A dirty integration of Leibniz algebras

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Abstract

In this paper we present an integration of any real finite-dimensional Leibniz algebra as a Lie rack which reduces in the particular case of a Lie algebra to the ordinary connected simply connected Lie group. The construction is not functorial.

Contents

1 Introduction

All manifolds considered in this manuscript are assumed to be Hausdorff and second countable.

Recall that a *pointed rack* is a pointed set (X, e) together with a binary operation $\rhd : X \times X \to X$ such that for all $x \in X$, the map $y \mapsto x \rhd y$ is bijective and such that for all $x, y, z \in X$, the self-distributivity and unit relations

$$x \rhd (y \rhd z) \;=\; (x \rhd y) \rhd (x \rhd z), \quad e \rhd x = x, \quad \text{and} \quad x \rhd e = e$$

are satisfied. Imitating the notion of a Lie group, the smooth version of a pointed rack is called *Lie rack*.

An important class of examples of racks are the so-called *augmented racks*, see [4]. An augmented rack is the data of a group G, a G-set X and a map $p : X \to G$ such that for all $x \in X$ and all $g \in G$,

$$p(g \cdot x) \;=\; gp(x)g^{-1}.$$

The set X becomes then a rack by setting $x \rhd y \;:=\; p(x) \cdot y$.

Lie racks are intimately related to *Leibniz algebras* $\mathfrak{h}$, i.e. a vector space $\mathfrak{h}$ with a bilinear bracket $[,] : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ such that for all $X, Y, Z \in \mathfrak{h}$, $[X, -]$ acts as a derivation:

$$[X, [Y, Z]] \;=\; [[X, Y], Z] + [Y, [X, Z]].$$

Indeed, Kinyon showed in [6] that the tangent space at $e \in H$ of a Lie rack H carries a natural structure of a Leibniz algebra, generalizing the relation between a Lie group and its tangent Lie algebra. Conversely, every (finite dimensional real or complex) Leibniz algebra $\mathfrak{h}$ may be integrated into a Lie rack (with underlying manifold $\mathfrak{h}$) using the rack product

$$X \rhd Y \;:=\; e^{\mathrm{ad}_X}(Y), \tag{1.1}$$

noting that the exponential of the inner derivation ad_X for each $X \in \mathfrak{h}$ is an automorphism. Although the assignment $\big(\mathfrak{h}, [\ ,\ ]\big) \to \big(\mathfrak{h}, 0, \rhd\big)$ is functorial since morphisms of Leibniz algebras are easily seen to go to morphisms of pointed Lie racks, the restriction to the category of all Lie algebras would not give the usual integration as a Lie group.

The purpose of the present paper is to construct an integration procedure which integrates real, finite-dimension Leibniz algebras into Lie racks in such a way that the restriction to Lie algebras gives the conjugation rack underlying the simply connected Lie group corresponding to a (real, finite-dimensional) Lie algebra.

This problem has been encountered by J.-L. Loday in 1993 [8] in the search of quantifying the lack of periodicity in algebraic K-Theory. Several attempts and constructions have been published since then. In 2010, Simon Covez [2] solves in his thesis the local integration problem by constructing a local Lie rack integrating a given (real, finite dimensional) Leibniz algebra in such a way that in the case of Lie algebras, one obtains the conjugation Lie rack underlying the usual (simply connected) Lie group integrating it. Other important contributions to the problem include Mostovoy's article [10] where he solves the problem in the framework of formal groups. The general problems is still open to our knowledge and our article constitutes another step towards its solution.

In this article, we construct a (global) Lie rack integrating a given (real, finite dimensional) Leibniz algebra $\mathfrak{h}$ in such a way that in the case of a Lie algebra, the construction yields the conjugation rack underlying the usual (simply connected) Lie group integrating it. More precisely, we work with augmented Leibniz algebras, i.e. Leibniz algebras $\mathfrak{h}$ with an action of a Lie algebra $\mathfrak{g}$ by derivations and an equivariant map $p : \mathfrak{h} \to \mathfrak{g}$ to $\mathfrak{g}$. We integrate the quotient Lie algebra $p(\mathfrak{h}) =: \mathfrak{g}'$ into a Lie group G' and integrate its action on $\mathfrak{h}$ such that the resulting (global) augmented Lie rack is an affine bundle over G' with typical fiber $\mathfrak{z} := \mathrm{Ker}(p)$. This is the content of our main theorem, Theorem 3.1.

Perhaps the most interesting point of the article is the fact that for this construction, we need an open neighbourhood in the Lie group G' on which the exponential is a diffeomorphism and which is invariant under (the connected component of the identity) $\mathrm{Aut}_0(G')$. It is not elementary to show that such a neighbourhood exists, and the proof of it will take the 10 pages of Appendix A. The arguments are rather classical and are inspired by Lazard-Tits [7] and Đoković-Hofmann [3]. On the other hand, it is the use of this neighbourhood which renders our construction non-functorial, which is a major drawback of the theory. This is why we call our integration *dirty*. For the moment, we do not know whether there exists a functorial construction of a Lie rack integrating a given Leibniz algebra (such that in the special case of a Lie algebra, we get back the conjugation rack underlying the usual simply connected Lie group).

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2 Augmented Leibniz algebras and Lie racks

2.1 (Augmented) Leibniz algebras

Let K be any unital commutative ring containing the rational numbers. We are mainly interested in the case $K = \mathbb{R}$. All modules in this section will be considered over K.

Recall that a *Leibniz algebra* over K is a K-module $\mathfrak{h}$ equipped with a linear map $[\ ,\ ] : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$, written $x \otimes y \mapsto [x, y]$ such that the *left Leibniz identity* holds for all $x, y, z \in \mathfrak{h}$

$$\big[x, [y, z]\big] = \big[[x, y], z\big] + \big[y, [x, z]\big] \tag{2.1}$$

A morphism of Leibniz algebras $\Phi : \mathfrak{h} \to \mathfrak{h}'$ is a K-linear map preserving brackets, i.e. for all $x, y \in \mathfrak{h}$ we have $\Phi\big([x, y]\big) = \big[\Phi(x), \Phi(y)\big]'$. Recall first that each Lie algebra over K is a Leibniz algebra giving rise to a functor $\mathfrak{i}$ from the category of all Lie algebras (over K), K**LieAlg**, to the category of all Leibniz algebras (over K), K**Leib**.

Furthermore, recall that each Leibniz algebra has two canonical K-submodules

$$\begin{aligned} Q(\mathfrak{h}) \;&:=\; \Big\{x \in \mathfrak{h} \mid \exists\ N \in \mathbb{N} \setminus \{0\},\ \exists\ \lambda_1, \ldots, \lambda_N \in K,\ \exists\ x_1, \ldots, x_N \\ &\qquad\qquad \text{such that } x = \sum_{r=1}^{N} \lambda_r [x_r, x_r]\Big\}, && (2.2) \\ \mathfrak{z}(\mathfrak{h}) \;&:=\; \big\{x \in \mathfrak{h} \mid \forall\ y \in \mathfrak{h} :\ [x, y] = 0\big\}. && (2.3) \end{aligned}$$

It is well-known and not hard to deduce from the Leibniz identity that both $Q(\mathfrak{h})$ and $\mathfrak{z}(\mathfrak{h})$ are two-sided abelian ideals of $(\mathfrak{h}, [\ ,\ ])$, that $Q(\mathfrak{h}) \subset \mathfrak{z}(\mathfrak{h})$, and that the quotient Leibniz algebras

$$\overline{\mathfrak{h}} := \mathfrak{h}/Q(\mathfrak{h}) \quad \text{and} \quad \mathfrak{h}/\mathfrak{z}(\mathfrak{h}) \tag{2.4}$$

are Lie algebras. Since the ideal $Q(\mathfrak{h})$ is clearly mapped into the ideal $Q(\mathfrak{h}')$ by any morphism of Leibniz algebras $\mathfrak{h} \to \mathfrak{h}'$ (which is a priori not the case for $\mathfrak{z}(\mathfrak{h})$!), there is an obvious functor $\mathfrak{h} \to \overline{\mathfrak{h}}$ from the category of all Leibniz algebras to the category of all Lie algebras. It is not hard to see and not important for the sequel that the functor $\mathfrak{h} \to \overline{\mathfrak{h}}$ is a left adjoint functor of the inclusion functor of the category of all Lie algebras in the category of all Leibniz algebras whence the former is a reflective subcategory of the latter, see e.g. [9, p.91] for definitions.

