= -[T, Q_1] - \left[\frac{1}{2}\rho_1 X_1, P_1 \right] = -[T, Q_1] - \frac{1}{2}\rho_1 X_2, \end{aligned}$$

we obtain

$$[T, Q_1] = -\frac{3}{2}\rho_1 X_2.$$

Hence, we have

$$\begin{cases} [T, J] = 0, & [Q_1, P_1] = -T, \\ [P_1, P_1] = \frac{1}{2}\rho_1 X_1, & [Q_1, Q_1] = \frac{1}{2}\rho_1 X_1, \\ [T, P_1] = \frac{3}{2}\rho_1 X_{2m+1}, & [T, Q_1] = -\frac{3}{2}\rho_1 X_2, \end{cases}$$

By using the next Leibniz identity

$$\begin{aligned} -\rho_1 X_{2m+2-k} &= [P_k, T] = [P_k, [P_k, Q_k]] = [[P_k, P_k], Q_k] - [[P_k, Q_k], P_k] \\ &= \left[\frac{1}{2}\rho_1 X_1 + \sum_{s=2}^{2m+2} \eta_{k,k,s} X_s, Q_k \right] - \left[T + \sum_{s=1}^{2m+2} \beta_{k,s} X_s, P_k \right] \\ &= \frac{1}{2}\rho_1 X_{2m+2-k} + \eta_{k,k,k+1} X_{2m+2} - [T, P_k] - \beta_{k,1} X_{k+1} + \beta_{k,2m+2-k} X_{2m+2}, \end{aligned}$$

we get

$$[T, P_k] = \frac{3}{2}\rho_1 X_{2m+2-k} - \beta_{k,1} X_{k+1} + (\beta_{k,2m+2-k} + \eta_{k,k,k+1}) X_{2m+2}, \quad 2 \leq k \leq m.$$

By applying the Leibniz identity to the elements $\{P_k, J, Q_k\}$ and $\{Q_k, J, P_k\}$, we get

$$\beta_{k,s} = \gamma_{k,s} = -2\eta_{k,k,2m+3-s}, \quad \beta_{k,t} = \gamma_{k,t} = 2\eta_{k,k,2m+3-t}, \quad \eta_{k,k,2m+2} = 0,$$

where $2 \leq k \leq m$, $2 \leq s \leq m+1$, $m+2 \leq t \leq 2m+1$.

By the next Leibniz identity applied to $\{P_k, J, P_k\}$, we have

$$\gamma_{k,1} = -\beta_{k,1}, \quad \eta_{k,k,s} = 0, \quad 2 \leq k \leq m, \quad 2 \leq s \leq 2m+1.$$

Now, by considering

$$\begin{aligned} \rho_1 X_{k+1} &= [Q_k, T] = [Q_k, [P_k, Q_k]] = [[Q_k, P_k], Q_k] - [[Q_k, Q_k], P_k] \\ &= [-T - \beta_{k,1} X_1, Q_k] - \left[\frac{1}{2}\rho_1 X_1, P_k \right] = -[T, Q_k] - \beta_{k,1} X_{2m+2-k} - \frac{1}{2}\rho_1 X_{k+1}, \end{aligned}$$

we get

$$[T, Q_k] = -\frac{3}{2}\rho_1 X_{k+1} - \beta_{k,1} X_{2m+2-k}.$$

By using the Leibniz identity for $\{T, P_k, Q_k\}$, we get

$$\beta_{k,1} = 0, \quad 2 \leq k \leq m.$$

So, we have

$$\begin{cases} [P_k, Q_k] = -[Q_k, P_k] = T, \\ [P_k, P_k] = [Q_k, Q_k] = \frac{1}{2}\rho_1 X_1, \\ [T, P_k] = \frac{3}{2}\rho_1 X_{2m+2-k}, \\ [T, Q_k] = -\frac{3}{2}\rho_1 X_{k+1}, \end{cases}$$

where $2 \leq k \leq m$.

By verifying Leibniz identity on elements, we obtain the following restrictions.