It is easy to observe that in both cases of the above Lie algebras, $\overline{\mathfrak{h}}$ and $\mathfrak{h}/\mathfrak{z}(\mathfrak{h})$, there is the following structure:

Definition 2.1 *A quintuple* $\big(\mathfrak{h}, p, \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \dot{\rho}\big)$ *is called a* $\mathfrak{g}$-augmented Leibniz algebra *iff the following holds:*

1. $\big(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}\big)$ *is a Lie algebra over* K.
2. $\mathfrak{h}$ *is a* K*-module which is a left* $\mathfrak{g}$*-module via the* K*-linear map* $\dot{\rho} : \mathfrak{g} \otimes \mathfrak{h} \to \mathfrak{h}$ *written* $\dot{\rho}_{\xi}(x) = \xi.x$ *for all* $\xi \in \mathfrak{g}$ *and* $x \in \mathfrak{h}$.
3. $p : \mathfrak{h} \to \mathfrak{g}$ *is a* K*-linear morphism of* $\mathfrak{g}$*-modules, i.e. for all* $\xi \in \mathfrak{g}$ *and* $x \in \mathfrak{h}$
$$p(\xi.x) = \big[\xi, p(x)\big]_{\mathfrak{g}}. \tag{2.5}$$

A morphism of augmented Leibniz algebras $\big(\mathfrak{h}, p, \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \dot{\rho}\big) \to \big(\mathfrak{h}', p', \mathfrak{g}', [\ ,\ ]'_{\mathfrak{g}}, \dot{\rho}'\big)$ *is a pair* (Φ, ϕ) *of* K*-linear maps where* $\phi : \mathfrak{g} \to \mathfrak{g}'$ *is a morphism of Lie algebras,* $\Phi : \mathfrak{h} \to \mathfrak{h}$ *is a morphism of Lie algebra modules over* ϕ*, i.e. for all* $x \in \mathfrak{h}$ *and* $\xi \in \mathfrak{g}$
$$\Phi(\xi.x) = \phi(\xi).\Phi(x). \tag{2.6}$$
Moreover the obvious diagram commutes, i.e.
$$p' \circ \Phi = \phi \circ p. \tag{2.7}$$

The following properties are immediate from the definitions:

Proposition 2.1 *Let* $\big(\mathfrak{h}, p, \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \dot{\rho}\big)$ *be an augmented Leibniz algebra. Define the following bracket on* $\mathfrak{h}$*:*
$$[x, y]_{\mathfrak{h}} := p(x).y. \tag{2.8}$$

1. $\big(\mathfrak{h}, [,]_{\mathfrak{h}}\big)$ *is a Leibniz algebra on which* $\mathfrak{g}$ *acts as derivations. If* (Φ, ϕ) *is a morphism of augmented Leibniz algebras, then* Φ *is a morphism of Leibniz algebras.*
2. *The kernel of* p, $\mathrm{Ker}(p)$*, is a* $\mathfrak{g}$*-invariant two-sided abelian ideal of* $\mathfrak{h}$ *satisfying* $Q(\mathfrak{h}) \subset \mathrm{Ker}(p) \subset \mathfrak{z}(\mathfrak{h})$.
3. *The image of* p, $\mathrm{Im}(p)$*, is an ideal of the Lie algebra* $\mathfrak{g}$.

Proof: We just check the Leibniz identity: Let $x, y, z \in \mathfrak{h}$, then, writing $[\ ,\ ]_{\mathfrak{h}} = [\ ,\ ]$,
$$\begin{aligned}
\big[x, [y, z]\big] &= p(x).\big(p(y).z\big) \\
&= p(x).\big(p(y).z\big) - p(y).\big(p(x).z\big) + p(y).\big(p(x).z\big) \\
&= \big[p(x), p(y)\big]_{\mathfrak{g}}.z + \big[y, [x, z]\big] \overset{(2.5)}{=} p\big(p(x).y\big).z + \big[y, [x, z]\big] \\
&= \big[[x, y], z\big] + \big[y, [x, z]\big].
\end{aligned}$$

$\square$

It follows that the class of all augmented Leibniz algebras forms a category $K\mathbf{LeibA}$, and there is an obvious forgetful functor from $K\mathbf{LeibA}$ to $K\mathbf{Leib}$ associating to $\big(\mathfrak{h}, p, \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \dot{\rho}\big)$ the Leibniz algebra $\big(\mathfrak{h}, [\ ,\ ]_{\mathfrak{h}}\big)$ where the Leibniz bracket $[\ ,\ ]_{\mathfrak{h}}$ is defined in eqn (2.8).

On the other hand there is a functor from $K\mathbf{Leib}$ to $K\mathbf{LeibA}$ associating to each Leibniz algebra $\big(\mathfrak{h}, [\ ,\ ]\big)$ the augmented Leibniz algebra $\big(\mathfrak{h}, p, \bar{\mathfrak{h}}, [\ ,\ ]_{\bar{\mathfrak{h}}}, \mathrm{ad}'\big)$ where $p : \mathfrak{h} \to \bar{\mathfrak{h}}$ is the canonical projection and the representation ad' of the Lie algebra $\bar{\mathfrak{h}}$ on the Leibniz algebra $\mathfrak{h}$ is defined by (for all $x, y \in \mathfrak{h}$)

$$\mathrm{ad}'_{p(x)}(y) := \mathrm{ad}_x(y) = [x, y]. \tag{2.9}$$

2.2 (Augmented) Lie racks

We now restrict to $K = \mathbb{R}$. Recall that a *pointed manifold* is a pair (M, e) where M is a differentiable manifold and e is a fixed element of M. Morphisms of pointed manifold are base point preserving smooth maps.

Recall that a *Lie rack* is a pointed manifold (M, e) equipped with a smooth map $\mathbf{m} : M \times M \to M$ of pointed manifolds (i.e. $\mathbf{m}(e, e) = e$) such that $\mathbf{m}(x, -) : M \to M$ is a diffeomorphism for all $x \in M$ and satisfying the following identities for all $x, y, z \in M$ where the standard notation is $\mathbf{m}(x, y) = x \rhd y$

$$\begin{aligned} e \rhd x &= x, && (2.10)\\ x \rhd e &= e, && (2.11)\\ x \rhd (y \rhd z) &= (x \rhd y) \rhd (x \rhd z) && (2.12)\end{aligned}$$

The last condition (2.12) is called the *self distributivity condition*. A morphims of Lie racks $\phi : (M, e, \mathbf{m}) \to (M', e', \mathbf{m}')$ is a map of pointed manifolds satisfying for all $x, y \in M$ the condition $\phi(x \rhd y) = \phi(x) \rhd' \phi(y)$. The class of all Lie racks forms a category called **LieRack**. Note that every pointed differentiable manifold (M, e) carries a *trivial Lie rack structure* defined for all $x, y \in M$ by

$$x \rhd_0 y := y, \tag{2.13}$$

and this assignment is functorial.
Moreover, any Lie group G becomes a Lie rack upon setting for all $g, g' \in G$

$$g \rhd g' := g g' g^{-1}, \tag{2.14}$$

again defining a functor from the category of Lie groups to the category of all Lie racks. Examples of racks which are not the conjugation rack underlying a group abound: Firstly, every conjugation class and every union of conjugation classes in a group (defining an immersed submanifold) in a Lie group is a Lie rack. Then, any Lie rack $(M, e, \rhd)$ can be *gauged* by any smooth map $f : (M, e) \to (M, e)$ of pointed manifolds satisfying for all $x, y \in M$

$$f(x \rhd y) = x \rhd f(y).$$

A straight-forward computation shows that the pointed manifold (M, e) equipped with the *gauged* multiplication $\rhd_f$ defined by

$$x \rhd_f y := f(x) \rhd y$$

is a Lie rack $(M, e, \rhd_f)$. We refer for more exotic examples to [4]. The following relation to Leibniz algebras is due to M. Kinyon [6]:

Proposition 2.2 *Let $(M, e, \mathbf{m})$ be a Lie rack and $\mathfrak{h} = T_eM$. Define the following bracket $[\ ,\ ]$ on $\mathfrak{h}$ by*

$$[x, y] = \frac{\partial}{\partial t} T_e L_{a(t)}(y) \bigg|_{t=0} \tag{2.15}$$

where $t \mapsto a(t)$ is any smooth curve defined on an open real interval containing 0 satisfying $a(0) = e$ and $(da/dt)(0) = x \in \mathfrak{h}$. Then we have the following

1. *$\big(\mathfrak{h}, [\ ,\ ]\big)$ is a real Leibniz algebra.*
2. *Let $\phi : (M, e, \mathbf{m}) \to (M', e', \mathbf{m}')$ be a morphism of Lie racks. Then $T_e\phi : \mathfrak{h} \to \mathfrak{h}'$ is a morphism of Leibniz algebras.*

Proof: Since for each $a \in M$ we have $L_a(e) = e$ it follows that the tangent map T_eL_a maps the tangent space T_eM to T_eM whence the curve $t \mapsto T_eL_{a(t)}$ is a curve of $\mathbb{R}$-linear maps $T_eM \to T_eM$ whence eqn (2.15) defines a well-defined real bilinear map $\mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$.

1. Let $x, y, z \in \mathfrak{h}$, and let $t \mapsto a(t)$ and $t \mapsto b(t)$ two smooth curves of an open interval (containing 0) into M such that $a(0) = e = b(0)$ and $(da/dt)(0) = x$, $(db/dt)(0) = y$. We compute

$$\begin{aligned}
&\big[x, [y, z]\big] \\
&\quad = \frac{\partial^2}{\partial s \partial t} \left(T_eL_{a(s)}\left(T_eL_{b(t)}(z)\right)\right)\bigg|_{s,t=0} = \frac{\partial^2}{\partial s \partial t} T_e\left(L_{a(s)} \circ L_{b(t)}\right)(z)\bigg|_{s,t=0} \\
&\quad \overset{(2.12)}{=} \frac{\partial^2}{\partial s \partial t} T_e\left(L_{a(s) \rhd b(t)} \circ L_{a(s)}\right)(z)\bigg|_{s,t=0} = \frac{\partial^2}{\partial s \partial t}\left(T_eL_{a(s)\rhd b(t)}\left(T_eL_{a(s)}(z)\right)\right)\bigg|_{s,t=0} \\
&\quad = \frac{\partial^2}{\partial s \partial t} T_eL_{a(s)\rhd b(t)}\bigg|_{s,t=0}\left(T_eL_{a(0)}(z)\right) \\
&\qquad + \frac{\partial}{\partial t} T_eL_{a(0)\rhd b(t)}\bigg|_{t=0}\left(\frac{\partial}{\partial s}\left(T_eL_{a(s)}(z)\right)\right)\bigg|_{s=0}.
\end{aligned}$$

Since $a(0)=e$ we have $T_eL_{a(0)}(z)=z$ and $a(0)\triangleright b(t)=b(t)$ whence the last term equals $\big[y,[x,z]\big]$. Since for each s the curve $t\mapsto a(s)\triangleright b(t)$ is equal to e at $t=0$ we get

$$\frac{\partial^2}{\partial s\partial t}T_eL_{a(s)\triangleright b(t)}\Big|_{s,t=0}\big(T_eL_{a(0)}(z)\big) \;=\; \left[\frac{\partial}{\partial s}T_eL_{a(s)}(y)\Big|_{s=0},z\right]=\big[[x,y],z\big]$$

proving the Leibniz identity.

2. Since ϕ maps e to e' its tangent map $T_e\phi$ maps T_eM to $T_{e'}M'$. We get for all $x,y\in\mathfrak{h}=T_eM$ where $t\mapsto a(t)$ is a smooth curve in M with $a(0)=e$ and $(da/dt)(0)=x$:

$$\begin{aligned}
T_e\phi\big([x,y]\big) &= T_e\phi\left(\frac{\partial}{\partial t}T_eL_{a(t)}(y)\Big|_{t=0}\right)=\frac{\partial}{\partial t}\Big(T_e\big(\phi\circ L_{a(t)}\big)(y)\Big)\Big|_{t=0}\\
&= \frac{\partial}{\partial t}\Big(T_e\big(L'_{\phi(a(t))}\circ\phi\big)(y)\Big)\Big|_{t=0}=\frac{\partial}{\partial t}T_{e'}L'_{\phi(a(t))}\Big|_{t=0}\big(T_e\phi(y)\big)\\
&= \big[T_e\phi(x),T_e\phi(y)\big].
\end{aligned}$$

□

Let $\mathbb{R}\mathbf{Leib}_{fd}$ denote the category of all finite-dimensional real Leibniz algebras. The preceding proposition shows that there is a functor $T_*\mathcal{R}$: $\mathbf{LieRack}\to\mathbb{R}\mathbf{Leib}_{fd}$ which associates to any Lie rack $(M,e,\triangleright)$ its tangent space $T_*\mathcal{R}(M):=T_eM$ at the distinguished point $e\in M$ equipped with the Leibniz bracket eqn (2.15).

Furthermore, recall that an *augmented Lie rack* (see [4]) (M,ϕ,G,ℓ) consists of a pointed differentiable manifold (M,e_M), of a Lie group G, of a smooth map $\phi:M\to G$ (of pointed manifolds), and of a smooth left G-action $\ell:G\times M\to M$ (written $(g,x)\mapsto\ell(g,x)=\ell_g(x)=gx$) such that for all $g\in G$, $x\in M$

$$ge_M = e_M, \tag{2.16}$$
$$\phi(gx) = g\phi(x)g^{-1}. \tag{2.17}$$

It is a routine check that the multiplication $\triangleright$ on M defined for all $x,y\in M$ by

$$x\triangleright y:=\ell_{\phi(x)}(y) \tag{2.18}$$

satisfies all the axioms (2.10), (2.11), and (2.12) of a Lie rack, thus making $(M,e_M,\triangleright)$ into a Lie rack such that the map ϕ is a morphism of Lie racks, i.e. for all $x,y\in M$

$$\phi(x\triangleright y)=\phi(x)\phi(y)\phi(x)^{-1}. \tag{2.19}$$

A morphism $(\Psi,\psi):(M,\phi,G,\ell)\to(M',\phi',G',\ell')$ of augmented Lie racks is a pair of maps of pointed differentiable manifolds $\Psi:M\to M'$ and

$\psi : G \to G'$ such that ψ is homomorphism of Lie groups and such that all reasonable diagrams commute, viz: for all $g \in G$

$$\phi' \circ \Psi = \psi \circ \phi, \tag{2.20}$$

$$\Psi \circ \ell_g = \ell'_{\psi(g)} \circ \Psi. \tag{2.21}$$

The class of all augmented Lie racks thus forms a category **LieRackA** with the obvious forgetful functor F : **LieRackA** $\to$ **LieRack**. Note that the trivial Lie rack structure of a pointed manifold (M, e) comes from an augmented Lie rack over the trivial Lie group $G = \{e\}$.

3 Dirty integration of Leibniz algebras

3.1 The main theorem

Let $\big(\mathfrak{h}, p, \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \dot{\rho}\big)$ be an augmented Leibniz algebra, let $\mathfrak{g}'$ be the Lie ideal $p(\mathfrak{h})$ of $\mathfrak{g}$, and let $\mathfrak{z} := \mathrm{Ker}(p)$ which –we recall– is a two-sided ideal of the Leibniz algebra $\mathfrak{h}$ lying in the left centre of $\mathfrak{h}$. Let furthermore G (resp. G') be a connected simply connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$ (resp. $\mathfrak{g}'$). Since G is connected and simply connected, its adjoint represention Ad_G preserves the ideal $\mathfrak{g}'$ of its Lie algebra $\mathfrak{g}$, whence there is a Lie group homomorphism $g \mapsto \mathrm{A}'_g$ of G into $\mathrm{Aut}_0(\mathfrak{g}')$, the component of the identity of the Lie group of all automorphisms of the Lie algebra $\mathfrak{g}'$. Since this latter Lie group is well known to be isomorphic to $\mathrm{Aut}_0(G')$, the connected component of the identity of the topological group of all Lie group automorphisms of G' (which also is a Lie group), see e.g. [5] for details, there is a unique Lie group homomorphism

$$\mathrm{I}' : G \to \mathrm{Aut}_0(G') : g \mapsto \big(g' \mapsto \mathrm{I}'_g(g')\big)$$

such that $T_e\mathrm{I}'_g = \mathrm{A}_g$ for all $g \in G$. Moreover, the injection $\mathfrak{g}' \to \mathfrak{g}$ induces a unique immersive Lie group homomorphism $\iota : G' \to G$ whose image is an analytic normal subgroup of G whence G acts on $\iota(G')$ by conjugations, and we have for all $g \in G$ and all $g' \in G'$

$$(\iota \circ \mathrm{I}'_g)(g') = g\iota(g')g^{-1}.$$

Next, let $\rho : G \times \mathfrak{h} \to \mathfrak{h}$ be the unique representation of G on $\mathfrak{h}$ such that for all $\xi \in \mathfrak{g}$ and $x \in \mathfrak{h}$

$$\frac{d}{dt}\rho_{\exp(t\xi)}(x)\Big|_{t=0} = \dot{\rho}_\xi(x). \tag{3.1}$$

We get for all $g \in G$ and $x \in \mathfrak{h}$:

$$p\big(\rho_g(x)\big) = \mathrm{A}'_g\big(p(x)\big). \tag{3.2}$$

The main theorem of this article reads:

Theorem 3.1 *With the above hypotheses and notations we have the following:*

1. *There is a smooth map $\mathsf{s} : G' \to \mathfrak{g}'$ having the following properties:*

$$\begin{aligned} \mathsf{s}(e) &= 0, & (3.3)\\ T_e\mathsf{s} &= \mathrm{id}_{\mathfrak{g}'}, & (3.4)\\ \forall\, g \in G,\ \forall\, g' \in G' : \quad \mathsf{s}\big(\mathrm{I}'_g(g')\big) &= \mathrm{A}'_g\big(\mathsf{s}(g')\big). & (3.5)\end{aligned}$$

2. *Consider $p : \mathfrak{h} \to \mathfrak{g}'$ as a fibre bundle over $\mathfrak{g}'$ (it is an* ***affine*** *bundle with typical fibre $\mathfrak{z}$ over $\mathfrak{g}'$), and form the pulled-back fibre bundle*

$$M := \mathsf{s}^*\mathfrak{h} = \big\{(x, g') \in \mathfrak{h} \times G' \mid p(x) = \mathsf{s}(g')\big\} \stackrel{\phi}{\to} G' \tag{3.6}$$

over G' having $(0, e_{G'})$ as a distinguished point. There is a canonical G-action ℓ on M induced by ρ on $\mathfrak{h}$ and by I' on G' such that $\big((M, (0, e_{G'}), \iota \circ \phi, G, \ell\big)$ is an augmented Lie rack.