Leibniz identity	Constraints
$\{P_k, P_s, T\}$	$\implies \eta_{k,s,1} = 0, \quad 1 \leq k, s \leq m, \quad k \neq s,$
$\{Q_k, Q_s, T\}$	$\implies \theta_{k,s,1} = 0, \quad 1 \leq k, s \leq m, \quad k \neq s,$
$\{P_k, Q_s, T\}$	$\implies \nu_{k,s,1} = 0, \quad 1 \leq k, s \leq m, \quad k \neq s,$
$\{Q_k, P_s, T\}$	$\implies \xi_{k,s,1} = 0, \quad 1 \leq k, s \leq m, \quad k \neq s.$

By applying the Leibniz identity to $\{P_k, P_s, J\}$, $\{Q_k, Q_s, J\}$, we get

$$[[P_k, P_s], J] = -[Q_k, P_s] - [P_k, Q_s], \quad [[Q_k, Q_s], J] = [Q_k, P_s] + [P_k, Q_s],$$

it follows that

$$[[P_k, P_s], J] = -[[Q_k, Q_s], J],$$

hence

$$\xi_{k,s,2m+2} = -\nu_{k,s,2m+2}, \quad \theta_{k,s,t} = -\eta_{k,s,t}, \quad 1 \leq k, s \leq m, \quad 2 \leq t \leq 2m+1, \quad k \neq s.$$

and

$$\begin{aligned} \nu_{k,s,t} + \xi_{k,s,t} &= -\eta_{k,s,2m+3-t} & 2 \leq t \leq m+1, \\ \nu_{k,s,t} + \xi_{k,s,t} &= \eta_{k,s,2m+3-t} & m+2 \leq t \leq 2m+1. \end{aligned}$$

Let us consider the identity

$$[[Q_k, P_s], J] = [Q_k, [P_s, J]] + [[Q_k, J], P_s] = -[Q_k, Q_s] + [P_k, P_s]$$

We have that $\theta_{k,s,2m+2} = \eta_{k,s,2m+2}$ and

$$\begin{aligned} \xi_{k,s,t} &= -2\eta_{k,s,2m+3-t}, & 2 \leq t \leq m+1, \\ \xi_{k,s,t} &= 2\eta_{k,s,2m+3-t}, & m+2 \leq t \leq m+1, \end{aligned}$$

and

$$\begin{aligned} \nu_{k,s,t} &= \eta_{k,s,2m+3-t}, & 2 \leq t \leq m+1, \\ \nu_{k,s,t} &= -\eta_{k,s,2m+3-t}, & m+2 \leq t \leq 2m+1. \end{aligned}$$

Analogously, by applying the Leibniz identity to $\{P_k, Q_s, J\}$, we get

$$\begin{aligned} \nu_{k,s,t} &= -2\eta_{k,s,2m+3-t}, & 2 \leq t \leq m+1, \\ \nu_{k,s,t} &= 2\eta_{k,s,2m+3-t}, & m+2 \leq t \leq 2m+1. \end{aligned}$$

We get that $\nu_{k,s,t} = 0$, $1 \leq k, s \leq m$, $2 \leq t \leq 2m+1$, $k \neq s$. It implies that $\eta_{k,s,t} = \xi_{k,s,t} = \theta_{k,s,t} = 0$ for $1 \leq k, s \leq m$, $2 \leq t \leq 2m+1$, $k \neq s$.

Hence, we have

$$\begin{aligned} [P_k, P_s] &= [Q_k, Q_s] = \eta_{k,s,2m+2} X_{2m+2}, & 1 \leq k, s \leq m, \quad k \neq s, \\ [P_k, Q_s] &= [Q_k, P_s] = \nu_{k,s,2m+2} X_{2m+2}, & 1 \leq k, s \leq m, \quad k \neq s. \end{aligned}$$

By equation (4) we have the following restrictions

$$\eta_{k,s,2m+2} = -\eta_{s,k,2m+2}, \quad \nu_{k,s,2m+2} = \nu_{s,k,2m+2}, \quad 1 \leq k, s \leq m, \quad k \neq s.$$

Finally, we apply the Leibniz identity to the elements $\{P_k, P_k, P_s\}$ with $k \neq s$ and we obtain $\rho_1 = 0$. We denote again $(\delta_{2m+2}, \eta_{k,s,2m+2}, \nu_{k,s,2m+2}) = (a_1, b_{k,s}, c_{k,s})$. $\square$

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