3. *The induced Leibniz algebra structure on the tangent space $T_{(0,e_{G'})}M$ is isomorphic to $\big(\mathfrak{h}, [\ ,\ ]_{\mathfrak{h}}\big)$.*

4. *In the particular case $\mathfrak{g} = \mathfrak{g}'$ and $G = G'$ the above construction gives a surjective projection $M \to G = G'$. If furthermore $\mathfrak{z} = \{0\}$ the above construction reduces to the usual conjugation Lie rack on $G = G'$.*

3.2 Proof of the main theorem

1. According to Proposition A.3 (which we separately show further down in the Appendix) there are two open neighbourhoods $\mathcal{U}'_{(\pi/2)\mathbf{i}} \subset \mathcal{U}'_{\pi\mathbf{i}}$ of $0 \in \mathfrak{g}'$ which are both $\mathrm{Aut}_0(\mathfrak{g}')$-invariant and on which the restriction of the exponential map $\exp_{G'}$ is a diffeomorphism onto the $\mathrm{Aut}_0(G')$-invariant open neighbourhoods $\mathcal{V}'_{(\pi/2)\mathbf{i}} \subset \mathcal{V}'_{\pi\mathbf{i}}$ of the unit element e' of G'. Moreover, again by Proposition A.3 there is an $\mathrm{Aut}_0(\mathfrak{g}')$-invariant bump function $\gamma' : \mathfrak{g}' \to [0, 1]$ whose support is in $\mathcal{U}'_{\pi\mathbf{i}}$ and which is equal to 1 on $\mathcal{U}'_{(\pi/2)\mathbf{i}}$. Let us define the following map $\mathsf{s} : G' \to \mathfrak{g}'$ by

$$\mathsf{s}(g') := \begin{cases} \gamma'\big(\exp_{G'}^{-1}(g')\big)\exp_{G'}^{-1}(g') & \forall\, g' \in \mathcal{V}'_{\pi\mathbf{i}} \\ 0 & \forall\, g' \notin \mathcal{V}'_{\pi\mathbf{i}} \end{cases} \tag{3.7}$$

It is clear that s is a well-defined smooth map $G' \to \mathfrak{g}'$. Moreover $\mathsf{s}(e_{G'}) = 0$ by the properties of the exponential map, and (setting $e = e_{G'}$) for all $\zeta \in \mathfrak{g}' = T_eG'$

$$\begin{aligned} T_e\mathsf{s}(\zeta) &= \frac{d}{ds}\left(\gamma'\big(\exp_{G'}^{-1}(\exp_{G'}(s\zeta))\big)\exp_{G'}^{-1}(\exp_{G'}(s\zeta))\right)|_{s=0} \\ &= \frac{d}{ds}\left(s\zeta\right)|_{s=0} = \zeta \end{aligned}$$

because the bump function γ' is constant equal to 1 near 0. Hence $T_e\mathsf{s} = \mathrm{id}_{\mathfrak{g}'}$. Finally, since for each $g \in G$ and $\zeta \in \mathfrak{g}'$

$$\mathrm{I}'_g\big(\exp_{G'}(\zeta)\big) = \exp_{G'}\left(\mathrm{A}'_g(\zeta)\right)$$

and since γ' is invariant under the action $g \mapsto \mathrm{A}'_g$ of G on $\mathfrak{g}'$ we get for all $g \in G$ and $g' \in G'$

$$\mathsf{s}\big(\mathrm{I}'_g(g')\big) = \mathrm{A}'_g\big(\mathsf{s}(g')\big),$$

proving the first statement of the theorem.

2. Since $p : \mathfrak{h} \to \mathfrak{g}'$ is a surjective linear map, it is a surjective submersion whose fibre over $0 \in \mathfrak{g}'$ is equal to $\mathrm{Ker}(p) = \mathfrak{z}$, and whose fibre over any $\zeta \in \mathfrak{g}'$ is the affine subspace $p^{-1}(\{\zeta\})$ of $\mathfrak{h}$. Choosing any vector space complement $\mathfrak{b}$ to $\mathfrak{z}$ in $\mathfrak{h}$ leads to differential geometric trivialization over the global chart domain $\mathfrak{g}'$ of $\mathfrak{g}'$. Hence $p : \mathfrak{h} \to \mathfrak{g}'$ is a fibre bundle over $\mathfrak{g}'$ with typical fibre $\mathfrak{z}$, and therefore the pull-back $\mathsf{s}^*\mathfrak{h} = M$ is a well-defined fibre bundle over G' with typical fibre $\mathfrak{z}$. Recall that the projection $\phi : M \to G'$ is given by the restriction of the projection on the second factor $\mathfrak{h} \times G' \to G'$ to the submanifold $M \subset \mathfrak{h} \times G'$.
Since $\mathsf{s}(e_{G'}) = 0 = p(0)$ it follows that the point $(0, e_{G'})$ is in M, and clearly $\phi(0, e_{G'}) = e_{G'}$.
There is a canonical diagonal G-action $\hat{\ell}$ on $\mathfrak{h} \times G'$ defined by $\hat{\ell}_g(x, g') = \big(\rho_g(x), \mathrm{I}'_g(g')\big)$. As for any $(x, g') \in M$ we have by definition $p(x) = \mathsf{s}(g')$, we get

$$\mathsf{s}\big(\mathrm{I}'_g(g')\big) = \mathrm{A}'_g\big(\mathsf{s}(g')\big) = \mathrm{A}'_g\big(p(x)\big) = p\big(\rho_g(x)\big)$$

proving that for any $(x, g') \in M$ the point $\hat{\ell}_g(x, g') \in M$, whence $\hat{\ell}$ restricts to a well-defined G-action ℓ on M. Clearly $\ell_g(0, e_{G'}) = (0, e_{G'})$. Moreover, for any $(x, g') \in M$ and $g \in G$ we have

$$\begin{aligned} (\iota \circ \phi)\big(\ell_g(x, g')\big) &= \iota\Big(\phi\big(\rho_g(x), \mathrm{I}'_g(g')\big)\Big) = \iota\big(\mathrm{I}'_g(g')\big) \\ &= g\iota(g')g^{-1} = g\big((\iota \circ \phi)(x, g')\big)g^{-1}, \end{aligned}$$

showing that $\big((M, (0, e_{G'}), \iota \circ \phi, G, \ell\big)$ is an augmented Lie rack.

3. According to the definition of the pull-back, the tangent space of M at $(e_{G'}, 0)$ is given by all the pairs $(x, \zeta) \in \mathfrak{h} \times \mathfrak{g}' = T_0\mathfrak{h} \times T_{e_{G'}}G'$ such that

$$T_{e_{G'}}\mathsf{s}(\zeta) = p(x), \quad \text{hence } \zeta = p(x)$$

because of $T_{e_{G'}}\mathsf{s} = \mathrm{id}_{\mathfrak{g}'}$. It follows that the linear map $\theta_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h} \times \mathfrak{g}'$ given by

$$\theta_{\mathfrak{h}}(x) = \big(x, p(x)\big)$$

is an isomorphism of the vector space $\mathfrak{h}$ onto the tangent space $T_{(e_{G'},0)}M$. We get for all $g \in G$ and for all $y \in \mathfrak{h}$

$$T_{(e_{G'},0)}\ell_g\big(y, p(y)\big) \;=\; \frac{d}{dt}\Big(\rho_g(ty), \mathrm{I}'_g\big(\exp\big(tp(y)\big)\big)\Big)\Big|_{t=0} = \Big(\rho_g(y), \mathrm{A}'_g\big(p(y)\big)\Big).$$

Now for all $x, y \in \mathfrak{h}$, we get for the Leibniz bracket on the tangent space $T_{(e_{G'},0)}M$ of the augmented Lie rack $\big((M, (0, e_{G'}), \iota \circ \phi, G, \ell\big)$

$$\begin{aligned}
\big[\theta_{\mathfrak{h}}(x), \theta_{\mathfrak{h}}(y)\big] &= \Big[\big((x, p(x)\big), \big(y, p(y)\big)\Big] \\
&= \frac{d}{dt} T_{(e_{G'},0)}\ell_{\exp\big(tp(x)\big)}\big(y, p(y)\big)\Big|_{t=0} \\
&= \frac{d}{dt}\Big(\rho_{\exp\big(tp(x)\big)}(y), \mathrm{A}'_{\exp\big(tp(x)\big)}\big(p(y)\big)\Big)\Big|_{t=0} \\
&= \Big([x,y]_{\mathfrak{h}}, \big[p(x), p(y)\big]_{\mathfrak{g}'}\Big) = \Big([x,y]_{\mathfrak{h}}, p\big([x,y]_{\mathfrak{h}}\big)\Big) = \theta_{\mathfrak{h}}\big([x,y]_{\mathfrak{h}}\big).
\end{aligned}$$

showing that the induced Leibniz structure form the augmented Lie rack is isomorphic with the Leibniz bracket $[\ ,\ ]_{\mathfrak{h}}$ on $\mathfrak{h}$.

4. This is immediate. □

A Automorphism invariant chart domains for the exponential map of a Lie group

A.1 Zeros of Polynomials

All the results in this section are classical, and the methods had been inspired e.g. by the article [11].

Let $\mathbb{C}[\lambda]^1_n$ denote the space of all *monic complex polynomials of degree* n. Since every such polynomial f is of the general form

$$f(\lambda) = \lambda^n + \sum_{r=1}^{n} a_r \lambda^{n-r} \tag{A.1}$$

with $a := (a_1, \ldots, a_n) \in \mathbb{C}^n$ it is clear that $\mathbb{C}[\lambda]_n^1$ is an affine space with associated complex vector space $\mathbb{C}^n$, whence $\mathbb{C}[\lambda]_n^1$ is homeomorphic to $\mathbb{C}^n$. We put the norm $||f|| = ||a|| := \sum_{j=1}^n |a_j|$ on it. Consider now the map $\mathcal{T} : \mathbb{C}^n \to \mathbb{C}[\lambda]_n^1$ given by

$$\mathcal{T}(z)(\lambda) := (\lambda - z_1) \cdots (\lambda - z_n) \tag{A.2}$$

so that the zeros of $\mathcal{T}(z)$ are given by $z_1, \ldots, z_n \in \mathbb{C}$ (where there can be repetitions, i.e. multiple roots). There is the well-known classical formula expressing the coefficients of $\mathcal{T}(z)$ in terms of Newton's elementary symmetric polynomials, i.e.

$$\mathcal{T}(z)(\lambda) = \lambda^n + \sum_{r=1}^{n} (-1)^r \Big(\sum_{1 \leq i_1 < \cdots < i_r \leq n} z_{i_1} \cdots z_{i_r} \Big) \lambda^{n-r}. \tag{A.3}$$

It follows that $\mathcal{T}$ is a complex analytic map, hence continuous. Moreover the Fundamental Theorem of Algebra states that $\mathcal{T}$ is surjective, and elementary algebra of polynomials shows that

$$\mathcal{T}(z) = \mathcal{T}(z') \quad \text{if and only if} \quad \exists\, \sigma \in S_n \text{ such that } z' = z.\sigma \tag{A.4}$$

where a permutation σ in the symmetric group S_n acts from the right on $\mathbb{C}^n$ in the usual way, i.e. $(z_1, \ldots, z_n).\sigma = (z_{\sigma(1)}, \ldots, z_{\sigma(n)})$. Let $\mathbb{C}^n/S_n$ be the space of all S_n-orbits equipped with the quotient topology, and let $\pi_n : \mathbb{C}^n \to \mathbb{C}^n/S_n$ be the canonical projection. Eqn (A.4) implies that the map $\mathcal{T}$ descends to a well-defined continuous map $T : \mathbb{C}^n/S_n \to \mathbb{C}[\lambda]_n^1$ which is bijective. It is classical, but a bit less well-known that T *is a homeomorphism*:

Proposition A.1 *With the above notations we have the following:*

1. *The map $\mathcal{T}$ is a closed continuous map.*
2. *The map T is a homeomorphism, and $\mathcal{T}$ is also an open map.*

Proof: We need first the following elementary estimate: Consider a monic polynomial $f \in \mathbb{C}[\lambda]_n^1$ in the form (A.1), and let μ be a root of f, then in case $|\mu| \geq 1$ we get from the equation $f(\mu) = 0$

$$\mu^n = -\sum_{r=1}^{n} a_r \mu^{n-r} \quad \text{hence} \quad |\mu|^n \leq \sum_{r=1}^{n} |a_r| |\mu|^{n-r} \quad \text{hence} \quad |\mu| \leq ||a||$$

where we have multiplied the second term by $|\mu|^{1-n} > 0$ and used the fact that $|\mu|^{1-r} \leq 1$ for all $1 \leq r \leq n$. This estimate implies the weaker estimate

$$|\mu| \leq \max\{1, ||a||\} \tag{A.5}$$

which clearly also holds for the other case $|\mu| \leq 1$.

1. Let F be a closed subset of $\mathbb{C}^n$, and consider a sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{T}(F)$ converging to a monic polynomial $f \in \mathbb{C}[\lambda]_n^1$ (where we use the above norm $||\ ||$ on the coefficients to define the convergence). We have to show that there is a $z \in F$ such that $f = \mathcal{T}(z)$. First, since each $f_k \in \mathcal{T}(F)$, there is a sequence $(z^{(k)})_{k\in\mathbb{N}}$ of elements of F such that for all $k \in \mathbb{N}$, we have $f_k = \mathcal{T}(z^{(k)})$. Denote by $a^{(k)} \in \mathbb{C}^n$ the coeffients of the polynomial f_k, and let $a \in \mathbb{C}^n$ be the vector of coeffients of the polynomial f. By hypothesis $a^{(k)} \to a$ when $k \to \infty$, hence the sequence of norms $||a^{(k)}||$ converges to $||a||$ and is therefore bounded. But the above estimate (A.5) implies that the sequence $(z^{(k)})_{k\in\mathbb{N}}$ of zero vectors of f_k is a bounded subset of $\mathbb{C}^n$. By the Bolzano-Weierstrass Theorem, there is a subsequence $\big(z^{(k_l)}\big)_{l\in\mathbb{N}}$ of the above sequence which converges to $z \in \mathbb{C}^n$. Since all the vectors $z^{(k_l)}$ are in the closed set F, it follows that the limit z lies also in F. By the continuity of $\mathcal{T}$ it follows that

$$f_{k_l} = \mathcal{T}\big(z^{(k_l)}\big) \to \mathcal{T}(z) \quad (l \to \infty),$$

and since the subsequence $(f_{k_l})_{l\in\mathbb{N}}$ converges to f, it follows by the uniqueness of limits that $f = \mathcal{T}(z)$, and $\mathcal{T}$ is a closed map.

2. We shall show that T is a closed map which will imply that its inverse is continuous: Let F be a closed subset of $\mathbb{C}^n/S_n$. Then $\pi_n\big(\pi_n^{-1}(F')\big) = F'$ because π_n is surjective, and

$$T(F') = T\Big(\pi_n\big(\pi_n^{-1}(F')\big)\Big) = \mathcal{T}\big(\pi_n^{-1}(F')\big)$$

is a closed subset of $\mathbb{C}[\lambda]_n^1$, because $\pi_n^{-1}(F')$ is a closed subset of $\mathbb{C}^n$ thanks to the continuity of π_n, and because $\mathcal{T}$ is a closed map. In order to prove the second half, observe that π_n is an open map: If $U \subset \mathbb{C}^n$ is open, then $\pi_n^{-1}\big(\pi_n(U)\big) = \bigcup_{\sigma\in S_n} U.\sigma$ is a union of the open sets $U.\sigma$ of $\mathbb{C}^n$, and therefore open whence $\pi_n(U)$ is open by definition of the quotient topology. Since T is a homeomorphism, it is an open map (its inverse is continuous), and therefore $\mathcal{T} = T \circ \pi_n$ is open as a composition of open maps. $\square$

The following corollary will be important in the proof of Theorem 3.1:

Corollary A.1 *Let U be an open and F be a closed subset of $\mathbb{C}$. Then for each positive integer n, the set of all those monic polynomials of degree n having all their roots in U (resp. in F) is an open (resp. closed) subset of $\mathbb{C}[\lambda]_n^1$.*

Proof: Apply the map $\mathcal{T}$ to the open (resp. closed) subset $U^{\times n}$ (resp. $F^{\times n}$) of $\mathbb{C}^n$ and use the fact that $\mathcal{T}$ is an open and closed map. $\square$

A.2 Injectivity domains of the exponential of linear maps

Let E be a real vector space of dimension n, and let $\mathcal{A} = \mathrm{Hom}_{\mathbb{R}}(E, E)$ be the real vector space of all $\mathbb{R}$-linear maps $E \to E$. Let $g : \mathbb{C} \to \mathbb{C}$ be a holomorphic function whose power series expansion around 0 has real coefficients, $g(z) = \sum_{r=0}^{\infty} g_r z^r$, $g_r \in \mathbb{R}$ for all $r \in \mathbb{N}$. Then for any $X \in \mathcal{A}$ the series $g(X) := \sum_{r=0}^{\infty} g_r X^r$ is well-known to converge to a well-defined element in $\mathcal{A}$ (where we write XY for the composition of linear maps $X \circ Y$, and set $X^0 := I := \mathrm{id}_E$). An important example is the exponential function

$$\exp : \mathcal{A} \to \mathcal{A} : X \mapsto \exp(X) := e^X := \sum_{r=0} \frac{1}{r!} X^r, \tag{A.6}$$

but also the series

$$h(X) := \frac{I - e^{-X}}{X} := \sum_{r=0}^{\infty} \frac{(-1)^r}{(r+1)!} X^r, \tag{A.7}$$

related to the derivative of the exponential function. For any strictly positive real number τ, let $S_{\tau\mathbf{i}}$ be the open strip

$$S_{\tau\mathbf{i}} := \{z = \alpha + \mathbf{i}\beta \in \mathbb{C} \mid |\beta| < \tau\}, \tag{A.8}$$

and let

$$\mathcal{A}_{\tau\mathbf{i}} := \{X \in \mathcal{A} \mid \text{all the eigenvalues of } X \text{ are in } S_{\tau\mathbf{i}}\}. \tag{A.9}$$

Proposition A.2 *For any positive integer n, we have the following:*

1. *For any positive real number $\tau > 0$, the subset $\mathcal{A}_{\tau\mathbf{i}}$ is an open subset of $\mathcal{A}$.*
2. *For all $X \in \mathcal{A}_{2\pi\mathbf{i}}$, the linear map $h(X)$ is invertible.*
3. *The restriction of the exponential map to the open subset $\mathcal{A}_{\pi\mathbf{i}}$ of $\mathcal{A}$ is injective.*

Proof: For any $X \in \mathcal{A}$, recall the Jordan decomposition

$$X = X_S + X_N$$

into its semisimple part $X_S \in \mathcal{A}$ and its nilpotent part $X_N \in \mathcal{A}$ where of course X_S is diagonalizable over $\mathbb{C}$ and X_N is nilpotent, and X_S and X_N are polynomials of X, hence commute with each other. The Jordan decomposition is well-known to be unique. Now take any holomorphic function $g : \mathbb{C} \to \mathbb{C}$ whose power series expansion around 0 has real coefficients. It follows that the function of two complex variables

$$(z, w) \mapsto \frac{g(z+w) - g(z)}{w}$$

is a well-defined holomorphic function $\mathbb{C}^2 \to \mathbb{C}$ (with $(z, 0)$ sent to $f'(z)$), whence the decomposition

$$g(X) = g\big(X_S + X_N\big) = g\big(X_S\big) + X_N \frac{g\big(X_S + X_N\big) - g\big(X_S\big)}{X_N}$$

is well-defined. If $\mu_1, \ldots, \mu_n \in \mathbb{C}$ are the eigenvalues of X hence of X_S, then $g(\mu_1), \ldots, g(\mu_n)$ are the eigenvalues of $g\big(X_S\big)$ (with the same eigenvectors) whence $g\big(X_S\big)$ is obviously semisimple, and since the second summand is equal to the product of the nilpotent map X_N and another map commuting with X_N, it follows that the preceding equation gives us the Jordan decomposition of $g(X)$:

$$g(X)_S = g\big(X_S\big), \quad \text{and} \quad g(X)_N = X_N \frac{g\big(X_S + X_N\big) - g\big(X_S\big)}{X_N} \tag{A.10}$$

1. Consider the map $\chi : \mathcal{A} \to \mathbb{C}[\lambda]_n^1$ which sends each $X \in \mathcal{A}$ to its characteristic polynomial $\lambda \mapsto \det(\lambda I - X)$. This map is clearly polynomial in the coefficients of X, and therefore continuous. By Corollary A.1, the set of all monic complex polynomials all of whose roots are in $S_{\tau\mathbf{i}}$ is the open set $\mathcal{T}(S_{\tau\mathbf{i}}^{\times n})$ of $\mathbb{C}[\lambda]_n^1$, and hence $\mathcal{A}_{\tau\mathbf{i}} = \chi^{-1}\big(\mathcal{T}(S_{\tau\mathbf{i}}^{\times n})\big)$ is an open subset of $\mathcal{A}$.
2. We get that

$$\det\big(h(X)\big) = \det\big(h(X)_S\big) = \det\big(h\big(X_S\big)\big).$$

Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of X_S (repetitions due to multiplicities may occur). Then

$$\det\big(h\big(X_S\big)\big) = h(\mu_1) \cdots h(\mu_n).$$

In case $\mu_r = 0$, then $h(\mu_r) = h(0) = 1 \neq 0$. Let $\mu_r = \alpha + \beta\mathbf{i} \neq 0$. Then if $h(\mu_r) = 0$, we get $\mu_r h(\mu_r) = 0$, hence

$$0 = 1 - e^{-\alpha - \beta\mathbf{i}}$$

which is the case iff $\alpha_r = 0$ and $\beta = 2k\pi$ for some integer k. Since by hypothesis $|\beta| < 2\pi$, we would necessarily have $\beta_r = 0$, contradiction! Hence all the complex numbers $h(\mu_r)$ are different from zero, and therefore $h(X)$ is invertible.
3. Let $X, Y \in \mathcal{A}$ such that $e^X = e^Y$. Then we get

$$e^{X_S} = \big(e^X\big)_S = \big(e^Y\big)_S = e^{Y_S}.$$

Let $\mu_1, \ldots, \mu_n \in \mathbb{C}$ be the eigenvalues of X_S and let $\nu_1, \ldots, \nu_n \in \mathbb{C}$ be the eigenvalues of Y_S (with possible repetitions due to multiple eigenvalues). Then both

$e^{\mu_1}, \ldots, e^{\mu_n}$ and $e^{\nu_1}, \ldots, e^{\nu_n}$ are the eigenvalues of $e^{X_S} = e^{Y_S}$, and we can assume after a possible permutation that

$$e^{\mu_1} = e^{\nu_1}, \;\; \ldots, \;\; e^{\mu_n} = e^{\nu_n}$$

Decomposing into real and imaginary part, i.e. $\mu_s = \alpha_s + \beta_s \mathbf{i}$ and $\nu_t = \gamma_t + \delta_t \mathbf{i}$ for all integers $1 \leq s, t \leq n$, we can conclude that there exist integers $k_1, \ldots, k_n$ such that for all $1 \leq s \leq n$

$$\gamma_s = \alpha_s, \quad \text{and} \quad \delta_s = \beta_s + 2\pi k_s.$$

Hence if both X and Y (and thus X_S, Y_S) are in $\mathcal{A}_{\mathbf{i}\pi}$, then

$$|\delta_s - \beta_s| \leq |\delta_s| + |\beta_s| < 2\pi$$

and if $e^{X_S} = e^{Y_S}$, then X_S and Y_S have the same eigenvalues and the same multiplicities. Let $\hat{\mu}_1, \ldots, \hat{\mu}_k \in \mathbb{C}$ be the k pairwise different eigenvalues of X_S and of Y_S. Then we can write

$$X_S = \sum_{r=1}^{k} \hat{\mu}_r P_r, \quad Y_S = \sum_{r=1}^{k} \hat{\mu}_r Q_r$$

where $P_r, Q_r : E \otimes_{\mathbb{R}} \mathbb{C} \to E \otimes_{\mathbb{R}} \mathbb{C}$ are the projections on the eigenspace associated to the eigenvalue $\hat{\mu}_r$ (recall that $P_r P_s = \delta_{rs} P_r$ and $Q_r Q_s = \delta_{rs} Q_r$ for all $1 \leq r, s \leq k$). We get

$$\sum_{r=1}^{k} e^{\hat{\mu}_r} P_r = e^{X_S} = e^{Y_S} = \sum_{r=1}^{k} e^{\hat{\mu}_r} Q_r,$$

and since all the k complex numbers $e^{\hat{\mu}_1}, \ldots, e^{\hat{\mu}_k}$ are pairwise different by the fact that $\hat{\mu}_1, \ldots, \hat{\mu}_k \in S_{\pi \mathbf{i}}$, it follows that for each $1 \leq r \leq k$ the projection P_r is the unique projection of $e^{X_S} = e^{Y_S}$ on the generalized eigenspace associated to the eigenvalue $e^{\hat{\mu}_r}$ whence $P_r = Q_r$, and therefore $X_S = Y_S$. Finally we have –according to eqn (A.10)–

$$\begin{aligned} e^X (I - e^{-X_N}) &= e^X X_N h(X_N) = X_N \frac{e^{X_S + X_N} - e^{X_S}}{X_N} = \left(e^X\right)_N \\ &= \left(e^Y\right)_N = Y_N \frac{e^{Y_S + Y_N} - e^{Y_S}}{Y_N} = e^Y Y_N h(Y_N) = e^Y (I - e^{-Y_N}) \end{aligned}$$

whence –since $e^X = e^Y$– it follows that $e^{X_N} = e^{Y_N}$. Since e^{X_N} is a polynomial in X_N, and $e^{X_N} - I$ is nilpotent, we may apply the logarithmic series $\log(1+z) = \sum_{r=0}^{\infty} (-z)^r/(r+1)$ to $e^{X_N} - I = e^{Y_N} - I$ to get $X_N = Y_N$. It follows that

$$X = X_S + X_N = Y_S + Y_N = Y,$$

and the restriction of the exponential function to $\mathcal{A}_{\pi \mathbf{i}}$ is injective. $\square$

A.3 Automorphism invariant chart domains for the exponential map of a Lie group

The results and techniques from this section are largely due to [7] and Definition 4.8 and Proposition 4.9 from [3].

Let $(\mathfrak{g}, [\ ,\ ])$ be a finite-dimensional real Lie algebra, and let G be a connected, simply connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$ (recall that G is unique up to isomorphism of Lie groups). Moreover, let $\mathrm{Aut}_0(G)$ be the connected component of the identity of the topological group (w.r.t. the compact-open topology) of all smooth Lie group automorphisms of G. Then it is known (see e.g. [5]) that $\mathrm{Aut}_0(G)$ is a connected Lie group isomorphic to the connected component of the identity, $\mathrm{Aut}_0(\mathfrak{g})$, of the Lie group $\mathrm{Aut}(\mathfrak{g})$ of all automorphisms of the Lie algebra $\mathfrak{g}$ where the canonical map $\mathrm{Aut}_0(G) \to \mathrm{Aut}_0(\mathfrak{g})$ sending an automorphism to its derivative at the unit element of e is known to be an isomorphism of Lie groups. Recall that the group I_G of all conjugations $g' \mapsto I_g(g') := gg'g^{-1}$ is a normal analytic subgroup of $\mathrm{Aut}_0(G)$ which is isomorphic (by the above canonical map) to the adjoint Lie group, Ad_G, of all adjoint represention Ad_g, $g \in G$, which is a normal analytic subgroup of $\mathrm{Aut}(\mathfrak{g})$. For any strictly positive real number τ, we set

$$\mathcal{U}_{\tau\mathbf{i}} := \{\xi \in \mathfrak{g} \mid \text{all the eigenvalues of } \mathrm{ad}_\xi \text{ lie in } S_{\tau\mathbf{i}}\}, \tag{A.11}$$

see eqn (A.8) for the definition of $S_{\tau\mathbf{i}}$, and recall for any $\xi \in \mathfrak{g}$ the definition of its adjoint representation $\mathrm{ad}_\xi : \eta \mapsto [\xi, \eta]$ for all $\eta \in \mathfrak{g}$. Furthermore, set

$$\mathcal{V}_{\tau\mathbf{i}} := \exp\left(\mathcal{U}_{\tau\mathbf{i}}\right) \subset G. \tag{A.12}$$

We shall prove the following

Proposition A.3 *With the above definitions and notations, we have the following: Let τ be any real number such that $0 < \tau \leq \pi$.*

1. *The subset $\mathcal{U}_{\tau\mathbf{i}}$ is an open $\mathrm{Aut}_0(\mathfrak{g})$-invariant neighbourhood of $0 \in \mathfrak{g}$ such that for all $\xi \in \mathcal{U}_{\tau\mathbf{i}}$ and for all η in the nilradical of $\mathfrak{g}$, the element $\xi + \eta$ still lies in $\mathcal{U}_{\tau\mathbf{i}}$.*

2. *The restriction of the exponential map to $\mathcal{U}_{\tau\mathbf{i}}$ is a diffeomorphism onto $\mathcal{V}_{\tau\mathbf{i}}$ which is an open $\mathrm{Aut}_0(G)$-invariant neighbourhood of the unit element $e \in G$.*

3. *Let τ' be any real number such that $0 < \tau' < \tau \leq \pi$. Then there is a smooth $\mathrm{Aut}_0(\mathfrak{g})$-invariant real-valued function $\gamma : \mathfrak{g} \to \mathbb{R}$ such that*

(a) $\gamma(\mathfrak{g}) \subset [0,1]$,

(b) For all $\xi \in \mathcal{U}_{\tau'\mathbf{i}}$*, we have* $\gamma(\xi) = 1$,

(c) The support of γ *is contained in* $\mathcal{U}_{\tau\mathbf{i}}$.

Proof: 1. Let $n := \dim(\mathfrak{g})$ and define the map $\tilde{\chi} : \mathfrak{g} \to \mathbb{C}[\lambda]^1_n$ for all $\xi \in \mathfrak{g}$ by

$$\tilde{\chi}(\xi)(\lambda) := \chi\big(\mathrm{ad}_\xi\big)(\lambda)$$

where we have written χ for the characteristic polynomial. $\tilde{\chi}$ clearly is a polynomial map, hence a smooth map. Next, for all $\vartheta \in \mathrm{Aut}_0(\mathfrak{g})$ and all $\xi \in \mathfrak{g}$, we get in $\mathcal{A} = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{g})$

$$\begin{aligned} \tilde{\chi}\big(\vartheta(\xi)\big)(\lambda) &= \chi\big(\mathrm{ad}_{\vartheta(\xi)}\big)(\lambda) = \det\big(\lambda I - \mathrm{ad}_{\vartheta(\xi)}\big) = \det\big(\lambda I - \vartheta \mathrm{ad}_\xi \vartheta^{-1}\big) \\ &= \det\big(\lambda I - \mathrm{ad}_\xi\big) = \chi\big(\mathrm{ad}_\xi\big)(\lambda) = \tilde{\chi}(\xi)(\lambda), \end{aligned}$$

whence $\tilde{\chi}$ is $\mathrm{Aut}_0(\mathfrak{g})$-invariant, and thus the set

$$\mathcal{U}_{\tau\mathbf{i}} = \tilde{\chi}^{-1}\big(\mathcal{T}(S^{\times n}_{\tau\mathbf{i}})\big)$$

is an open $\mathrm{Aut}_0(\mathfrak{g})$-invariant subset of $\mathfrak{g}$. Let $\mathfrak{n} \subset \mathfrak{g}$ be the nilradical of $\mathfrak{g}$, i.e. the largest nilpotent ideal of $\mathfrak{g}$. Then it is well-known that there is a positive integer N such that for any $\eta_1, \ldots, \eta_N \in \mathfrak{n}$, the product of linear maps $\mathrm{ad}_{\eta_1} \cdots \mathrm{ad}_{\eta_N}$ vanishes. More generally, let $W_{N,m}$ let be any product of $N+m$ adjoint representations ad_{ξ_i} with $1 \leq i \leq N+m$ and $\xi_i \in \mathfrak{g}$ for all $1 \leq i \leq N+m$ such that N adjoint representations are of the type ad_{η_j} with $\eta_1, \ldots, \eta_N \in \mathfrak{n}$. It is easy to see by induction on the positive integer m that $W_{N,m} = 0$: Indeed, this is clear for $m = 0$, and the induction step is simple if the first or the last position of $W_{N,m+1}$ is not in $\mathrm{ad}_{\mathfrak{n}}$, because then $W_{N,m+1}$ contains $W_{N,m}$ as a factor and has to vanish. In the other cases, it is not hard to see that the N ad_{η_j}'s can all be moved to the right by getting commutators with the remaining ad_{ξ_i}'s which are adjoint representations of elements of the nilradical. Each commutator reduces the word length by maintaining the number N of elements in the nilradical, and the term vanishes by induction. The remaining final term has a factor $\mathrm{ad}_{\eta_1} \cdots \mathrm{ad}_{\eta_N}$ and also has to vanish, thus proving the induction. As a consequence, for any $\xi_1, \ldots, \xi_m \in \mathfrak{g}$, any integer $1 \leq s \leq m$ and for all $\eta \in \mathfrak{n}$, the linear map $\mathrm{ad}_{\xi_1} \cdots \mathrm{ad}_{\xi_s} \mathrm{ad}_\eta \mathrm{ad}_{\xi_{s+1}} \cdots \mathrm{ad}_{\xi_m}$ is nilpotent whence for all integers $1 \leq r \leq n$

$$\mathrm{trace}\Big(\big(\mathrm{ad}_{\xi+\eta}\big)^r\Big) = \mathrm{trace}\Big(\big(\mathrm{ad}_\xi\big)^r\Big)$$

implying –thanks to the Waring identities, see e.g. [1, p.430]–

$$\tilde{\chi}(\xi+\eta)(\lambda) = \det\big(\lambda I - \mathrm{ad}_\xi - \mathrm{ad}_\eta\big) = \det\big(\lambda I - \mathrm{ad}_\xi\big) = \tilde{\chi}(\xi)(\lambda)$$

which shows the invariance of $\tilde{\chi}$ under translation by elements in the nilradical which proves the statement.

2. For any $\xi, \eta \in \mathfrak{g}$ recall the formula for the derivative of the exponential map:

$$T_\xi \exp(\eta) = \frac{d}{dt}\exp(\xi + t\eta)\Big|_{t=0} = T_e L_{\exp(\xi)}\left(\frac{I - e^{\mathrm{ad}_\xi}}{\mathrm{ad}_\xi}(\eta)\right) = T_e L_{\exp(\xi)}\big(h\big(\mathrm{ad}_\xi\big)(\eta)\big)$$

where for any $g \in G$ $L_g : G \to G$ denotes the usual left multiplication map in G, and we have used the function h, see eqn (A.7). Since $T_e L_{\exp(\xi)} : \mathfrak{g} \to T_{\exp(\xi)}G$ is a linear isomorphism, it remains to check the linear map $h\big(\mathrm{ad}_\xi\big)$. But according to the second statement of Proposition A.2, it follows that $h\big(\mathrm{ad}_\xi\big)$ is a linear bijection for all $\xi \in \mathcal{U}_{\tau i}$. Hence the restriction of exp to $\mathcal{U}_{\tau i}$ is a local diffeomorphism. By the Inverse Function Theorem, the image $\mathcal{V}_{\tau i} = \exp\big(\mathcal{U}_{\tau i}\big)$ must be an open set of G containing e: Indeed, let $g \in \mathcal{V}_{\tau i}$. Then there is $\xi \in \mathcal{U}_{\tau i}$ with $g = \exp(\xi)$. There is an open neighbourhood of ξ (which we can choose to be in the open set $\mathcal{U}_{\tau i}$) on which the restriction of the exponential map is invertible, i.e. there is an open neighbourhood of g and a local inverse of exp. It follows that this second neighbourhood is still contained in $\mathcal{V}_{\tau i}$ whence $\mathcal{V}_{\tau i}$ is an open set of G containing $e = \exp(0)$. Next, let $\xi, \eta \in \mathcal{U}_{\tau i}$ such that $\exp(\xi) = \exp(\eta)$: It follows that

$$e^{\mathrm{ad}_\xi} = \mathrm{Ad}_{\exp(\xi)} = \mathrm{Ad}_{\exp(\eta)} = e^{\mathrm{ad}_\eta}.$$

Again the third statement of Proposition A.2 implies that

$$\mathrm{ad}_\xi = \mathrm{ad}_\eta \quad \text{hence } \exists\ \zeta \in \mathfrak{z}(\mathfrak{g}) \text{ such that } \eta = \xi + \zeta$$

where $\mathfrak{z}(\mathfrak{g}) = \mathrm{Ker}(\mathrm{ad})$ is the centre of $\mathfrak{g}$. It follows that

$$\exp(\xi) = \exp(\eta) = \exp(\xi + \zeta) = \exp(\xi)\exp(\zeta)$$

whence

$$\exp(\zeta) = e.$$

Since G is a connected simply connected Lie group, ζ has to vanish by the following beautiful argument of [7, Section 2.3]: Suppose that there is a nonvanishing $\zeta \in \mathfrak{z}(\mathfrak{g})$ such that $\exp(\zeta) = e$. It follows that

$$S := \{\exp(t\zeta) \in G \mid t \in \mathbb{R}\}$$

is a circle subgroup S in the connected component Z_0 of the centre Z of G. Consider the connected simply connected Lie group $G' = G \times G$. There is a 2-torus $S \times S$ in the connected component of the centre of G'. Let $\mathfrak{d}$ be a one-dimensional subspace in the 2-dimensional Lie algebra of $S \times S$ such that its image under the exponential map is a *dense* one-dimensional central subgroup $D \subset S \times S$. Since $\mathfrak{d}$ is in the centre of $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{g}$, the quotient Lie algebra $\mathfrak{g}'' := \mathfrak{g}'/\mathfrak{d}$ is well-defined. Let $q : \mathfrak{g}' \to \mathfrak{g}''$ be the canonical projection which is a homomorphism of Lie algebras. Let G'' be a connected simply connected Lie group whose Lie algebra is $\mathfrak{g}''$. Then there is a unique smooth Lie group homomorphism $\kappa : G' \to G''$ such that its derivative at the unit element (e, e) of G' is equal to q. It follows that κ is a surjective submersion. Now for all $\zeta' \in \mathfrak{d}$, we get

$$\kappa\big(\exp_{G'}(\zeta')\big) = \exp_{G''}\big(q(\zeta')\big) = \exp_{G''}\big(0\big) = e'',$$

whence D lies in the kernel of κ. The latter is a closed subgroup of G', and contains thus the closure of D which is equal to the torus $S \times S$. It follows that the kernel of κ is at least two-dimensional contradicting the fact that the dimension of G'' is equal to the dimension of G' minus 1. Therefore $\zeta = 0$, and the restriction of exp to $\mathcal{U}_{\tau\mathbf{i}}$ is injective, hence a diffeomorphism onto its image $\mathcal{V}_{\tau\mathbf{i}}$. Finally, let $\theta \in \mathrm{Aut}_0(G)$ and $g \in \mathcal{V}_{\tau\mathbf{i}}$. Then there is a unique $\xi \in \mathcal{U}_{\tau\mathbf{i}}$ with $\exp(\xi) = g$. We get

$$\theta(g) = \theta\big(\exp(\xi)\big) = \exp\big(T_e\theta(\xi)\big) \in \mathcal{V}_{\tau\mathbf{i}}$$

since $\mathcal{U}_{\tau\mathbf{i}}$ is invariant by the Lie algebra automorphism $T_e\theta$ whence $\mathcal{V}_{\tau\mathbf{i}}$ is invariant by Lie group automorphisms in $\mathrm{Aut}_0(G)$.
3. Consider the following subsets of $\mathbb{C}[\lambda]^1_n$:

$$\mathcal{W}_{\tau'\mathbf{i}} := \mathcal{T}\big(S^{\times n}_{\tau'\pi}\big), \quad \overline{\mathcal{W}}_{\tau'\mathbf{i}} := \mathcal{T}\big(\overline{S_{\tau'\pi}}^{\times n}\big), \quad \mathcal{W}_{\tau\mathbf{i}} := \mathcal{T}\big(S^{\times n}_{\tau\pi}\big)$$

where $\overline{S_{\tau'\pi}}$ is the closure of $S_{\tau'\pi}$, i.e. the closed strip of all those complex numbers whose imaginary part lies in the interval $[-\tau'\pi, \tau'\pi]$. The obvious inclusions $S_{\tau'\pi} \subset \overline{S_{\tau'\pi}} \subset S_{\tau\pi}$ imply the inclusions $\mathcal{W}_{\tau'\mathbf{i}} \subset \overline{\mathcal{W}}_{\tau'\mathbf{i}} \subset \mathcal{W}_{\tau\mathbf{i}}$ where $\mathcal{W}_{\tau'\mathbf{i}}$ and $\mathcal{W}_{\tau\mathbf{i}}$ are open subsets and $\overline{\mathcal{W}}_{\tau'\mathbf{i}}$ is closed (in fact, the closure of $\mathcal{W}_{\tau'\mathbf{i}}$). It follows that the two open sets $\mathcal{W}_{\tau\mathbf{i}}$ and $\mathbb{C}[\lambda]^1_n \setminus \overline{\mathcal{W}}_{\tau'\mathbf{i}}$ cover $\mathbb{C}[\lambda]^1_n$. Let $(\gamma', 1-\gamma')$ be a smooth partition of unity subordinate to the open cover $\big(\mathcal{W}_{\tau\mathbf{i}}, \mathbb{C}[\lambda]^1_n \setminus \overline{\mathcal{W}}_{\tau'\mathbf{i}}\big)$ of $\mathbb{C}[\lambda]^1_n$. It follows that all the values of the smooth real-valued function γ' lie in the interval $[0, 1]$, that its support is contained in $\mathcal{W}_{\tau\mathbf{i}}$, and that the support of $1 - \gamma'$ is contained in $\mathbb{C}[\lambda]^1_n \setminus \overline{\mathcal{W}}_{\tau'\mathbf{i}}$, whence $\gamma'(f) = 1$ for all $f \in \overline{\mathcal{W}}_{\tau'\mathbf{i}}$, in particular for all $f \in \mathcal{W}_{\tau'\mathbf{i}}$. It follows that the composed function $\gamma = \gamma' \circ \tilde{\chi} : \mathfrak{g} \to \mathbb{R}$ has all the properties in the statement since clearly $\mathcal{U}_{\tau\pi} = \tilde{\chi}^{-1}\big(\mathcal{W}_{\tau\pi}\big)$, $\mathcal{U}_{\tau'\pi} = \tilde{\chi}^{-1}\big(\mathcal{W}_{\tau'\pi}\big)$, and $\tilde{\chi}$ is $\mathrm{Aut}_0(\mathfrak{g})$-invariant. $\square$

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